INTRODUCTORY PUTNAM MEETING

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All relevant information can be found at

http://math.stanford.edu/~vakil/stanfordputnam.html

including (soon) an updated version of this handout.

1. Background

The William Lowell Putnam Mathematical Competition is the intercollegiate continent-wide mathematical competition. The sixty-second Putnam will be held on December 1. (If you can't take it at the regular time for religious reasons, this is no problem, but please let me know right away.)

Last year, Stanford placed seventh. (The top 5: Duke, MIT, Harvard, Caltech, Toronto.) Travis Kopp, who was a senior, won an honorable mention. The year before, we placed eighth, and had ten students in the top roughly 200.

The competition consists of a morning session (3 hours, 6 questions A1-A6), a pizza lunch (from the math department), and an afternoon session (3 hours, 6 more questions B1-B6). The A's are *very* roughly in order of difficulty, as are the B's.

Each question is worth 10 points, so there are 120 points total.

This is probably unlike any test or contest you've taken before. It is intended more as a challenge than a contest. Just under 3000 students will take it, and they are among the best and brightest in the continent. The median score will likely be 0, 1, or 2 out of 120; for the last two years, the median has been 0.

So these are hard problems, and the strategy is different. The challenge is to sit down for three hours, look over a list of six problems, and try to figure one out and write it up. They are hard not because they have many parts, or have lots of computation; they solutions are very short, but ingenious. For sample questions, see the attached 1998 competition. The median on this competition was 10. (It was only the second time since '88 that the median has been above 3).

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They are all proof questions, meaning that you have to not just give an answer, but explain why it's true in a rigorous manner, not just beyond a reasonable doubt.

Grading is quite strict. There are ten points per problem, the scores are almost all 0, 1, 2, 8, 9, or 10. 8 is essentially correct with small gaps, and 2 is for very serious progress. So don't try to just get part marks on many problems, because you won't. Instead, you try to figure out a problem, and then write it up very very well.

Stanford will name a team of three in advance.

Why it's worth writing the Putnam.

- for the challenge
- a different kind of thinking than homework problems, much more akin to mathematical research
- it's worth seeing what these problems are like
- (can help in applying to math grad school)
- perhaps most important: the way of thinking you pick up will make understanding more advanced mathematical ideas that much easier

2. What you have to do

1. Sign up if you *might* take it! Name, e-mail address (you'll get e-mail from me soon). I have to submit Stanford's slate in less than a week (although a few additions are possible up until some time in November). If you end up being busy on December 1 and can't write, that's fine.
2. Shortly before the Putnam, I'll e-mail you to say where it is; there will also be posters around the math department.
3. *(Optional)* In preparation, depending on interest, I will likely run a problem solving seminar. What we do will depend on who is there, but no background will be assumed. Possibly: Half-hour on a technique, an hour of problems. Possibilities: Weekly. Dinner (pizza). Mid-evening or weekend.
4. For more experienced people: we may have a more advanced seminar (or portion of the seminar), or just weekly meetings in small groups. There are a lot of resources around the dept, especially people (e.g. many gold medalists from the International Mathematical Olympiad who are graduate students or faculty, who are happy to talk with you).

How to prepare. Talk to me. Glance at Loren Larson's "Problem Solving through Problems". (There are many other good books too.) Look at old Putnam problems; the pre-1985 Putnams are in two books (look more at the second one). The more recent Putnams are easily available on the web; see the Stanford Putnam webpage for links. (I have better solutions — again, just talk to me.)

3. **The Fifty-Ninth William Lowell Putnam Mathematical Competition, December 5, 1998**

A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

A3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

A4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

A5. Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$E \subseteq \bigcup_{j=1}^{n} 3D_j.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

A6. Let A, B, C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

B1. Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

B2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

B3. Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers.

B4. Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

B5. Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

B6. Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

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