PUTNAM PROBLEM SOLVING SEMINAR WEEK 5: FUN WITH CALCULUS (AND LENNY NG)

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The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Problem of the Week: volume of n-dimensional spheres.

Let $S_n(R)$ be the "n-dimensional sphere of radius R". For example, $S_3(R)$ is the sphere of radius R; $S_2(R)$ is the (interior of the) circle of radius R; $S_1(R)$ is a line segment of length $2R$ (why?).

1. Make a table of values of the "volume" $V_n(R)$ and "surface area" $A_n(R)$ of $S_n(R)$ for $n = 2, 3$, then 1 and 0. (This is an example of "generalizing downward", and will require some creative thinking.)

n	0	1	2	3	4	5
$V_n(R) =$ "volume" of $S_n(R)$						
$A_n(R) =$ "surface area" of $S_n(R)$						

2. Why is $S_n(R)$ a constant multiple of R^n? Why is $A_n(R)$ a constant multiple of R^{n-1}?

3. Why is $A_n(R) = \frac{d}{dr} S_n(R)$?

In fact,

$$S_n(R) = \frac{\pi^{n/2}}{(n/2)!} r^n. \tag{1}$$

4. "But wait!" you exclaim — "we don't know the meaning of $(n/2)!$ when n is odd!" So using the table, define $(1/2)!$. Then define $n! = n(n-1)!$ even when n is a half-integer.

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Check your answer by verifying that the resulting formula for the volume of the 3-sphere still works.

5. Prove equation (1) by induction as follows. Prove it for $n = 1$ and 2. Then prove it for n assuming it holds for $n-2$, by showing that

$$\begin{aligned} \operatorname{vol} S_n(R) &= \iiint_{\vec{x}\in n\text{-sphere of radius } R} 1 \\ &= \iint_{\vec{y}\in\text{circle of radius } R} \left(\iiint_{\vec{z}\in(n-2)\text{-sphere of radius } \sqrt{R^2-|\vec{y}|^2}} 1 \right) \end{aligned}$$

and computing the nested integrals on the right. (At some point, polar coordinates may help.)

6. (This is easier than many of the earlier ones.) There is a less ad hoc definition of $n!$ when n isn't an integer. If s is a non-negative real number, define $g(s) = \int_0^\infty x^s e^{-x}\,dx$. Prove that (i) $g(0) = 1$, (ii) $g(s) = sg(s-1)$ if $s \geq 1$, assuming that $g(s-1)$ exists. Be careful with convergence! Hence $g(s) = s!$ when s is an integer.

This function, shifted by one, is called the "gamma function". There is a neat argument that $g(1/2)$ is the value you found in problem 5; thus $g(s) = s!$ even when s is a half-integer.

Putnam Problems.

1987A3. For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.
(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

1987B1. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}.$$

1995A2. For what pairs (a,b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}} - \sqrt{\sqrt{x}-\sqrt{x-b}} \right) dx$$

converge?

1991B2. Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

1997B2. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

1998A3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

1991A5. Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 \leq y \leq 1$.

1993A5. Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx$$

is a rational number.

1994B3. Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

1997A3. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right) dx.$$

This handout can (soon) be found at

http://math.stanford.edu/~vakil/stanfordputnam/

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