PUTNAM PROBLEM SOLVING SEMINAR
WEEK 3: COMPLEX NUMBERS

ALOK AGGARWAL AND RAVI VAKIL

The Rules. There are way too many problems here to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

If the problem has a 2002 in it, what happens if you replace 2002 by 1, or 2, or 3? What is important about 2002 — is it that it is even, not divisible by 3, etc.?

The Problems. *The first problems relate to the introductory complex number mini-lecture.*

A1. If a, b and n are positive integers, prove there exist integers x and y such that

$$(a^2+b^2)^n = x^2+y^2.$$

A2.

(a) Show that $\arctan 1 + \arctan 2 + \arctan 3 = \pi$.

(b) Show that $4\ \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

A3. Evaluate:

$$\binom{2001}{0} - \binom{2001}{2} + \binom{2001}{4} - \binom{2001}{6} + \cdots - \binom{2001}{1998} + \binom{2001}{2000}$$

A4.

(a) Find constants $a_0, a_1, \dots, a_6$ so that

$$\cos^6\theta = a_6\cos 6\theta + a_5\cos 5\theta + \cdots + a_1\cos\theta + a_0.$$

(b) Express $\cos 5\theta$ in terms of $\cos\theta$.

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A5. Suppose $\cos(\theta) = 1/\pi$. Evaluate:

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}$$

A6.

(a) For which ordered pairs of real numbers b and c do both roots of the quadratic equation $z^2 + b\,z + c = 0$ lie inside the unit disk $|z| < 1$ in the complex plane?

(b) Draw a reasonably accurate graph of the region in the real bc-plane for which the above condition holds. Identify precisely the boundary curves of this region.

Most problems in sections B and C are from previous Putnam contests. The B questions are mainly related to analysis, while the C questions deal with geometry. Many problems, particularly in geometry, can be solved without complex numbers; feel free to try other methods!

B1. Let $f(z) = |z^{1000} - z^5 + 1|$ where z is a complex number on the unit circle. Find, with proof, the maximum and minimum values of $f(z)$.

B2. For integer $n \geq 2$ show that

$$\sin\frac{\pi}{n}\,\sin\frac{2\pi}{n}\,\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

B3. For positive integer n define

$$S_n = \binom{3n}{0} + \binom{3n}{3} + \cdots + \binom{3n}{3n}.$$

(a) Find a closed-form expression for S_n.

(b) Prove that:

$$\lim_{n\to\infty} S_n^{1/3n} = 2.$$

B4. Prove that if z satisfies $11\,z^{10} + 10i\,z^9 + 10i\,z - 11 = 0$ then $|z| = 1$.

B5. Let $n = 2m$ where m is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Find a finite set of integers $\{a_k\}$ such that

$$\frac{1}{1-\theta} = a_k\theta^k + a_{k-1}\theta^{k-1} + \cdots + a_1\theta + a_0.$$

B6. Let k be a positive integer, let $m = 2^k + 1$, and let $r \neq 1$ be a complex root of $z^m - 1 = 0$. Prove that there exist polynomials $P(z)$ and $Q(z)$ with integer coefficients such that

$$(P(r))^2 + (Q(r))^2 = -1$$

B7. Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

B8. Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \leq m \leq 10$ is $I_m \neq 0$?

B9. Let $G_n = x^n \sin nA + y^n \sin nB + z^n \sin nC$, where x, y, z, A, B, C are real and $A+B+C$ is an integral multiple of π. Prove that if $G_1 = G_2 = 0$, then $G_n = 0$ for all positive integral n.

B10. Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ where $a_k \in \mathbb{R}$. Let $m \geq 0$ and $n \geq 1$ be integers. Find a method for calculating

$$S(m,n) = \sum_{k=0}^{\infty} a_{m+kn}.$$

Hint: Consider substituting n^{th} roots of unity for x.

Comment: This result is useful when the terms of a summation correspond to the series expansion of a known function. For example:

(i) Let $f(x) = (1+x)^{2002}$, $m = 0$, and $n = 4$ to evaluate:

$$\binom{2002}{0} + \binom{2002}{4} + \binom{2002}{8} + \cdots + \binom{2002}{2000}.$$

(ii) Let $f(x) = e^x$, $m = 1$, and $n = 3$ to evaluate:

$$1 + \frac{1}{4!} + \frac{1}{7!} + \frac{1}{10!} + \frac{1}{13!} + \cdots$$

C1. Suppose $ABCD$ is a (convex) plane quadrilateral. Construct a square with side AB, outwards (i.e. not overlapping with the quadrilateral). Do the same thing with the other three sides. If L and M are the line segments joining the midpoints of opposite squares, show that L and M are perpendicular, and have the same length.

C2. Prove that if the points in the complex plane corresponding to two distinct complex numbers z_1 and z_2 are two vertices of an equilateral triangle, then the third vertex corresponds to $-\omega z_1 - \omega^2 z_2$, where ω is an imaginary cube root of unity.

C3. A regular n–sided polygon is inscribed in a unit circle. Find the product of the lengths of all its sides and diagonals.

C4. $A_1, A_2, \ldots, A_n$ are vertices of a regular polygon inscribed in a circle of radius R and center O. P is a point on OA_1 extended beyond A_1. Show that:

$$\prod_{k=1}^{n} PA_k = OP^n - r^n.$$

C5. Given a point P on the circumference of a unit circle and the vertices $A_1, A_2, \ldots, A_n$ of an inscribed regular polygon of n sides, prove that $PA_1^2 + PA_2^2 + \cdots + PA_n^2$ is a constant.

C6. Let C be a circle with center O, and Q a point inside C different from O. Show that the area enclosed by the locus of the centroid of triangle OPQ as P moves about the circumference of C is independent of Q.

C7. In a triangle ABC the points D, E, and F trisect the sides so that $BC = 3\,BD$, $CA = 3\,CE$, and $AB = 3\,AF$. Show that triangles ABC and DEF have the same centroid.

Problem References

A1. H.G. Dworschak, "Eureka," Vol. 2, No. 3, March 1976, pg. 50
A2. Larson, "Problem Solving through Problems," pg. 118
A4. Larson, "Problem Solving through Problems," pg. 116
A6. Putnam 1975 A2
B2. R. Honsberger, "Erdos to Kiev: Problems of Olympiad Caliber," pg. 75
B3. Crux Mathematicorum, 1987, 138
B4. Putnam 1989 A3
B5. Putnam 1975 A4
B6. Putnam 1983 B6
B7. Putnam 1991 B2
B8. Putnam 1985 A5
B9. 1980 USA Olympiad
C2. Putnam 1959 A2
C4. Putnam 1955 A2
C5. Larson, "Problem Solving Through Problems," pg. 312
C6. Gilbert, Krusemeyer, and Larson, "The Wohascum County Problem Book," Problem 34
C7. 1978 USA Olympiad

This handout can (soon) be found at

http://math.stanford.edu/˜vakil/stanfordputnam/

E-mail address: `alok@stanford.edu, vakil@math.stanford.edu`