# ARTICLE SKELETON

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Abstract. This is the abstract.

Contents



Everything in the first four sections should be in good shape.

## 1. Miscellanea

**1.1.** *Terminology.* By *lattice* $L$ in a real vector space $V$, we mean a $\mathbb{Z}$-submodule of $V$ with an inner product such that $L \times_{\mathbb{Z}} \mathbb{R} = V$. The fundamental volume of $L$ is denoted vol. If two nonzero numbers $a$ and $b$ have quotient that is an integral power of 2, we say $a \sim b$. If two nonzero numbers $a$ and $b$ have quotient that is an integral power of 3, we say $a \approx b$.

**1.2.** *Acknowledgements.* Thanks Jordan and Kiran.

## 2. Representation theory

Throughout this section, let $V$ be a real vector space with an inner product $\langle \cdot, \cdot \rangle$ and an $S_4$-action preserving the inner-product. Let $\tau$ be the number of standard

*Date*: Thursday March 16, 2:30 pm.
Partially supported by NSF Grant DMS–9970101.
1991 MATHEMATICS SUBJECT CLASSIFICATION: PRIMARY **(???)**, SECONDARY **(???)**

| $S_4$ | $S_3$ | $D_4$ | $K_4$ |
|---|---|---|---|
| triv | triv | triv | triv |
| sgn | sgn | sgn | sgn |
| $X_2$ | std | triv $\oplus$ sgn | triv $\oplus$ sgn |
| std | triv $\oplus$ std | $W_1 \oplus W_2$ | triv $\oplus X_1 \oplus X_1'$ |
| std $\otimes$ sgn | sgn $\oplus$ std | $W_1' \oplus W_2$ | sgn $\oplus X_1 \oplus X_1'$ |

TABLE 1. Decomposing irreducible representations of $S_4$ **(chad)**

irreducible representations in the representation $V$. Let $L$ be a lattice in $V$ such that the $S_4$-action is rational with respect to $L$.

We refer to the irreducible representations of $S_4$ as $\mathrm{Irr}_{S_4} = \{\mathrm{triv}, \mathrm{sgn}, X_2, \mathrm{std}, \mathrm{std} \otimes \mathrm{sgn}\}$ (with corresponding representations $R_{\mathrm{triv}}, \ldots, R_{\mathrm{std}\otimes\mathrm{sgn}}$). **(Manjul, I changed $V_2$ to $X_2$.)** Let $V = V_{\mathrm{triv}} \oplus \cdots \oplus V_{\mathrm{std}}$ be the decomposition of $V$ into irreducible components, and let $L_\lambda = V_\lambda \cap L$ for $\lambda \in \mathrm{Irr}_{S_4}$. (Thus $\dim V_{\mathrm{std}} = 3\tau$.) Define $V_{\mathrm{triv},X_2} = V_{\mathrm{triv}} \oplus V_{X_2} \subset V$, and $L_{\mathrm{triv},X_2} = V_{\mathrm{triv},X_2} \cap L$; define $V_{\mathrm{triv},X_2,\mathrm{std}}$ and $L_{\mathrm{triv},X_2,\mathrm{std}}$ similarly.

As all irreducible representations of $S_4$ are rational, the intersection of $L$ with any $V_\lambda$ is a lattice in $V_\lambda$. The sublattice of $L$ generated by elements in the $V_\lambda$ is of finite index in $L$.

**2.1.** *Orthogonality.* By Schur's lemma, $\langle v, w\rangle = 0$ if $v \in V_\lambda$, $w \in V_{\lambda'}$, $\lambda \neq \lambda'$. Reason: $V_\lambda$ is (noncanonically) self-dual (as a representation), so the inner-product restricted to $V_\lambda \times V_{\lambda'}$ lies in $((V_\lambda \otimes V_{\lambda'})^*)^G = \mathrm{Hom}(V_\lambda, V_{\lambda'})^G = 0$. Hence $V = \oplus_\lambda V_\lambda$ is a direct-sum decomposition as inner-product spaces (i.e. the inner product on $V$ is the direct sums of the restricted inner products on $V_\lambda$).

Up to scalar, there is only one $S_4$-invariant inner product on an irreducible representation of $S_4$, also by Schur.

Table 1 shows how the irreducible representations of $S_4$ decompose into irreducible representations of the subgroups $S_3$, $D_4$, and $K_4 = \{e, (12), (34), (12)(34)\}$ (*not* the Klein 4-group).

**2.2.** The one-dimensional subspaces of $R_{X_2}$ fixed by $D_4$ and $K_4$ are the same, so $V_{X_2}^{D_4} = V_{X_2}^{K_4}$.

**2.3.** The vector spaces $R_{\mathrm{std}}^{S_3}$ and $R_{\mathrm{std}}^{K_4}$ are both one-dimensional, but are not the same. We analyze the restriction of some lattice $L_{\mathrm{std}}$ (preserved by the representation) to these subspaces. Let $v_1 - v_2$, $v_2 - v_3$, $v_3 - v_4$ be the usual basis of $R_{\mathrm{std}}$. Then $R_{\mathrm{std}}^{S_3}$ is generated by $q = 3v_1 - v_2 - v_3 - v_4$ (for the $S_3$ corresponding to $\{2,3,4\}$), and $R_{\mathrm{std}}^{K_4}$ is generated by $r = v_1 + v_2 - v_3 - v_4$ (for the $K_4$ corresponding to $\{(12),(34)\}$. Note that $q = r + (23)r + (24)r$ and $2r = q + (12)q$, and $|r| = \sqrt{3}|q|$ with respect to the norm induced by any $S_4$-invariant inner product.

Hence the linear transformation $e+(12)$ gives an isomorphism of $V_{\mathrm{std}}^{S_3}$ onto $V_{\mathrm{std}}^{K_4}$, which scales the inner product by $(2/\sqrt{3})^2$, and **(ghana)**

$$(e+(12))L_{\mathrm{std}}^{S_3} \subset L_{\mathrm{std}}^{K_4} \subset \frac{1}{2}(e+(12))L_{\mathrm{std}}^{S_3}. \tag{1}$$

Therefore $\mathrm{vol}(L_{\mathrm{std}}^{S_3})/\mathrm{vol}(L_{\mathrm{std}}^{K_4}) = \sqrt{3}/2^s$ where $s=0$ or 1.

If $V=V_{\mathrm{std}}$ is a representation containing only standard representations, and $L_{\mathrm{std}}$ is a lattice preserved by the action, then (1) still holds and $e+(12)$ still scales the inner product by $(2/\sqrt{3})^2$, so **(haiti)**

$$\mathrm{vol}(L_{\mathrm{std}}^{S_3})/\mathrm{vol}(L_{\mathrm{std}}^{K_4}) = \sqrt{3}^{\tau}/2^s, \tag{2}$$

where $\tau$ is the number of standards appearing, and $s \le \tau$. **(Say more here.)**

**2.4.** *Theorem.* — **(cuba)** *(a)* $\dim(V^{S_4}) + \dim(V^{K_4}) = \dim(V^{S_3}) + \dim(V^{D_4})$,

*(b)* $\sqrt{3}^{\tau}\,\mathrm{vol}(L^{S_4})\,\mathrm{vol}(L^{K_4}) \sim \mathrm{vol}(L^{S_3})\,\mathrm{vol}(L^{D_4})$.

*Proof.* Part (a) is immediate from Table 1our earlier discussion. For (b), from Table 1, we may ignore the irreducibles sgn and $\mathrm{std}\otimes\mathrm{std}$, and restrict attention to $L_{\mathrm{triv},X_2,\mathrm{std}}$.

The projection $\rho_{\mathrm{std}} : V \to V_{\mathrm{std}}$ is given by $\frac{1}{8}\sum_{g\in S_4}\chi_{\mathrm{std}}(g)g$, so

$$L_{\mathrm{triv},X_2} \oplus L_{\mathrm{std}} \subset L_{\mathrm{triv},X_2,\mathrm{std}} \subset \frac{1}{8}(L_{\mathrm{triv},X_2} \oplus L_{\mathrm{std}}).$$

Thus the index of $L_{\mathrm{triv},X_2} \oplus L_{\mathrm{std}}$ in $L_{\mathrm{triv},X_2,\mathrm{std}}$ is a power of 2, so for any subgroup $G$ of $S_4$, $\mathrm{vol}(L^G) \sim \mathrm{vol}(L_{\mathrm{triv},X_2}^G)\,\mathrm{vol}(L_{\mathrm{std}}^G)$.

From Table 1, $L_{\mathrm{triv},X_2}^{S_4} = L_{\mathrm{triv},X_2}^{S_3}$ and $L_{\mathrm{std}}^{S_4} = L_{\mathrm{std}}^{D_4} = 0$. From Section 2.2, $L_{\mathrm{triv},X_2}^{K_4} = L_{\mathrm{triv},X_2}^{D_4}$. From (2), $\sqrt{3}^{\tau}\,\mathrm{vol}(L_{\mathrm{std}}^{K_4}) \sim \mathrm{vol}(L_{\mathrm{std}}^{S_3})$. The result follows. □

**2.5.** *Notes to ourselves.* Properties of $S_4$ we use: $G$'s irreducible representations are rational and self-dual.

More precise version of the Theorem: $2^t\sqrt{3}^{\tau}\,\mathrm{vol}(L^{S_4})\,\mathrm{vol}(L^{K_4}) \sim 2^s\,\mathrm{vol}(L^{S_3})\,\mathrm{vol}(L^{D_4})$, where $t \le 2\dim V_{\mathrm{triv},X_2,\mathrm{std}}$ and $s\le\tau$. Here is an example to show that $t$ need not be 0: consider the representation $(R_{\mathrm{triv}}\oplus R_{\mathrm{std}})\oplus(R_{\mathrm{triv}}\oplus R_{X_2})$, with generators $v_1$, $v_2$, $v_3$, $v_4$, $v_{12+34}$, $v_{13+24}$, $v_{14+23}$, $(v_1+v_2+v_{12+34})/2$.

## 3. Number theory

For the purpose of this section, fix the following diagram of number fields:

$$
\begin{tikzcd}[column sep=small]
 & M^{24} \arrow[dl] \arrow[dr] & \\
F^4 \arrow[dr] & & L^6 \arrow[d] \\
 & & K^3 \arrow[dl] \\
 & Z^1 &
\end{tikzcd}
\tag{3}
$$

**(iceland)** The superscripts refer to the degree of the extension of $Z$, $Z \subset M$ is a Galois $S_4$ extension, and $F$ (resp. $L$, $K$) is the field fixed by some $S_3$ (resp. $K_4$, $D_4$). Note that this diagram can be constructed given only the data of $Z \subset F$ (under the assumption that the Galois group of this field extension is $S_4$), or $Z \subset K \subset L$ (under the same assumption that the Galois group of $Z \subset L$ is $S_4$). Let $d$ be the degree of $Z$ over $\mathbb{Q}$.

Let $r_1$ (resp. $s_1$) be the number of real (resp. complex) embeddings of $Z$, $h_1$ the class number, $R_1$ the regulator, $D_1$ the discriminant, and $w_1$ the number of roots of unity. Similarly, let the corresponding variables with subscript 4 (resp. 3, 6, 24) be the corresponding numbers for $F$ (resp. $K$, $L$, $M$).

**3.1.** *Roots of unity.* Note that $w_1 = w_3 = w_4 = w_6$, as any root of unity of $M$ that isn't in $Z$ must generate the discriminant field (as the abelianlization of $S_4$ is $\mathbb{Z}/2$), which isn't contained in $Z$, $F$, $K$ or $L$.

**3.2.** Suppose $A \subset B$ is a galois field extension, with Galois group $G$. If $V$ is the vector space in which the unit lattice of $B$ naturally sits, then $V^G$ can be identified with the vector space in which the unit lattice of $A$ usually sits.

Applying this to (3) and using Proposition 2.4 (a),

$$(r_1 + s_1) + (r_6 + s_6) = (r_3 + s_3) + (r_4 + s_4).$$

Clearly $(r_1 + r_6) + 2(s_1 + s_6) = (r_3 + r_4) + 2(s_3 + s_4)$ (as both sides are $7d$), so $r_1 + r_6 = r_3 + r_4$ and $s_1 + s_6 = s_3 + s_4$.

By standard properties of the Artin L-function and Table 1,

$$\zeta_Z(s)\zeta_L(s) = \zeta_K(s)\zeta_F(s). \tag{4}$$

**(spain)** Using the functional equations for the four $\zeta$-functions, $D_1 D_6 = D_3 D_4$. (The arguments are identical to those given in [H] in the case $Z = \mathbb{Q}$, so we omit them.)

Comparing residues at $s = 1$ in (4),

$$\frac{2^{r_1}(2\pi)^{s_1}R_1h_1}{w\sqrt{D_1}} \cdot \frac{2^{r_6}(2\pi)^{s_6}R_6h_6}{w\sqrt{D_6}} = \frac{2^{r_3}(2\pi)^{s_3}R_3h_3}{w\sqrt{D_3}} \cdot \frac{2^{r_4}(2\pi)^{s_4}R_4h_4}{w\sqrt{D_4}}.$$

Hence, as $r_1 + r_6 = r_3 + r_4$ etc., **(egypt)**

$$(R_1h_1)(R_6h_6) = (R_3h_3)(R_4h_4). \tag{5}$$

## 4. Class number equality

**4.1.** *Theorem.* — **(iran)** *$h_1h_6 \sim h_3h_4$, or equivalently, $h_4/h_1 = h_6/h_3$.*

**4.2.** *Remark.* Up to powers of 2 and 3, it is straightforward to show that $h_1h_6 = h_3h_4$ (by the representation theory of $S_4$ to finite abelian groups, for example), and even that the relative class groups of $Z \subset F$ and $K \subset L$ are isomorphic away from 2 and 3. Thus the main content of Theorem 4.1 is the 3-part of the statement. (Our argument does not show that the relative class groups are isomorphic at 3, only that they have the same 3-order.)

**4.3.** The rest of this section will be devoted to proving Theorem 4.1.

By (5), it suffices to prove that $R_1R_6 \sim R_3R_4$.

Let $W_{24}$ be the inner product space with standard basis associated to the $r_{24}+s_{24}$ infinite places of $M$. If $U$ is the unit group of $M$, let $\phi : U \to W_{24}$ be the group homomorphism

$$u \mapsto (\log |u^{(1)}|, \ldots, \log |u^{(r_{24}+s_{24})}|).$$

Let $L_{24}$ be the lattice in $W_{24}$ generated by the image of $U$, and $(1, \ldots, 1)$. The regulator is given ([L] p. 109) by

$$R_{24} = \frac{2^{r_{24}}}{r_{24} + 2s_{24}} \operatorname{vol}(L_{24}).$$

The galois group $S_4$ of $Z \subset M$ acts on $R_{24}$ by permuting the coordinates.

Define $W_i$, $L_i$ similarly for $i = 1, 3, 4, 6$. There are natural linear maps $\psi_i : W_i \to W_{24}$ restricting to $L_i \to L_{24}$. Note that $R_1R_6 \sim R_3R_4 \Leftrightarrow \operatorname{vol}(L_1)\operatorname{vol}(L_6) \sim \operatorname{vol}(L_3)\operatorname{vol}(L_4)$; we will prove the latter.

**4.4. Assumption R.** We'll need to assume that the index of $\psi_i(L_i)$ in $L_{24} \cap W_{24}^G$ (where $G$ is the appropriate subgroup of $S_4$) is a power of 2.

It will suffice to assume that the discriminant extension of $Z$ isn't $Z[\omega]$, where $\omega$ is a cube root of unity. Details, should we need it, are in an e-mail, currently posted on Manjul's page in Ravi's account.

**4.5.** By Theorem 2.4, $\sqrt{3}^{\tau}\operatorname{vol}(L_{24}^{S_4})\operatorname{vol}(L_{24}^{K_4}) \sim \operatorname{vol}(L_{24}^{S_3})\operatorname{vol}(L_{24}^{D_4})$, where $\tau$ is the number of standard irreducible representations in $W_{24}$. From Table 1, $\tau = \dim(W_4) - \dim(W_1) = r_4 + s_4 - r_1 - s_1$. Hence, by **(Assumption R)**, **(nepal)**

$$\sqrt{3}^{r_4+s_4-r_1-s_1}\operatorname{vol}(\psi_1(L_1))\operatorname{vol}(\psi_6(L_6)) \sim \operatorname{vol}(\psi_3(L_3))\operatorname{vol}(\psi_4(L_4)). \tag{6}$$

We next compute the scaling factors $\operatorname{vol}(\psi_i(L_i))/\operatorname{vol}(L_i)$. Note that $W_i$ is naturally a direct sum of components coming from each place of $Z$, so we consider four cases.

*Case $s_1$.* Each of the $s_1$ complex places of $Z$ splits into 3, 4, and 6 complex places of $K$, $F$, and $L$ respectively.

The splitting of the real places of $Z$ over $K$, $L$, and $M$ turns out to be determined by the splitting over $F$, which can happen in 3 ways.

*Case $\alpha$.* Suppose $\alpha$ of the real places of $Z$ split into 4 real places of $F$. Then each one splits into 3, 6, and 24 real places of $K$, $L$, and $M$ respectively.

*Case $\beta$.* Suppose $\beta$ of the real places of $Z$ split into 2 real and 1 complex place of $F$. Then each one splits into 1 real and 1 complex place of $K$, 2 real and 2 complex places of $L$, and 12 complex places of $M$.

*Case $\gamma$.* Suppose $\gamma$ of the real places of $Z$ split into 2 complex places of $F$. Then each one splits into 3 real places of $K$, 2 real and 2 complex places of $L$, and 12 complex places of $M$.

Note that **(france)**

$$\begin{aligned} r_4 + s_4 - r_1 - s_1 &= (4\alpha + 2\beta) + (\beta + 2\gamma + 4s_1) - (\alpha + \beta + \gamma) - s_1 \\ &= 3\alpha + 2\beta + \gamma + 3s_1. \end{aligned}$$

We now compute the scaling factors contributed by each of the four cases.

*Scaling factor in Case $s_1$.* As each complex place of $Z$ splits into 24 complex places of $M$, the corresponding basis element of $W_1$ is sent to the sum of 24 basis elements of $M$. Hence the contribution to $\operatorname{vol}(\psi_1(L_1))/\operatorname{vol}(L_1)$ is $\sqrt{24}^{s_1} \sim \sqrt{6}^{s_1}$. Each of the correponding three complex places of $F$ splits into 8 complex places of $M$, so the contribution to $\operatorname{vol}(\psi_3(L_3))/\operatorname{vol}(L_3)$ is $(\sqrt{8}^3)^{s_1} \sim \sqrt{2}^{s_1}$. Similarly, the contribution to $\operatorname{vol}(\psi_4(L_4))/\operatorname{vol}(L_4)$ is $(\sqrt{6}^4)^{s_1} \sim \sqrt{3}^{4s_1}$, and the contribution to $\operatorname{vol}(\psi_6(L_6))/\operatorname{vol}(L_6)$ is $(\sqrt{4}^6)^{s_1} \sim 1$.

*Scalaing factor in Case $\alpha$.* Similarly, the contributions in this case are $\sqrt{6}^{\alpha}$, $\sqrt{2}^{\alpha}$, $\sqrt{3}^{4\alpha}$, and 1.

*Scaling factor in Case $\beta$.* Each such real place of $Z$ splits into 12 complex places of $M$, so this case contributes $\sqrt{12}^{\beta} \sim \sqrt{3}^{\beta}$ to $\operatorname{vol}(\psi_1(L_1))/\operatorname{vol}(L_1)$. The corresponding real place of $K$ splits into 4 complex places of $M$, and the complex place of $F$ splits into 8 places, contributing $(\sqrt{4}\sqrt{8})^\beta \sim \sqrt{2}^\beta$ to $\operatorname{vol}(\psi_3(L_3))/\operatorname{vol}(L_3)$. The 2 corresponding real places of $F$ each split into 3 places of $M$, and the corresponding complex place of $F$ splits into 6 places of $M$, so this case contributes $(\sqrt{3}^2\sqrt{6})^\beta$ to $\operatorname{vol}(\psi_4(L_4))/\operatorname{vol}(L_4)$. The 2 corresponding real places of $L$ each split into 2 places of $M$, and the 2 corresponding complex place of $L$ splits into 4 places of $M$, so this case contributes $(\sqrt{2}^2\sqrt{4}^2)^\beta \sim 1$ to $\operatorname{vol}(\psi_6(L_6))/\operatorname{vol}(L_6)$.

*Scaling factor in Case* $\gamma$. By a similar argument, this case contributes $\sqrt{12}^\gamma \sim \sqrt{3}^\gamma$ to $\operatorname{vol}(\psi_1(L_1))/\operatorname{vol}(L_1)$, $(\sqrt{4}^3)^\gamma \sim 1$ to $\operatorname{vol}(\psi_3(L_3))/\operatorname{vol}(L_3)$, $(\sqrt{6}^2)^\gamma \sim \sqrt{3}^{2\gamma}$ to $\operatorname{vol}(\psi_4(L_4))/\operatorname{vol}(L_4)$, and $(\sqrt{4}^2\sqrt{8}^2)^\gamma \sim 1$ to $\operatorname{vol}(\psi_6(L_6))/\operatorname{vol}(L_6)$.

Therefore, from (6) and (7),

$$\sqrt{3}^{3\alpha+2\beta+\gamma+3s_1}\operatorname{vol}(\psi_1(L_1))\operatorname{vol}(\psi_6(L_6)) \sim \operatorname{vol}(\psi_3(L_3))\operatorname{vol}(\psi_4(L_4))$$

from which

$$\begin{aligned} & \sqrt{3}^{3\alpha+2\beta+\gamma+3s_1}\cdot\sqrt{6}^{s_1}\sqrt{6}^{\alpha}\sqrt{3}^{\beta}\sqrt{3}^{\gamma}\operatorname{vol}(L_1)\cdot\operatorname{vol}(L_6) \\ \sim\ & \sqrt{2}^{s_1}\sqrt{2}^{\alpha}\sqrt{2}^{\beta}\operatorname{vol}(L_3)\cdot\sqrt{3}^{4s_1}\sqrt{3}^{4\alpha}\sqrt{54}^{\beta}\sqrt{3}^{2\gamma}\operatorname{vol}(L_4) \\ \Leftrightarrow\ & \operatorname{vol}(L_1)\operatorname{vol}(L_6) \sim \operatorname{vol}(L_3)\operatorname{vol}(L_4), \end{aligned}$$

as desired. □

## 5. The quadratic-cubic variation

For this section, suppose you have a diagram of fields

$$\begin{tikzcd} & F^6 \arrow[dl] \arrow[dr] & \\ F^2 \arrow[dr] & & F^3 \arrow[dl] \\ & F^1 & \end{tikzcd}$$

**5.1.** *Assumptions.* (1) Suppose every root of unity of $F^6$ actually lies in $F^1$. (2) Suppose $F^2$ can't be obtained from $F^1$ by adjoining a trace 0 root of unity.

**5.2.** Define $r_i$, $s_i$ similarly as before. Suppose $\alpha$ is the number of real places of $F^1$ that split completely in $F^6$, and $\beta$ is the remaining number of real places.

Let $\tau = 2s_1 + 2\alpha + \beta$. Note that $\tau > 0$; it is also the number of standards in the representation of $S_3$ on the unit lattice of $F^6$.

**5.3.** *Theorem.* — $h_{F^6/F^2} = 2^\tau h^2_{F^3/F^1}$.

The rest of this section will be spent proving this.

| $S_3$ | $S_2$ | $A_3$ | $(1)$ |
|---|---|---|---|
| triv | triv | triv | triv |
| sgn | sgn | triv | triv |
| std | triv $\oplus$ sgn | $\chi_1 \oplus \overline{\chi}_1$ | triv $\oplus$ triv |

TABLE 2. Decomposing irreducible representations of $S_3$

**5.4. Representation theory.**

We first make a table showing how irreducibles of $S_3$ restrict to subgroups.

Notice that there are two possible lattices $L$ in the standard representation of $S_3$: the usual triangular lattice, and the triangular lattice with centroids added. The fundamental domains differ in area by 3.

Thus

$$\frac{\sqrt{3}}{4}\operatorname{vol}(L_{\mathrm{std}}^{S_2})^2 \approx \operatorname{vol}(L_{\mathrm{std}}.$$

From this we get the analog of Theorem 2.4 (b):

**5.5.** *Theorem.* —

$$\frac{\operatorname{vol}(L)}{\operatorname{vol}(L^{A_3})} \approx \left(\frac{\sqrt{3}}{4}\right)^{\tau} \left(\frac{\operatorname{vol}(L^{S_2})}{\operatorname{vol}(L^{S_3})}\right)^2.$$

*Proof.* This is true if the lattice splits into irreducibles. For the general case, note that the projection to the standard component of a representation is given by $\frac{1}{3}\sum \chi_{\mathrm{std}}(g)g$, so it is true for all lattices. □

**5.6. Number theory.**

From degrees of extension over $\mathbb{Q}$, we get:

$$(r_6 + 2s_6) - (r_2 + 2s_2) = 2((r_3 + 2s_3) - (r_1 - 2s_1)).$$

Using the above table on the unit lattice, we get:

$$(r_6 + s_6 - 1) - (r_2 + s_2 - 1) = 2((r_3 + s_3 - 1) - (r_1 - s_1 - 1)).$$

Hence $r_6 - r_2 = 2(r_3 - r_1)$ and $s_6 - s_2 = 2(s_3 - s_1)$.

The $\zeta$-function argument is as before: use group theory to show that $\zeta_6/\zeta_2 = (\zeta_3/\zeta_1)^2$. Thus

$$\left(\frac{R_6 h_6}{R_2 h_2}\right) = \left(\frac{R_3 h_3}{R_1 h_1}\right)^2$$

assuming there is no roots of unity issue. Hence the relative class number is always a perfect square, up to 2 or 3 (which wasn't hard to begin with).

**5.7.** Now let's check unit lattice issues to see if problems come up at 2.

Suppose (as mentioned earlier) that $w_6 = w_1$.

The groups $S_3$ and $\mathbb{Z}/2$ have abelianization $\mathbb{Z}_2$. So if $gu = \alpha u$, then $\alpha = -1$, which means $u$ has trace 0. (Hence the assumption that $F^3$ isn't obtained from $F$ by adding a unit with trace 0.)

The group $A_3$ has abelianization $\mathbb{Z}_3$. So if $gu = \alpha u$, then $u^3$ is fixed by $g$ and thus lies in $F^2$. So we have no 2-issues here.

**5.8.** Now we consider splittings of places of $F^1$.

*Case $s_1$:* A complex place in $F^1$ splits completely into complex places in the other number fields..

*Case $\alpha$:* where a real place in $F^1$ splits into 3 real places of $F^3$. Then $(r_2, s_2) = (4, 0)$ and $(r_6, s_6) = (0, 3)$.

*Case $\beta$:* where a real place in $F^1$ splits into one real and one complex case in $F^3$. The $(r_2, s_2) = (0, 1)$ and $(r_6, s_6) = (0, 3)$.

From the table, $\tau = r_3 + s_3 - r_1 - s_1$. Hence we get the analog of (6):

$$\mathrm{vol}(L_6)\,\mathrm{vol}(\psi_1(L_1))^2 \approx \left(\frac{\sqrt{3}}{4}\right)^{r_3+s_3-r_1-s_1} \mathrm{vol}(\psi_2(L_2))\,\mathrm{vol}(\psi_3(L_3))^2.$$

Now

$$r_3 + s_3 - r_1 - s_1 = (3\alpha + \beta) + (3s_1 + \beta) - \alpha - \beta - s_1 = 2\alpha + \beta + 2s_1.$$

*Case $s_1$.* $\mathrm{vol}(\psi_1(L_1))/\,\mathrm{vol}(L_1) = \sqrt{6}^{s_1}$, $\mathrm{vol}(\psi_2(L_2))/\,\mathrm{vol}(L_2) = (\sqrt{3}^2)^{s_1} \approx 1$, $\mathrm{vol}(\psi_3(L_3))/\,\mathrm{vol}(L_3) = (\sqrt{2}^3)^{s_1} = \sqrt{2}^{3s_1}$.

*Case $\alpha$.* By identical calculations, we get $\mathrm{vol}(\psi_1(L_1))/\,\mathrm{vol}(L_1) = \sqrt{6}^{\alpha}$, $\mathrm{vol}(\psi_2(L_2))/\,\mathrm{vol}(L_2) = (\sqrt{3}^2)^{\alpha} \approx 1$, $\mathrm{vol}(\psi_3(L_3))/\,\mathrm{vol}(L_3) = (\sqrt{2}^3)^{\alpha} = \sqrt{2}^{3\alpha}$.

*Case $\beta$.* $\mathrm{vol}(\psi_1(L_1))/\,\mathrm{vol}(L_1) = \sqrt{3}^{\beta}$, $\mathrm{vol}(\psi_2(L_2))/\,\mathrm{vol}(L_2) = \sqrt{3}^{\beta}$, $\mathrm{vol}(\psi_3(L_3))/\,\mathrm{vol}(L_3) = \sqrt{2}^{\beta}$.

Hence

$$\sqrt{2}^{8\alpha+4\beta+8s_1}\,\mathrm{vol}(L_6)\sqrt{6}^{2s_1+\alpha}\sqrt{3}^{2\beta}\,\mathrm{vol}(L_1)^2 \approx \sqrt{3}^{2\alpha+\beta+2s_1}\sqrt{3}^{\beta}\,\mathrm{vol}(L_2)\sqrt{2}^{6s_1+6\alpha}\sqrt{2}^{2\beta}\,\mathrm{vol}(L_3)^2$$

$$\sqrt{2}^{10\alpha+10s_1+4\beta}\sqrt{3}^{2\alpha+2s_1+2\beta}v_6v_1^2 = \sqrt{2}^{6\alpha+6s_1+2\beta}\sqrt{3}^{2\alpha+2s_1+2\beta}v_2v_3^2$$

$$2^{2\alpha+2s_1+\beta}v_6v_1^2 = v_2v_3^2.$$

From [L], $\mathrm{vol}(L_i) = R_i(r_i + 2s_i)/2^{r_i}$, so

$$2^{2\alpha+2s_1+\beta} R_6 R_1^2 \times 6 \approx R_2 R_3^2 \times 2 \times 3^2$$

$$2^{2\alpha+2s_1+\beta} R_6 R_1^2 \approx R_2 R_3^2.$$

Thus

$$h_6/h_2 = 2^{2\alpha+s_1+\beta}(h_3/h_1)^2 = 2^{\tau}(h_3/h_1)^2.$$

Ravi has double-checked this, although he may have introduced typos.

## 6. Things to think about in this write-up

Perhaps mention as an extended remark that our lattice results gives $D_1 D_6 \sim D_3 D_4$ immediately, but the equality at 2 indicates that something very special is happening with the lattices.

Manjul will think about the sieve.

$S_6$-statement; outer automorphism. $D_{4n}$ statements. $WD_4$: arithmetic Donagi. Check what is happening in binary-cubic situation. Make sure everything is kosher for $d > 1$.

March 16: Look at $A_4$ case; this will be trickier. By e-mail: divisibility by 3 issue.

**6.1.** *Things possibly to add.* The character table of $S_4$:

| $S_4$ | (1) | (12) | (123) | (1234) | (12)(34) |
|---|---|---|---|---|---|
| triv | 1 | 1 | 1 | 1 | 1 |
| sgn | 1 | -1 | 1 | -1 | 1 |
| $X_2$ | 2 | 0 | -1 | 0 | 2 |
| std | 3 | 1 | 0 | -1 | -1 |
| std $\otimes$ sgn | 3 | -1 | 0 | 1 | -1 |

## 7. Loose ends and things to think about later

**7.1.** *2-census.* Why is there no $\tau$ in the discriminant case?

**7.2.** *Recillas-Donagi.* Our lattice argument is partially bogus, and yet the result is still true.

**7.3.** *Order question.*

## References


[B] A. M. Bailey, *On the density of discriminants of quartic fields.* J. Reine Angew. Math. **315** (1980), 190–210. **(Cited.)**

[H] H. Heilbronn. On the 2-classgroup of cubic fields. 1971 *Studies in Pure Mathematics (Presented to Richard Rado)* pp. 117–119 *Academic Press, London.* **(Cited.)**

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