

PUTNAM PROBLEM SOLVING SEMINAR WEEK 1

RAVI VAKIL

The Rules. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on. If you would like to practice with the Pigeonhole Principle or Induction (a good idea if you haven't seen these ideas before), try those problems.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

If the problem has a 2002 in it, what happens if you replace 2002 by 1, or 2, or 3? What's important about 2002 — is it that it is even, not divisible by 3, etc.?

The problems.

The first problems are for practice with the pigeonhole principle.

1. Prove that there are two people in the U.S. right now with the same amount of hair on their heads (not including bald people!).
2. Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct integers in A whose sum is 104.
3. Show that if there are n people at a party, then two of them know the same number of people (among those present).
4. Five points lie in an equilateral triangle of size 1. Show that two of the points lie no farther than $1/2$ apart. Can the " $1/2$ " be replaced by anything smaller? Can it be improved if the "five" is replaced by "six"?
5. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point.
6. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e. for any real number x and any $\epsilon > 0$, there is an element $y \in S$ such that $|x - y| < \epsilon$. (Hint: first let $x = 0$.)

Date: Tuesday, October 15, 2002.

7. For every n in the set $\mathbb{Z}^+ = \{1, 2, \dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathbb{Z}^+$.
8. Prove that there is some integral power of 2 that begins 2002....
9. Given any $n + 1$ integers between 1 and $2n$, show that one of them is divisible by another.
10. Prove that in any group of six people there are either three mutual friends or three mutual strangers. (Hint: Represent the people by the vertices of a regular hexagon. Connect two vertices with a red line segment if the couple represented by these vertices are friends; otherwise, connect them with a blue line segment. Consider one of the vertices, say A . At least three line segments emanating from A have the same color. There are two cases to consider.)
11. Follow-up to the previous question: Find some n so that in any group of n people there are either four mutual friends or four mutual strangers.
12. A polygon in the plane has area 1.2432. Show that it contains two distinct points (x_1, y_1) and (x_2, y_2) that differ by (a, b) , where a and b are integers.

The next few problems are for practice with induction.

13. Show that $1 + 3 + 5 + \dots + (2n - 1) = n^2$.
14. Show that for all positive integers n , $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer.
15. Show that $1 + 1/\sqrt{2} + 1/\sqrt{3} + \dots + 1/\sqrt{n} < 2\sqrt{n}$.
16. Prove that all even perfect squares are divisible by 4. Prove that all odd perfect squares leave a remainder of 1 upon division by 8. (This is a useful fact to know!) What are the possible remainders when you divide a perfect square by 3?

The last problems are not on any particular topic.

17. In this problem, “knights” always tell the truth and “knaves” always lie. In (a)–(c), all persons are either knights or knaves.
- (a) There are two persons, A and B . A says “At least one of us is a knave.” What are A and B ?
- (b) A says, “Either I am a knave or B is a knight.” What are A and B ?
- (c) Now we have three people, A , B , and C . A says “All of us are knaves.” B says “Exactly one of us is a knight.” What are A , B , and C ?
- (d) Now we have a third type of person, called “normal”, who sometimes lies and sometimes tells the truth. A says “I am normal.” B says “That is true.” C says, “I am not normal.” Exactly one of A , B , C is a knight, one is a knave, and one is normal. What are A , B , and C ?

18. Mr. and Mrs. Adams recently attended a party at which there were three other couples. Various handshakes took place. No one shook hands with his/her own spouse, no one shook hands with the same person twice, and of course, no one shook his/her own hand. After all the handshaking was finished, Mr. Adams asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mrs. Adams shake?

19. (a) In how many zeros does $10000!$ end? (b) What is its last nonzero digit?

20. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x - and y -axes with diagonal pq . What is the probability that no point of R lies outside of C ?

This handout, and other useful things, can (soon) be found at

<http://math.stanford.edu/~vakil/stanfordputnam.html>

E-mail address: `vakil@math.stanford.edu`