TWELVE POINTS ON THE PROJECTIVE LINE, BRANCHED COVERS, AND RATIONAL ELLIPTIC FIBRATIONS

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ABSTRACT. The following divisors in the space $\mathrm{Sym}^{12}\mathbb{P}^1$ of twelve points on $\mathbb{P}^1$ are actually the same: ($\mathcal{A}$) The possible locus of the twelve nodal fibers in a rational elliptic fibration (i.e. a pencil of plane cubic curves); ($\mathcal{B}$) degree 12 binary forms that can be expressed as a cube plus a square; ($\mathcal{C}$) the locus of the twelve tangents to a smooth plane quartic from a general point of the plane; ($\mathcal{D}$) the branch locus of a degree 4 map from a hyperelliptic genus 3 curve to $\mathbb{P}^1$; ($\mathcal{E}$) the branch locus of a degree 3 map from a genus 4 curve to $\mathbb{P}^1$ induced by a theta-characteristic; and several more.

The corresponding moduli spaces are smooth, but they are not all isomorphic; some are finite étale covers of others. We describe the web of interconnections between these spaces, and give monodromy, rationality, and Prym-related consequences. Enumerative consequences include: (i) the degree of this locus is 3762 (e.g. there are 3762 rational elliptic fibrations with nodes above 11 given general points of the base); (ii) if $C \to \mathbb{P}^1$ is a cover as in ($\mathcal{D}$), then there are 135 different such covers branched at the same points; (iii) if $p \in \mathbb{P}^2 \supset C$ is a configuration as described in ($\mathcal{C}$), then there are 120 essentially different such configurations with the same tangent lines.

Some parts of this story are well-known, and some other parts were known classically (to Zeuthen, Zariski, Coble, Mumford, and others). The unified picture is surprisingly intricate and connects many beautiful constructions, including Recillas' trigonal construction and Shioda's E_8-Mordell-Weil lattice.

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1. Introduction

In this paper, we explore the links between various moduli spaces defined in Section 2 (and sketched in the abstract). Although some of the connections are well-known or classical, for completeness of exposition we describe them in detail; some classical references are discussed in Section 7.

1.1. *Outline of the paper.* In Section 3, we interpret points of $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ as 2-torsion information on rational elliptic fibrations (corresponding to points of $\mathcal{A}$); the Mordell-Weil lattice plays a central role. In Section 4, we relate $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ using Recillas' trigonal construction, and interpret points of $\mathcal{C}$ and $\mathcal{D}$ as theta-characteristics of a genus 4 curve corresponding to a point of $\mathcal{E}$. As consequences, we give monodromy and Prym-related results, including a theorem of Mumford on hyperelliptic Pryms. In the short Section 5, we note that some of the connections are already visible in the discriminant of a quartic or cubic polynomial. In Section 6, we study the actual locus of 12 points on the projective line, the divisor $\mathcal{Z} \subset \operatorname{Sym}^{12}\mathbb{P}^1 \cong \mathbb{P}^{12}$, and compute its degree to be 3762. In Section 8, we suggest further questions about two-torsion of elliptic fibrations.

These results can also be seen as related to Persson's enumeration of the possible singular fibers of rational elliptic surfaces ([P], [Mi]); he studies how the singular fibers can degenerate, while we study the locus of possible configurations. (Note that [P] makes use of some of the standard constructions below.)

Many of the constructions in this paper are reminiscent of [B] Ch. VI, but there does not appear to be a precise connection.

1.2. *Acknowledgements.* The advice of I. Dolgachev has been invaluable. In particular, the authour is grateful to him for suggesting that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ might be related via Recillas' construction, and for pointing out the beautiful classical reference [Co]. The author would also like to thank D. Allcock, A. Bertram, A. J. de Jong, N. Elkies, J. Harris, B. Hassett, and R. Smith for helpful conversations.

2. Definitions and Results

For convenience, we work over the complex numbers. (All results are true over an algebraically closed field of characteristic greater than 5, except possibly Theorem 6.2 in characteristics 11 and 19.) All moduli spaces are Deligne-Mumford stacks unless otherwise indicated.

We now describe various loci of twelve points on the projective line, and the relationships between them. The constructions are invariant under $\operatorname{Aut}(\mathbb{P}^1) = PGL(2)$, and there are natural quotients in each case. Often the quotient is the more "natural" object, but for consistency, we will deal with the "framed" construction.

(If X is abstractly isomorphic to $\mathbb{P}^1$, a *frame* for X is a choice of isomorphism to $\mathbb{P}^1$.)

Most of these constructions have natural compactifications (or at least partial compactifications). It would be interesting to understand how these relationships specialize to the boundary; the reader will note that they often specialize well to the divisor where two of the twelve points come together. Many of the results of [E] (about six points on $\mathbb{P}^1$) can be understood in this context by letting the points come together in six pairs (for example the results relating to genus 4 curves with vanishing theta characteristic, and facts relating to the E_8-lattice, including the numbers 136 and 120). Many more of the results of [E] have a similar flavor, but the connections are not clear.

2.1. Degree 1 Del Pezzo surfaces and rational elliptic fibrations. We recall relevant facts relating degree 1 Del Pezzo surfaces, rational elliptic fibrations, and pencils of plane cubics. (The fundamental sources are [S], especially Section 10, and [Ma] Ch. IV.) By genus 1 fibration, we will mean a flat family of reduced genus 1 curves over a smooth curve, with smooth total space and at worst multiplicative reduction. By elliptic fibration, we will mean a genus 1 fibration with a choice of section.

A degree 1 Del Pezzo surface X has a pencil of sections of the canonical sheaf, with one base point p_0. Then the pencil $\mathrm{Bl}_{p_0} X \to \mathbb{P}^1$ is generically an elliptic fibration, with the proper transform E_0 of p_0 as zero-section. An open subset of the moduli space of degree 1 Del Pezzo surfaces correspond to fibrations where there are twelve singular fibers; in this case the fibration is an elliptic fibration. Let $\mathcal{A}$ be the moduli space of such rational elliptic fibrations, along with a *frame* for the base $\mathbb{P}^1$. A straight-forward dimension count shows that $\dim \mathcal{A} = 11$: 16 from choosing a pencil of plane cubics, +3 for a frame for the pencil, -8 from $\mathrm{Aut}(\mathbb{P}^2)$.

The structure of the group of sections of a rational elliptic fibration is isomorphic to the E_8-lattice (via the Mordell-Weil height pairing). There are 240 sections disjoint from E_0, corresponding to the 240 minimal vectors of the lattice. Given any nine mutually disjoint sections including E_0, those sections can be blown down, expressing the fibration as the total space of plane cubics with distinct base points (corresponding to the erstwhile sections). There are $\#W(E_8) = 2^{14}3^5 5^2 7$ ways to choose the (ordered) octuple of disjoint sections (not including than E_0). Conversely, any pencil of cubics with 12 nodal fibers has distinct base points, and the total space is a rational elliptic fibration. From this description, $\mathcal{A}$ is smooth.

Given a representation of a blow-up of a degree 1 Del Pezzo surface X as the total space of a pencil of cubics, we use the following convention for the homology of X: H is the hyperplane class of the plane, and $E_0, \dots, E_8$ are the base points of the pencil (with E_0 corresponding to the exceptional divisor of the blow-up of the Del Pezzo surface).

We will make repeated use of the following fact about 2-torsion on a smooth elliptic fibration.

2.2. *Lemma. — Suppose B is a curve, $b \in B$ is a closed point, and that $X \to B$ is an elliptic fibrational with smooth total space, such that the fiber X_b above b has multiplicative reduction. Let $\mathcal{L}$ be a line bundle on X of relative degree 2, and let C be the closure of the locus in X of points p such that in each smooth fiber, $\mathcal{O}(2p)$ is linearly equivalent to the restriction of $\mathcal{L}$. Then $C \to B$ is simply branched above b (i.e. the total ramification index above b is 1), and the branch point is the node of X_b.*

Proof. The 4-section C meets X_b at two points away from the node, and hence with multiplicity two at the node. Thus C must be smooth at the node, and $C \to B$ is branched with ramification index 1 there. □

2.3. Weighted projective space. Consider the weighted projective space

$$\mathbb{P} = \mathbb{P}(\underbrace{3, \dots, 3}_{5}, \underbrace{2, \dots, 2}_{7})$$

with projective coordinates $[a_0; \dots; a_4; b_0; \dots; b_6]$. Let $\mathcal{B}$ be the open subscheme of $\mathbb{P}$ where the homogeneous degree 12 polynomial

$$(a_0 x^4 + a_1 x^3 y + \dots + a_4 y^4)^3 + (b_0 x^6 + b_1 x^5 y + \dots + b_6 y^6)^2$$

has distinct roots. Then $\mathcal{B}$ is non-empty (as $[1, 0, \dots, 0, 1] \in \mathcal{B}$), and is clearly of dimension 11, nonsingular (as the only singular points of $\mathbb{P}$ are where $a_0 = \dots = a_4 = 0$ or $b_0 = \dots = b_6 = 0$), and rational.

2.4. Branched covers of the projective line. Recall that a *simply branched* cover of $\mathbb{P}^1$ is defined to be one where the total ramification index is 1 above each point of $\mathbb{P}^1$; the Hurwitz scheme is a moduli space for simply branched connected covers. The loci $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ defined below are locally closed subvarieties of the Hurwitz scheme. By the Riemann-Hurwitz formula, there are 12 branch points in each case, giving a morphism (via the branch map) to the space of twelve points in $\mathbb{P}^1$.

2.5. *In the locus of genus 3 degree 4 covers of $\mathbb{P}^1$, there is a codimension 1 set mapped to $\mathbb{P}^1$ canonically.* Let $\mathcal{C}$ be the locus of smooth genus 3 degree 4 covers of $\mathbb{P}^1$ branched over twelve distinct points, such that the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ is (isomophic to) the canonical bundle. We call this a *canonical pencil*.

2.6. Note that the source curve C cannot be hyperelliptic. Otherwise, if $h : C \to \mathbb{P}^1$ is the hyperelliptic map, then the canonical bundle is (isomorphic to) $h^*\mathcal{O}_{\mathbb{P}^1}(2)$, and the sections are pullbacks of sections of $h^*\mathcal{O}_{\mathbb{P}^1}(2)$. Thus any canonical pencil on C factors through some 2-to-1 $j : \mathbb{P}^1 \to \mathbb{P}^1$; as j is branched above 2 points, $h \circ j$ is not simply branched above those two points.

There are various equivalent formulations of the definition of $\mathcal{C}$. For example, let V be the (rank 3) Hodge bundle over $\mathcal{M}_3$. Let $\mathbb{G}_f(2, V)$ be the space of *framed* pencils in $\mathbb{P}V$ over $\mathcal{M}_3$;

$$\dim \mathbb{G}_f(2,V) = \dim \mathcal{M}_3 + \dim G(2,3) + \dim \operatorname{Aut} \mathbb{P}^1 = 11.$$

Then $\mathcal{C}$ is an open substack (actually a scheme, as we saw above) of $\mathbb{G}_f(2,V)$ and rational (as $\mathcal{M}_3$ is rational, [K]), smooth, and of dimension 11.

An alternative formulation comes from the fact that if C is not hyperelliptic, then the canonical model of C is a smooth quartic curve in $\mathbb{P}^2$. Thus $\mathcal{C}$ is a quotient of an open subset of $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) \times \mathbb{P}^2$ by $\operatorname{Aut} \mathbb{P}^2 = PGL(3)$ (with the additional data of a frame for the pencil): the locus in $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) \times \mathbb{P}^2$ is the choice of a smooth quartic $Q \subset \mathbb{P}^2$, and a point p not on any flex line or bitangent line of Q, nor on Q. Then the corresponding cover of $\mathbb{P}^1$ can be recovering by projecting Q from p. In this guise, $\mathcal{C}$ is clearly smooth of dimension 11, but rationality is not obvious.

2.7. *In the locus of genus 3 degree 4 covers of $\mathbb{P}^1$, there is a codimension 1 set where the source curve is hyperelliptic.* Let $\mathcal{D}$ be the locus of smooth genus 3 degree 4 covers of $\mathbb{P}^1$ branched over twelve distinct points, such that the source curve is hyperelliptic. By Section 2.6, the pencil is not a canonical pencil (i.e. the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ is not the canonical bundle). It is not hard to see that $\mathcal{D}$ is smooth of dimension 11: the hyperelliptic locus is smooth of dimension 5; the choice of non-canonical degree 4 line bundle gives 3 more dimensions; such a line bundle has exactly 1 pencil by Riemann-Roch; and there are 3 dimensions of choice of frame for the pencil.

2.8. *In the locus of genus 4 degree 3 covers of $\mathbb{P}^1$, there is a codimension 1 set where the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ is a theta-characteristic.* In $\mathcal{M}_4$, there is a divisor $\mathcal{M}_4^1$ corresponding to curves C with theta-characteristics with 2 sections (vanishing theta-characteristic), smooth away from the hyperelliptic locus $\mathcal{H}_4$ (where it has 10 sheets corresponding to the Weierstrass points). Note that no (smooth) hyperelliptic genus 4 curve C can have a theta-characteristic inducing a base-point-free pencil, or indeed any other degree 3 base-point-free pencil. (Otherwise, let D_1 be the divisor class of such a pencil, and let D_2 be the hyperelliptic divisor class. Define

$$\phi : C \stackrel{(D_1,D_2)}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1.$$

Then $\phi_*[C]$ is in class (2,3) on $\mathbb{P}^1 \times \mathbb{P}^1$, so ϕ cannot carry C multiply onto its image. Hence ϕ carries C birationally onto its image. But any curve in class (2,3) has arithmetic genus 2, and the genus of C is 4, giving a contradiction.)

A point of $\mathcal{M}_4^1 \setminus \mathcal{H}_4$ corresponds to a curve C whose canonical model is the intersection of a cone and a cubic hypersurface in $\mathbb{P}^3$; the pencil is given by the $\mathbb{P}^1$ parametrizing the rulings of the cone. Let $\mathcal{E}$ be the open subset of such pencils, with framings, such that the induced triple cover of $\mathbb{P}^1$ is (simply) branched at 12 distinct points. It is not hard to see that $\mathcal{E}$ is non-empty; in any case, it will follow from Proposition 3.2. Clearly $\mathcal{E}$ is smooth of dimension 11, and as $\mathcal{M}_4^1$ is rational ([D] p. 14), $\mathcal{E}$ is as well.

2.9. Twelve points on the projective line: polynomials that are a cube plus a square. All of the loci described above have natural morphisms to the moduli space of twelve points on the projective line, $\mathrm{Sym}^{12}\mathbb{P}^1 \cong \mathbb{P}^{12}$, and all have the same image. For concreteness, let $\mathcal{Z}$ be the locus of points in $\mathrm{Sym}^{12}\mathbb{P}^1$ (considered as homogeneous degree 12 polynomials in x and y) corresponding to polynomials that can be expressed as the sum of a square and a cube. Then $\mathcal{Z}$ is the image of $\mathcal{B}$ and hence locally closed.

2.10. *Other loci.* There are various other descriptions of twelve points on $\mathbb{P}^1$ that turn out to describe the same locus as those above. As they are either simple variants of the above, or are not well-defined moduli problems, we will not dwell on them.

- In light of loci $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, a natural locus to consider is that of hyperelliptic genus 5 curves branched over 12 points in the locus $\mathcal{Z}$. A better description is to use the bijection between simply branched double covers (see [EEHS] Theorem 6(1); when $\mathbb{P}^1$ is replaced by $\mathbb{Q}$, this is just part of the classical method of solving the cubic), and étale triple covers of double covers (or equivalently a choice of 3-torsion of a double cover, modulo $\{\pm 1\}$). Under this "elliptic-trigonal" correspondence, there is a locus $\mathcal{F}$ corresponding to $\mathcal{E}$; via the isomorphism of $\mathcal{E}$ with $\mathcal{B}$, it is essentially the moduli space of representations of hyperelliptic genus 5 curves as

$$z^2 = f^2 + g^3 \tag{1}$$

  where f and g are binary forms in x and y of degree 6 and 4 respectively. (Note that by specifying the form of the equation (1) of the genus 5 curve, one automatically specifies an étale triple cover.) The author is unaware of a geometric way of describing this locus.
- Project a smooth cubic surface in $\mathbb{P}^3$ from a general point p, and let B be the branch divisor in $\mathbb{P}^2$. (B turns out to be a genus 4 curve mapped canonically to $\mathbb{P}^2$.) Any conic in the plane meets the sextic at twelve points, and hence describes a set of 12 points on $\mathbb{P}^1$. (This is related to $\mathcal{E}$ as follows: let C be the intersection of the cubic surface and the cone over the conic with vertex p.)
- In [Z2] Section 8, Zariski describes a locus of sextic plane curves B', smooth except for six cusps, and the six cusps lie on a conic. Any conic meets this sextic at twelve points. (The sextic is the curve B in the previous example. Zariski showed that the fundamental group of the complement of B' is $\mathbb{Z}/2 * \mathbb{Z}/3$; the cubic surface can be recovered from the natural index 3 subgroup of this group.)

2.11. Relationship between moduli spaces. Surprisingly, it is possible to describe the relationship between almost any pair of the spaces $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ (and through $\mathcal{E}$, $\mathcal{Z}$) described above. Table 1 summarizes where the links are explained.

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$
$\mathcal{A}$	—	2.12	3.1	3.3	3.2
$\mathcal{B}$	2.12	—	5	5	5
$\mathcal{C}$	3.1	5	—		4.1
$\mathcal{D}$	3.3	5		—	4.1
$\mathcal{E}$	3.2	5	4.1	4.1	—

TABLE 1. Sections where links are described

We will repeatedly make use of the fact that if $\pi : X \to Y$ is a morphism of smooth varieties of the same dimension (over $\mathbb{C}$) such that every point of Y has n pre-images, then π is finite étale of degree n.

2.12. *Theorem.* — *There are natural isomorphisms $\mathcal{A} \cong \mathcal{B} \cong \mathcal{E}$, and natural finite étale morphisms $\mathcal{C} \to \mathcal{A}$, $\mathcal{D} \to \mathcal{A}$ of degree 120 and 135 respectively.*

The numbers 120 and 135 are related to the E_8 lattice; this is made explicit in Section 4.

Proof. The isomorphsm $\mathcal{A} \cong \mathcal{B}$ follows from the classical theory of Weierstrass models of rational elliptic fibrations (see for example [MS] Section 3, especially Theorem 1'). The rest of the Theorem follows from Propositions 3.1 — 3.3 below. $\square$

Let M_1 be the moduli space of degree 1 Del Pezzo surfaces. If we take quotients of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$ by $PGL(2)$, we have immediately:

2.13. *Corollary.* — *M_1, $\mathcal{B}/PGL(2)$, and $\mathcal{M}_4^1$ are birational (and the isomorphic open sets are given quite explicitly). As $\mathcal{M}_4^1$ is rational, M_1 and $\mathcal{B}/PGL(2)$ are as well.*

Of course, this is well-travelled ground: M_1 is well-known to be rational precisely because it is birational to $\mathcal{M}_4^1$, see e.g. [D] for example. But Heckmann and Looijenga have used the link to $\mathcal{B}$ to give a new proof of the rationality of M_1, or equivalently $\mathcal{M}_4^1$ or $\mathcal{B}/PGL(2)$ ([HL]).

We also get a degree 120 rational map from pencils in the Hodge bundle over $\mathcal{M}_3$ to $\mathcal{M}_4^1$ (and an interpretation of this map in terms of the E_8 lattice, see Section 4).

2.14. Ball quotients. Deligne and Mostow ([DM]) have shown that the moduli space of unordered 12-tuples of points on a line (up to automorphisms of the line $PGL(2)$) is a quotient of (a compactification of) the complex 9-ball by an (arithmetic) lattice in $PU(1,9)$. Heckman and Looijenga have shown ([HL]) that the lift of this divisor in the locus of twelve points is geodesically embedded in the

9-ball (and hence that this locus is the quotient of a complex 8-ball by an arithmetic group.

3. Relating branched covers to elliptic fibrations via 2-torsion

The strategy we use to show a relationship between $\mathcal{C}$ (resp. $\mathcal{D}$, $\mathcal{E}$) and $\mathcal{A}$ is as follows. To construct an elliptic fibration $X \to \mathbb{P}^1$ (with zero-section s) from a branched cover $C \to \mathbb{P}^1$, we embed C in a rational surface Y that is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (the Hirzebruch surface $\mathbb{F}_0$, $\mathbb{F}_1$, or $\mathbb{F}_2$), so that C (possibly union another curve) is a 4-section of the $\mathbb{P}^1$-bundle. We then double-cover Y branched over C (and possibly another curve) to construct X, a genus 1 fibration over $\mathbb{P}^1$. (In each case X turns out to be rational.) We then reconstruct the zero-section s to express $X \to \mathbb{P}^1$ as an elliptic fibration.

To construct a cover from an elliptic fibration $X \to \mathbb{P}^1$ (with zero-section s), we choose a line bundle on X of relative degree 2, and construct a 4-section that over each point over $\mathbb{P}^1$ consists of points p such that $2p$ is linearly equivalent to the restriction of the line bundle to the fiber.

3.1. *Proposition. — There is a natural degree 120 finite étale morphism $\mathcal{C} \to \mathcal{A}$.*

We refer the interested reader also to [DO] Chapter IX.

Proof. We first describe the morphism $\mathcal{C} \to \mathcal{A}$. For simplicity of exposition, we define this morphism pointwise. The data of a point of $\mathcal{C}$ is a smooth quartic Q in $\mathbb{P}^2$ along with a point $p \in \mathbb{P}^2$ not on a bitangent line or flex line of Q, nor on Q. Let $Y = \mathrm{Bl}_p \mathbb{P}^2$ (fibered over $\mathbb{P}^1$ via the pencil of lines through p, so $Y \cong \mathbb{F}_1$); by abuse of notation we consider Q to be a subscheme of Y (and a 4-section of the fibration).

Let X be the double cover of Y branched over Q. Such a double cover exists, as Q is divisible by 2 in $\operatorname{Pic} Y$. (If the two pre-images in X of p are blown down, by a classical result we have a degree 2 Del Pezzo surface. For a modern reference see [CD] Proposition 0.3.5. The 56 (-1)-curves of the Del Pezzo surface as the preimages of the 28 bitangents of Q.) Then X is a rational genus 1 fibration over $\mathbb{P}^1$. To make it an elliptic fibration, we choose as zero-section either pre-image of the exceptional divisor of p. (The choice is irrelevant as they are interchanged under the involution of the double cover $X \to Y$.) The nodal fibers of $X \to \mathbb{P}^1$ correspond to the branch points of $Q \to \mathbb{P}^1$, and indeed the nodes correspond to the ramification points.

We next reverse the process. Represent a point of $\mathcal{A}$ as a pencil of cubics (with total space $X \to \mathbb{P}^1$), with one of the base points corresponding to E_0. Then E_1 (corresponding to another base point) is a section disjoint from E_0. Let $E = E_2 + \cdots + E_8$ be the sum of exceptional divisors corresponding to the remaining base points. Let Q' be the locus of points p that, in each fiber, satisfy $2p = E_0 + E_1$ (in the group law of the fiber). Then Q' is a smooth curve mapping 4-to-1 to $\mathbb{P}^1$, simply branched at 12 points (the nodes of the fibration, by Lemma 2.2); hence Q'

has genus 3. (It is not yet clear that Q' is connected.) In $\operatorname{Pic} X$,

$$Q' = aH + b(E_0 + E_1) + cE$$

for some integers a, b, c. As $Q' \cap E_0 = Q' \cap E_1 = \emptyset$, $b = 0$. As Q' is a 4-section of the fibration,

$$4 = Q' \cdot (3H - E_0 - E_1 - E) = -Q' \cdot K_X = 3a + 7c.$$

As $p_a(Q') = 3$, by adjunction,

$$4 = Q' \cdot (Q' + K_X) = (Q')^2 - 4,$$

so $8 = (Q')^2 = a^2 - 7c^2$. Substituting $3a + 7c = 4$, we find $(a-6)^2 = 0$, so $a = 6$, $c = -2$. By adjunction,

$$K_{Q'} = (3H - E)|_{Q'} = \mathcal{O}_{\mathbb{P}^1}(1)|_{Q'}$$

so the branched cover $Q' \to \mathbb{P}^1$ is indeed a canonical pencil. Hence Q' is connected. Observe that this construction commutes with the morphism $\mathcal{C} \to \mathcal{A}$ described earlier.

The only choice involved was that of the section E_1 disjoint from E_0; there are 240 such sections (Section 2.1). However, they come in 120 pairs: if $(X, E_0, E_1) \to \mathbb{P}^1$ is one such pair, and E_{-1} is the section corresponding to $-E_1$ in the group law of the elliptic fibration, then there is an isomorphism

$$\begin{tikzcd} (X, E_0, E_1) \arrow[rr, "\sim"] \arrow[dr] & & (X, E_{-1}, E_0) \arrow[dl] \\ & \mathbb{P}^2 & \end{tikzcd}$$

(corresponding to translation by E_{-1} in the group law of the elliptic fibration). As the construction of Q' above depended only on the data $(X, E_0 \cup E_1) \to \mathbb{P}^1$, we see that for each point of $\mathcal{A}$ there are 120 preimages in $\mathcal{C}$. $\square$

3.2. *Proposition. — There is a natural isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{A}$.*

This is classical (see for example [D] p. 13 or [CD] p. 36), but we give a proof that highlights the importance of the genus 4 curve.

Proof. We first describe the morphism $\mathcal{E} \to \mathcal{A}$. Consider a point of $\mathcal{E}$, or equivalently a cone in $\mathbb{P}^3$ transversely intersecting a cubic surface, along with a frame for the $\mathbb{P}^1$ parametrizing the rulings of the cone. Let Y be the blow-up of the cone at the vertex, isomorphic to the Hirzebruch surface $\mathbb{F}_2$. Let E be the exceptional divisor of the blow-up. Let C be the intersection of the cone and the cubic, pulled back to Y. The induced morphism $C \to \mathbb{P}^1$ is precisely the triple cover in the definition of $\mathcal{E}$. Let X be the double cover of Y branched over $C + E$. Such a cover exists as $C + E$ is even in $\operatorname{Pic} Y$. (Reason: If h is the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$ to $Y \cong \mathbb{F}_2$, and f is a ruling of $\mathbb{F}_2$, then $C + E = (3h) + (h - 2f) = 2(2h - f)$.) Then X is a genus 1 fibration. The choice of (preimage of) E as 0-section makes X an elliptic fibration. Moreover, the blowdown of X along E is a degree 1 Del Pezzo surface (classical; see [CD] Proposition 0.3.6 for a modern reference). Finally, the nodes of the fibers coincide with the ramification points of $C \to \mathbb{P}^1$, so $X \to \mathbb{P}^1$ corresponds to a point of $\mathcal{A}$.

To reverse the process, represent a point of $\mathcal{A}$ as the total space of a pencil of cubics $X \to \mathbb{P}^1$, with one of the base points corresponding to E_0. Let $E = E_1 + \cdots + E_8$ be the sum of the exceptional divisors corresponding to the remaining base points. Let C'' be the closure of the locus of points p that, in each fiber, satisfy $2p = 2E_0$. Then by Lemma 2.2 C'' is a smooth curve with E_0 as a component; let $C' = C'' \setminus E_0$. Thus C' is a smooth curve mapping 3-to-1 to $\mathbb{P}^1$, simply branched at 12 points (the nodes of the fibration); hence C' has arithmetic genus 4 (although it is not yet clear that C' is connected).

In $\operatorname{Pic} X$, $C' = aH + bE + cE_0$. As $C' \cap E_0 = \emptyset$, $c = 0$. As C' is a 3-section of the fibration,

$$3 = C' \cdot (3H - E_0 - E) = -C' \cdot K_X = 3a + 8b.$$

As $p_a(C') = 4$, by adjunction,

$$6 = (C')^2 + K_X \cdot C' = (C')^2 - 3,$$

so $a^2 - 8b^2 = 9$, from which $(a-9)^2 = 0$, so $a = 9$, $b = -3$. By adjunction, $K_{C'} = 2(3H - E)|_{C'} = \mathcal{O}_{\mathbb{P}^1}(2)|_{C'}$, so the morphism $C' \to \mathbb{P}^1$ indeed corresponds to a theta-characteristic. Hence C' is connected. Observe that this construction commutes with the morphism $\mathcal{E} \to \mathcal{A}$ described earlier. $\square$

In the course of the proof, we recovered the (presumably classical) result that the non-trivial 2-torsion points of fibers of a rational elliptic fibration are transitively permuted by monodromy.

3.3. *Proposition. — There is a natural degree 135 finite étale morphism $\mathcal{D} \to \mathcal{A}$.*

This proposition was known to Mumford (in the guise of the morphism $\mathcal{D} \to \mathcal{E}$); see the discussion after Proposition 4.2.

Proof. We begin by describing the "reverse" of the morphism $\mathcal{D} \to \mathcal{A}$, i.e., given a point of $\mathcal{A}$ and some choices, how to construct a point of $\mathcal{D}$.

Consider a point of $\mathcal{A}$ represented as a pencil of cubics $X \to \mathbb{P}^1$ with section E_0 corresponding to one of the base points (and $E_1, \ldots, E_8$ correspond to the rest). Then $H - E_0$ is a divisor on X of relative degree 2. Let C' be the closure of the locus of points p that, in each fiber, satisfy $2p = H - E_0$ (in the group law of the fiber). Then in $\operatorname{Pic} X$, $C' = aH + bE_0 + cE$, where $E = E_1 + \cdots + E_8$. One can check (as in the proofs of Propositions 3.1 and 3.2) that $a = 5$, $b = -3$, $c = -1$, so the image of C' in $\mathbb{P}^2$ is a quintic with a triple point (at the image p_0 of E_0). Thus C' is visibly hyperelliptic (just project from p_0), so we have described a point of $\mathcal{D}$.

(The value $b = -3$ corresponds to the enumerative fact that three times in the pencil of cubics there is a flex at p_0; equivalently, there are three fibers of the elliptic fibration where the divisor $H-3E_0$ is trivial. We will use this fact in the discussion after Theorem 4.4.)

By examining this construction, we will see how to construct the morphism $\mathcal{D} \to \mathcal{A}$. Consider the morphism

$$\phi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

given by the divisors $(H - E_0, 3H - (E_0 + \cdots + E_8))$; ϕ maps C to the first $\mathbb{P}^1$ via the hyperelliptic map. Clearly ϕ is surjective of degree 2; it is branched over C. The image of E_i $(1 \leq i \leq 8)$ is a fiber of the first projection; we see as an added bonus that the points $E_i \cap C$ are the Weierstrass points of C.

Thus any double-cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a smooth divisor of type $(4, 2)$ is a rational genus 1 fibration (with the second projection as the morphism of the fibration, assuming that the second morphism expresses the branch locus as a simply branched cover of $\mathbb{P}^1$), as this property is preserved under deformation.

We are now ready to describe the morphism $\mathcal{D} \to \mathcal{A}$. Consider a point of $\mathcal{D}$, or equivalently a hyperelliptic genus 3 curve C (with hyperelliptic divisor class D_1), a degree 4 divisor class D_2 (not the canonical divisor class), and a frame for the pencil of sections of D_2. Then there is a morphism

$$\phi : C \stackrel{(D_1,D_2)}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1,$$

determined up to automorphisms of the first $\mathbb{P}^1$. Let Y be the target $\mathbb{P}^1 \times \mathbb{P}^1$, and let pr_i be the projection to the ith factor. Consider Y as a fibration over (framed) $\mathbb{P}^1$ via the second projection pr_2. Then ϕ is a closed immersion (first note that C is birational onto its image, then that the arithmetic genus of the image is 3); for convenience, we identify C with its image.

Let X be the double cover of Y branched over the image of C; via the fibration $pr_2 : Y \to \mathbb{P}^1$, X is a genus 1 rational fibration over $\mathbb{P}^1$. We now recover a distinguished section E_0 to express X as an elliptic fibration.

Let $F_1, \dots, F_8$ be the fibers of $\mathbb{P}^1$ passing through the hyperelliptic points of C. The preimages of F_i on X are two rational curves (meeting at a node, the hyperelliptic point of C); call them E_i and E_i'. By making these labellings, we have made $8! \times 2^7$ choices. (We have actually made $8! \times 2^8$ choices. But exchanging E_i with E_i' for all i gives the same configuration up to the involution of the double cover $X \to Y$.) Note that $E_i \cdot E_j = 0$ if $i \neq j$, and

$$E_i^2 = E_i \cdot (E_i + E_i' - E_i') = E_i \cdot (pr_1^*(pt) - E_i') = -1.$$

Having chosen $E_1, \dots, E_8$, we can now determine E_0 uniquely. The orthogonal complement of $\{E_1, \dots, E_8\}$ in $H^2(X, \mathbb{Z})$ is $\mathbb{Z}H \oplus \mathbb{Z}E_0$; by diagonalizing (to get $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$) we can find E_0. (More precisely, diagonalize to determine the class of E_0 in $H^2(X, \mathbb{Z})$ up to sign. Then note that E_0 is effective to determine the class of E_0 in $H^2(X, \mathbb{Z})$. E_0 is the only effective representative in its class.)

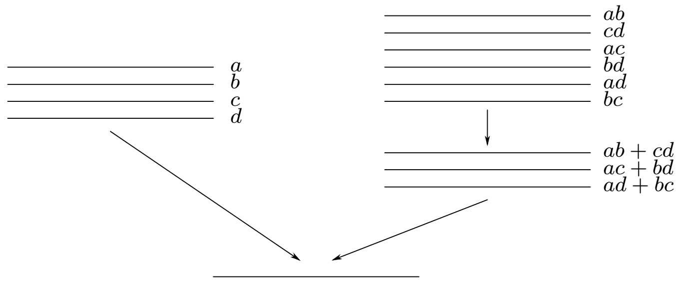

FIGURE 1. One direction of the Recillas correspondence

In short, given a rational elliptic fibration with choice of sections $E_1, \ldots, E_8$ (there are $\#W(E_8)$ possible choices by Section 2.1), we recover a point of D along with one of $8! \times 2^7$ choices. Hence each point of $\mathcal{D}$ has

$$\#W(E_8)/(8! \times 2^7) = 135$$

preimages. $\square$

4. Recillas' trigonal construction, and theta-characteristics of the genus 4 curve

We recall Recillas' beautiful construction ([R]) giving a bijection between connected simply branched quadruple covers of $\mathbb{P}^1$ to connected unramified double covers of connected simply branched triple covers of $\mathbb{P}^1$. The triple cover is simply branched over the same points as the quadruple cover. Furthermore, the Jacobian of Q is isomorphic to the Prym of $S \to C$. See [Do] for an excellent exposition (the Prym result is a variant of Donagi's tetragonal construction).

For simplicity, we give a description of the bijection over the complex numbers, although the construction works over any algebraically closed field of characteristic at least 5. Given the data of a simply branched quadruple cover $Q \to \mathbb{P}^1$, one recovers $S \to C \to \mathbb{P}^1$ (where $S \to C$ is an unramified double cover, and $C \to \mathbb{P}^1$ is a simply branched triple cover) as follows. After choosing a base point and making branch cuts to the branch points of Q, label the sheets of Q a, b, c, d (and remember how monodromy around the branch points of $Q \to \mathbb{P}^1$ permutes the sheets). Then $C \to \mathbb{P}^1$ is obtained by considering three sheets labelled $\{\{a,b\},\{c,d\}\}$, $\{\{a,c\},\{b,d\}\}$, $\{\{a,d\},\{b,c\}\}$ (each is a set of sets; for simplicity call these $ab+cd$, $ac+bd$, and $ad+bc$), where the action of the monodromy group on the three sheets is as induced by the action on the four sheets of Q. Also, $S \to \mathbb{P}^1$ is obtained by considering six sheets labelled $\{a,b\}$, $\{c,d\}$, $\{a,c\}$, $\{b,d\}$, $\{a,d\}$, $\{b,c\}$; the morphism $S \to C$ is as one would expect. One can quickly check that $C \to \mathbb{P}^1$ is simply branched and that $S \to C$ is étale. This construction is summarized pictorially in Figure 1.

The opposite direction is more elaborate. Let the three sheets of C be called x, y, and z, and let the six sheets of S be called x', x'', y', y'', z', z'' (with x' and x'' the

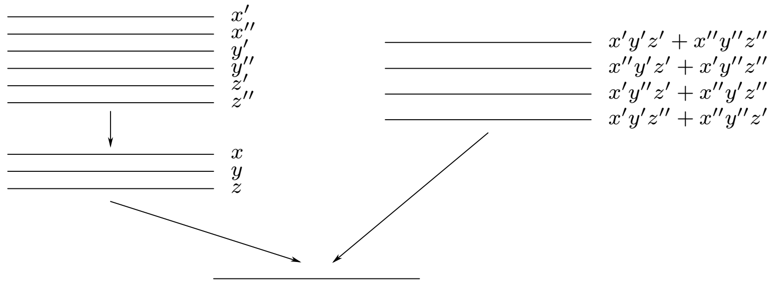

FIGURE 2. The other direction of the Recillas correspondence

preimages of of x, etc.). Then the monodromy group of $S \to \mathbb{P}^1$ is a subgroup of the symmetries of an octahedron: consider an octahedron with vertices $x', \dots, z''$, with x' opposite x'', y' opposite y'', and z' opposite z''. Then the octahedron has four pairs of opposite faces; this gives the four sheets of Q over $\mathbb{P}^1$. This construction is summarized pictorially in Figure 2.

4.1. *Proposition. — The morphisms $\mathcal{C} \to \mathcal{E}$ and $\mathcal{D} \to \mathcal{E}$ (obtained by composing the morphisms and isomorphisms of Propositions 3.1, 3.2, and 3.3) are obtained by Recillas' trigonal construction.*

Proof. Consider a point of $\mathcal{C}$, corresponding to a quadruple (canonical) cover $Q \to \mathbb{P}^1$, and any point q of $\mathbb{P}^1$ that is not a branch point of $Q \to \mathbb{P}^1$. Following the proof of Proposition 3.1, embed Q in a rational elliptic fibration, where the points of Q above q are distinct points a, b, c, d such that $2a = 2b = 2c = 2d$ in the group law of the fiber. Then to each unordered pair of elements of $\{a, b, c, d\}$ we can associated their difference, which is a non-zero two-torsion point of the fiber (the order of the pair doesn't matter). Also, the complementary pair is associated to the same two-torsion point. Hence we have identified the set $\{ab+cd, ac+bd, ad+bc\}$ (which appears in Recillas' construction) with the set of non-zero two-torsion points of the fiber (which agrees with the description of $\mathcal{E}$ given in Proposition 3.2).

The proof for $\mathcal{D}$ is identical. $\square$

The morphism can be related to the beautiful geometry of the genus 4 curve in the description of $\mathcal{E}$. Consider a given point of $\mathcal{E}$, corresponding to a curve C with vanishing theta-characteristic θ inducing the base-point-free pencil $C \to \mathbb{P}^1$. As C has genus 4, C has exactly $\#H^1(C, \mathbb{Z}/2) - 1 = 255$ connected étale double covers, so all such are accounted for by the tetragonal covers of the form $\mathcal{C}$ and $\mathcal{D}$.

Now C has 120 odd theta-characteristics and 136 even ones (including θ). Translating by θ, we have 120 "odd" 2-torsion points of $\operatorname{Pic} C$ (all non-zero), and 135 non-zero "even" 2-torsion points. The non-zero 2-torsion points parametrize double covers of C. Globalizing this construction over $\mathcal{E}$, we have an étale degree 255 morphism $\mathcal{E}' \to \mathcal{E}$; $\mathcal{E}'$ is disconnected, and splits into two pieces (not a priori connected) $\mathcal{E}'_{\text{odd}}$ and $\mathcal{E}'_{\text{even}}$, of degree 120 and 135 over $\mathcal{E}$ respectively. Hence we must have $\mathcal{E}'_{\text{odd}} = \mathcal{C}$ and $\mathcal{E}'_{\text{even}} = \mathcal{D}$. As a consequence we have:

4.2. *Proposition.* —

(a) *The monodromy group of odd (resp. even) non-zero 2-torsion of the Picard group of the universal curves over $\mathcal{M}_4^1 \setminus \mathcal{H}_4$ is full, i.e. $\#W(E_8)/\{\pm 1\}$. The same statement is true with non-zero 2-torsion of Picard replaced by the non-vanishing theta-characteristics.*
(b) *The Jacobian of any genus 3 curve $C_{\mathcal{C}}$ is the Prym of a rational 2-parameter family of genus 4 curves with vanishing theta-characteristic (parametrized by an open subset of $\mathbb{P}H^0(C_{\mathcal{C}}, K_{C_{\mathcal{C}}})^\vee$).*
(c) *The Jacobian of a hyperelliptic curve $C_{\mathcal{D}}$ is the Prym of a 3-parameter family of genus 4 curves with vanishing theta-characteristic (parametrized by an open subset of $\operatorname{Pic}^4 C_{\mathcal{D}}$).*

Proof. All that remains to be proved is the monodromy statement on theta-characteristics (or equivalently 2-torsion of Picard) on the universal curve over $\mathcal{M}_4^1 \setminus \mathcal{H}_4$. But if L is the universal E_8-Mordell-Weil-lattice over $\mathcal{A}$, then the monodromy group on L is full (i.e. $W(E_8)$), and $L/L(2)$ is now identified with the 2-torsion of the universal curve. $\square$

In light of the above discussion, we see that in Proposition 3.3 we have recovered a result of Mumford. (We are grateful to R. Smith for pointing this out.) Mumford has shown (Theorem (c) of [M] p. 344): If $\tilde{C} \to C$ is an étale double cover of a non-hyperelliptic genus 4 curve C (corresponding to $L \in \operatorname{Pic}(C)[2]$, with Prym variety P (with theta divisor Ξ), then (P, Ξ) is a (genus 3) Jacobian, and Ξ is singular iff (P, Ξ) is a hyperelliptic Jacobian iff C has a vanishing theta-characteristic θ such that $\theta \otimes L$ is an even theta-characteristic. (His method of proof is different.)

4.3. Explicit construction of 2-torsion of a genus 4 curve with vanishing theta-characteristic, and relation to $\mathcal{C}$ and $\mathcal{D}$. Given the explicitness of the above constructions, one should expect to see all of the theta-characteristics and 2-torsion of a genus 4 curve in $\mathcal{M}_4^1 \setminus \mathcal{H}_4$ in a particularly straightforward way. For concreteness, fix such a curve C (with theta-characteristic θ inducing pencil $\pi : C \to \mathbb{P}^1$) and embed it in the total space X of a pencil of cubics as described in the proof of Proposition 3.2. Then X is the blow-up of $\mathbb{P}^2$ (with hyperplane class H) at 9 points, with exceptional divisors $E_0, \dots, E_8$, and C is in class $9H - 3(E_1 + \cdots + E_8)$. X is an elliptic fibration (with zero-section E_0), and C is the non-zero 2-torsion of the fibration. Note that there is a natural involution ι of $X \to \mathbb{P}^1$ preserving C, which is the inverse in the group law of the fibration.

4.4. *Theorem. — The restriction map*

$$\sigma : \operatorname{Pic} X \to C \tag{2}$$

surjects onto $\mathbb{Z}\theta \oplus \operatorname{Pic} C[2]$.

We can describe σ explicitly. Note that $E_0 \cap C = \emptyset$, so $\sigma(E_0) = 0$. From the proof of Proposition 3.2, $\sigma(3H - (E_1 + \cdots + E_8)) = \theta$.

The genus 3 curve $C_{\mathcal{C}}$ in X described in the proof of Proposition 3.1 lies in class $6H - 2(E_2 + \cdots + E_8)$, so $C \cdot C_{\mathcal{C}} = 12$. But C and $C_{\mathcal{C}}$ both pass through the 12 nodes of the elliptic fibration, at the points where both $\pi : C \to \mathbb{P}^1$ and $C_{\mathcal{C}} \to \mathbb{P}^1$ ramify. Hence $\sigma(C_{\mathcal{C}})$ is the ramification divisor of π. By the Riemann-Hurwitz formula, $K_C = \pi^* K_{\mathbb{P}^1} + \sigma(C_{\mathcal{C}})$, so

$$\sigma(6H - 2(E_1 + \cdots + E_8)) = \sigma(-6H + 2(E_0 + \cdots + E_8) + 6H - 2(E_2 + \cdots + E_8)$$
$$\Longrightarrow \sigma(6H - 2(E_1 + \cdots + E_8) - 2E_1) = 0.$$

Hence $\sigma(3H - 2E_1 - (E_2 + \cdots + E_8)) \in \operatorname{Pic} C[2]$; this corresponds to the double cover associated to the tetragonal cover $C_{\mathcal{C}} \to \mathbb{P}^1$. (It would be interesting to understand this correspondence more explicitly.) Also $\theta = \sigma(3H - (E_1 + \cdots + E_8))$, so $\sigma(E_1)$ is a theta-characteristic, and by symmetry $\sigma(E)$ is a theta-characteristic for any of the 240 exceptional curves on X not meeting E_0 (i.e. the exceptional curves on the degree 1 Del Pezzo surface that is X blown down along E_0). As $\sigma(E) = \sigma(\iota(E))$ (where ι is the involution described just before the statement of Theorem 4.4), they come in pairs; they are the 120 odd theta-characteristics.

The hyperelliptic genus 3 curve $C_{\mathcal{D}}$ described in the proof of Proposition 3.3 lies in class $5H - 3E_0 - (E_1 + \cdots + E_8)$, so $C_{\mathcal{D}} \cdot C = 21$. Note that $C_{\mathcal{D}}$ and C meet at the three non-zero 2-torsion points in the three fibers where $H = 3E_0$ (restricted to the fiber). (These three fibers were mentioned in the proof of Proposition 3.3.) This is because in such a fiber, $H - E_0 = 2E_0$, and the restriction of $C_{\mathcal{D}}$ (resp. C) to the fiber are those points p such that $2p = H - E_0$ (resp. $2p = 2E_0$ and $p \neq 0$) in the group law of the fiber. Thus $C_{\mathcal{D}}$ meets C transversely at these 9 points, and at the 12 nodes of the fibration, so

$$\sigma(C_{\mathcal{D}}) = \sigma(C_{\mathcal{C}}) + 3\sigma(3H - (E_0 + \cdots + E_8)) \tag{3}$$

From the previous paragraph,

$$\begin{aligned} \sigma(C_{\mathcal{C}}) &= \sigma\left((6H - 2(E_2 + \cdots + E_8)) + (6H - 4E_1 - 2(E_2 + \cdots + E_8))\right) \\ &= \sigma(12H - 4(E_1 + \cdots + E_8)) \end{aligned}$$

Substituting this into (3):

$$\sigma(16H - 6(E_1 + \cdots + E_8)) = 0.$$

Thus $\sigma(8H - 3(E_1 + \cdots + E_8)) \in \operatorname{Pic} C[2]$, and this corresponds to the double cover associated to the tetragonal curve $C_{\mathcal{D}} \to \mathbb{P}^1$.

Proof of Theorem 4.4. $\sigma(E_0) = 0$, $\sigma(3H - (E_1 + \cdots + E_8))$ and $\sigma(E_i)$ $(i > 0)$ are theta-characteristics, and $\sigma(8H - 3(E_1 + \cdots + E_8)) \in \operatorname{Pic} C[2]$; hence the image of σ is contained in $\mathbb{Z}\theta \oplus \operatorname{Pic} C[2]$. On the other hand, as all theta-characteristics are in the image of σ, the image of σ contains $\mathbb{Z}\theta \oplus \operatorname{Pic} C[2]$. ∎

4.5. Summary: A unified picture of $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$. The results of this section can be summarized in a single picture essentially due to Dolgachev and Ortland ([DO] Section VII.5). Above the moduli space $\mathcal{A}$ of framed degree 1 Del Pezzo surfaces X, there is a cover corresponding to the Mordell-Weil lattice modulo even vectors. This is the same cover corresponding to 2-torsion in the Picard group of the genus 4 curve parametrized by $\mathcal{E}$. The author is grateful to N. Elkies for pointing out that the Weil-pairing on 2-torsion corresponds in the second incarnation to half of the Mordell-Weil pairing in the first incarnation.

This degree 256 cover splits into $\mathcal{C}$, $\mathcal{D}$, and the zero-section.

5. Explicit description of the morphisms $\mathcal{C} \to \mathcal{B}$, $\mathcal{D} \to \mathcal{B}$, $\mathcal{E} \to \mathcal{B}$ via discriminants

(This section is classical, although we hope the presentation is of interest.) Although the morphisms $\mathcal{C} \to \mathcal{B}$, $\mathcal{D} \to \mathcal{B}$, $\mathcal{E} \to \mathcal{B}$ are now clear via elliptic fibrations (Section 3), it is enlightening to see them in terms of the elementary algebra of discriminants. Let $f(a,b)$ be the binary quartic with indeterminate coefficients

$$f(a,b) = p_0a^4 + p_1a^3b + p_2a^2b^2 + p_3ab^3 + p_4b^4. \tag{4}$$

Then the discriminant of f is $\Delta f = 4u_2^3 + 27u_3^2$ where

$$\begin{aligned} u_2 &= p_1p_3 - 4p_0p_4 - p_2^2/3, \\ u_3 &= p_1^2p_4 + p_0p_3^2 - 8p_0p_2p_4/3 - p_1p_2p_3/3 + 2p_2^3/27. \end{aligned}$$

If the p_i are binary forms of degree 2 in x and y, Δf is of degree 12, and (as it is a square plus a cube) corresponds to a point in $\mathcal{B}$. Then (4) describes a hyperelliptic curve mapping 4-to-1 onto $\mathbb{P}^1$ (and specifically a (4,2)-class on $\mathbb{P}^1 \times \mathbb{P}^1$, which appeared in the proof of Proposition 3.3).

If the p_i are binary forms of degree i in x and y, then Δf is again of degree 12, and again corresponds to a point of $\mathcal{B}$; then (4) describes a plane quartic with the data of an additional point in the plane (more precisely a curve on the Hirzebruch surface $\mathbb{F}_1$), and we recover the construction of the proof of Proposition 3.1.

The multiplicities of 120 and 135 in the above cases are not clear.

The same idea works with cubics. Let

$$f(a,b) = q_0a^3 + q_1a^2b + q_2ab^2 + q_3b^3. \tag{5}$$

Then the discriminant of f is $\Delta f = (4u_2^3 + 27u_3^2)/q_0^2$ where

$$\begin{aligned} u_2 &= q_0q_2 - q_1^2/3, \\ u_3 &= q_0^2q_3 - q_0q_1q_2/3 + 2q_1^3/27. \end{aligned}$$

If the q_i are binary forms of degree $2i$ in x and y, and $q_0 = 1$, then Δf has degree 12, hence corresponds to a point of $\mathcal{B}$, and (4) describes the restriction of a cubic hypersurface to a quadric cone in $\mathbb{P}^3$, so we recover the construction of the proof of Proposition 3.2.

6. Twelve points on the projective line: the locus $\mathcal{Z}$

6.1. *Theorem. —*

(a) *The morphism $\pi : \mathcal{B} \to \mathcal{Z}$ is unramified and proper.*
(b) *π is birational, but not an isomorphism.*
(c) *π is the normalization of $\mathcal{Z}$.*

(Clearly (c) implies (b); however, we use (b) to prove (c).)

Hence $\mathcal{Z}$ is rational of dimension 11. Also, this result shows that the title of the paper is somewhat misleading: the locus $\mathcal{Z}$ (of twelve points on the projective line) is the "wrong" moduli space to study, and the "right" spaces are $\mathcal{A}$, $\mathcal{B}$, etc.

Proof. (a) We need only show that $\pi : \mathcal{B} \to \operatorname{Sym}^{12} \mathbb{P}^1$ is unramified and proper. Properness is immediate: if $\mathbb{P}$ is the weighted projective space from the definition of $\mathcal{B}$, then $\mathbb{P} \to \operatorname{Sym}^{12} \mathbb{P}^1$ is proper, and $\mathcal{B}$ was defined as the subset of $\mathbb{P}$ disjoint from the preimage of the discriminant locus Δ, so $\mathcal{B} \to \operatorname{Sym}^{12} \mathbb{P}^1 \setminus \Delta$ is proper.

To check that π is unramified, note that by the "hyperelliptic-trigonal" correspondence of Section 2.10, $\mathcal{B}$ can be associated with a locally closed subscheme of the étale cover $X \to \operatorname{Sym}^{12} \mathbb{P}^1 \setminus \Delta$, where X is the non-zero three-torsion (modulo $\{\pm 1\}$) of the hyperelliptic curve branched at those 12 points.

(One can also check that π is unramified directly, by explicitly describing a general point and tangent vector of $\mathcal{B}$, and computing the image in $\operatorname{Sym}^{12} \mathbb{P}^1$.)

(b) Assume that $\mathcal{B} \to \mathcal{Z}$ has degree greater than one. Then as $\dim \mathcal{B} = 11$, the parameter space P of homogeneous polynomials f_1, g_1, f_2, g_2 (in two variables) such that

(i) $f_1^3 + g_1^2 = f_2^3 + g_2^2$,
(ii) $\deg f_i = 4$, $\deg g_i = 6$,
(iii) $g_1^2 \neq g_2^2$ (or equivalently $f_1^3 \neq f_2^3$)

is of dimension at least 12. But there are 10 dimensions of choices of coefficients of f_1 and f_2. Once f_1 and f_2 are given, there is a one-parameter family of choices of g_1 and g_2 (through $(g_2 + g_1)(g_2 - g_1) = f_1^3 - f_2^3 \neq 0$, as the roots of $f_1^3 - f_2^3$ must be split between $(g_2 + g_1)$ and $(g_2 - g_1)$; the one parameter comes from the choice of leading coefficient of $(g_2 + g_1)$). Hence $\dim P = 11$, giving a contradiction. Thus $\mathcal{B}$ is birational.

As $(x^6)^2 + (y^4)^3 = (y^6)^2 + (x^4)^3$ has 12 distinct roots, $\mathcal{B} \to \mathcal{Z}$ isn't injective and thus isn't an isomorphism.

(c) From (a), π is proper and (as $\mathcal{B}$ is smooth and π is unramified) quasifinite, so π is finite. As π is birational and $\mathcal{B}$ is normal, π is the normalization of $\mathcal{Z}$. ☐

As a side benefit, we see from the proof that at any point of $\mathcal{Z}$, the branches are smooth.

Note that the analogous argument works for $\operatorname{Sym}^n \mathbb{P}^1$ when $n = 6k$, $k \geq 2$ (i.e. a general homogeneous polynomial of degree n that is expressible as the sum of a cube and a square, is so expressible in only one way), but the proof of (b) (and hence (c)) breaks down when $k = 1$. Indeed, the general sextic can be written as a cube plus a square in 40 essentially different ways ([E] Theorem 3 i), a result of Clebsch). Also, the proof that π is unramified in (a) must be done explicitly.

6.2. *Theorem.* — *The degree of $\mathcal{Z}$ in $\mathbb{P}^{12}$ is 3762.*

Proof. In [V] Section 9.1, it was shown that the number of genus 3 canonical covers of $\mathbb{P}^1$ (i.e. points of $\mathcal{C}$) with 11 fixed branch points is 3762×120. The result then follows Theorems 2.12 and 6.1.

Alternatively, Zariski computed degree $\deg \mathcal{Z} = 3762$ via the locus $\mathcal{F}$ described in Section 2.10; see Section 7.2. □

It would be interesting to derive this degree more directly.

6.3. *Remark.* W. Lang has proved that the degree of the locus $\mathcal{A}$ in characteristic 2 is 1 — remarkably, the condition for twelve points to appear as the discriminant locus of a rational elliptic fibration is linear ([L]).

7. Classical results

Some of the above links have been described classically.

7.1. *Zeuthen.* In [Ze] (p. XXII), Zeuthen solves an enumerative problem that, in modern language, translates to: given 11 points on a line, how many canonical covers are there branched at those 11 points? He gives the correct answer ($451440 = 120 \times 3762$), but it is unclear how he obtained this. (This fact falls out as a side benefit of the proof of [V] Theorem 8.1.) More precisely, he asks a different question, to which he gives an incorrect answer, without throwing off his calculation of the characteristic number of quartic curves. The reason for his error, in modern terms, is that he did not suspect that the degree of $\mathcal{C} \to \mathcal{Z}$ was 120. This is discussed in [V] Section 9.1.

7.2. *Zariski.* In [Z1], Zariski calculates the answer to a similar question: he defines the locus Γ_{12} in $\operatorname{Sym}^{12} \mathbb{P}^1$ as the image of $\mathcal{C}$ (our $\mathcal{Z}$), and computes $\deg \Gamma_{12} = 3762$ using the genus 5 hyperelliptic locus $\mathcal{F}$ (see Section 2.10). He also discusses the loci $\mathcal{B}$ and $\mathcal{D}$. (The quintic plane curve with triple point from the proof of Prop. 3.3 appears explicitly on p. 319.) He did not seem to be aware that the degree of the morphisms $\mathcal{C} \to \Gamma_{12}$, $\mathcal{D} \to \Gamma_{12}$ he describes are greater than 1.

7.3. *Coble.* Many of the constructions above have appeared, at least implicitly, in [Co] (mainly Section 51, but also Sections 50 and 58).

$\mathcal{C}$: On p. 210, the sextics with seven nodes from the proof of Proposition 3.1 appear, and they are shown to correspond to the 120 pairs of (-1)-curves on a degree 1 Del Pezzo surface exchanged by an involution, and also with odd theta-characteristics. On p. 219, this is connected to the data of plane quartics with another point in the plane.

$\mathcal{D}$: On p. 108, the hyperelliptic curve of genus 3 appears, with branch points corresponding to the 8 exceptionals on a degree 1 Del Pezzo surface. On p. 212, the quintic with a triple point from the proof of Proposition 3.3 appears, although it is not identified with the hyperelliptic genus 3 curve mentioned earlier.

$\mathcal{E}$: On p. 209, a construction from the proof of Proposition 3.2 appears, the plane nonic with triple points at the 8 given points. The morphism $\mathcal{A} \to \mathcal{E}$ is explicitly described. On p. 220, Coble remarks (section 58) that Schottky gives the coordinates of the 8 points and the equation of the nonic explicitly from modular functions on the space $\mathcal{M}_4^1$.

8. Further questions

8.1. *Interpreting these results in terms of elliptic fibrations.* Given an elliptic fibration over some base B, the non-zero two-torsion of the fibration is a triple cover of B (and this can be tweaked to give quadruple covers as 2-torsion information, see the proofs of Propositions 3.1 and 3.3). One natural question is: what triple (or quadruple) covers can arise as two-torsion of an elliptic fibration?

Also, if the family is not isotrivial, it can be reconstructed from the n-torsion of the family if $n > 2$. A second natural question is: to what extent is this true if $n = 2$?

In the case of rational elliptic fibrations, both questions are answered completely. For the first, striking explicit geometric conditions are given. And for the second, one can recover the fibration using the geometric conditions in the answer to the first.

It would be interesting to extend this analysis to other elliptic fibrations.

(a) The next reasonable geometric example would be elliptic K3-surfaces, which have 24 singular fibers. This analog of $\mathcal{A}$ corresponds to a codimension 3 locus in $\operatorname{Sym}^{24}\mathbb{P}^1$, and is of course the same as the analog of $\mathcal{B}$. (Another codimension 3 locus in the parameter space of 24 points on the projective line are genus 5 curves mapped to $\mathbb{P}^1$ by a canonical pencil, but there's no obvious reason why this should be the same locus!)

(b) Could any sense be made of this question arithmetically, e.g. over $\mathbb{Q}$?

References

[B] A. Beauville, *Variétés de Prym et Jacobiannes Intermédiaires*, Ann. Scient. Éc. Norm. Sup., **4** (1977) no. 10, pp. 309–391.

[Co] A. Coble, *Algebraic geometry and theta functions*, A.M.S., Providence RI 1961 (reprinting of 1929 edition).

[CD] F. Cossec and I. Dolgachev, *Enriques surfaces I*, Progress in Math. **76**, Birkhäuser, Berlin, 1989.

[DM] P. Deligne and G. D. Mostow, *Commensurabilities among Lattices in $PU(1,n)$*, Annals of Math. Studies **132**, Princeton U.P., Princeton, 1993.

[D] I. Dolgachev, *Rationality of fields of invariants*, in *Algebraic geometry, Bowdoin, 1985* pp. 3–16, Proc. Sympos. Pure Math. 46, Part 2, AMS, Providence, R.I., 1987.

[DL] I. Dolgachev and A. Libgober, *On the fundamental group of the complement to a discriminant variety*, in *Algebraic geometry: Proceedings, U. of Ill. at Chicago Circle 1980*, Lecture Notes in Math. **862**, Springer-Verlag, Berlin, 1981.

[DO] I. Dolgachev and D. Ortland, *Point sets in projective spaces and theta functions*, Astérisque **165**, Soc. Math. de France, Paris, 1988.

[Do] R. Donagi, *The fibers of the Prym map*, Contemp. Math. **136** (1992) 55–125.

[EEHS] D. Eisenbud, N. Elkies, J. Harris, and R. Speiser, *On the Hurwitz scheme and its monodromy*, Compositio Mathematica **77** (1991) 95–117.

[E] N. D. Elkies, *The identification of three moduli spaces*, preprint 1999, math.AG/9905195.

[K] P. Katsylo, *Rationality of the moduli variety of curves of genus 3*, Comment. Math. Helv. **71** (1996), no. 4, 507–524.

[HL] G. Heckman and E. Looijenga, manuscript in preparation.

[L] W. Lang, personal communication.

[Ma] Yu. Manin, *Cubic forms* 2nd ed., North-Holland, New York, 1986.

[Mi] R. Miranda, *Persson's list of singular fibers for a rational elliptic surface*, Math Z. **205** (1990) no. 2, 191–211.

[MS] D. Mumford and K. Suominen, *Introduction to the theory of moduli*, p. 171–222, in *Algebraic Geometry, Oslo 1970*, F. Oort ed., Wolters-Noordhoff Publ., Groningen 1972.

[M] D. Mumford, *Prym varieties I*, in *Contributions to analysis*, pp. 325–350. Academic Press, New York, 1974.

[P] U. Persson, *Configurations of Kodaira fibers on rational elliptic surfaces*, Math. Z. **205** (1990) no. 1, 1–47.

[R] S. Recillas, *Jacobians of curves with g^1_4's are the Pryms of trigonal curves*, Bol. Soc. Mat. Mexicana **19** (1974), no. 1, 9–13.

[S] T. Shioda, *On the Mordell-Weil Lattices*, Comment. Math. Univ. St. Paul. **39** (1991), no. 2, 211–240.

[V] R. Vakil, *Characteristic numbers of quartic plane curves*, Can. J. Math., to appear. Also available at http:www-math.mit.edu/˜vakil/preprints.html.

[Z1] O. Zariski, *On Hyperelliptic θ-Functions with Rational Characteristics*, Amer. J. Math. **50** (1928), 315–344.

[Z2] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math., **51** (1929) pp. 305–328.

[Ze] H.G. Zeuthen, *Almindelige Egenskaber ved Systemer af plane Kurver*, Kongelige Danske Videnskabernes Selskabs Skrifter — Naturvidenskabelig og Mathematisk, **10** (1873) 285–393. Danish with French summary.

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