SCHUBERT INDUCTION

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ABSTRACT. We describe a Schubert induction theorem, a tool for analyzing intersections on a Grassmannian over an arbitrary base ring. The key ingredient in the proof is the Geometric Littlewood-Richardson rule of [V2].

Schubert problems are among the most classical problems in enumerative geometry of continuing interest. As an application of Schubert induction, we address several long-standing natural questions related to Schubert problems, including: the "reality" of solutions; effective numerical methods; solutions over algebraically closed fields of positive characteristic; solutions over finite fields; a generic smoothness (Kleiman-Bertini) theorem; and monodromy groups of Schubert problems. These methods conjecturally extend to the flag variety.

CONTENTS

1. INTRODUCTION

The Schubert induction theorem (Theorem 3.4) is a tool for studying intersections on a Grassmannian over an arbitrary base, by deformations. One motivation for such a result are *Schubert problems*, among the most classical problems in enumerative geometry of continuing interest. It is perhaps surprising that many natural questions about Schubert problems remain open. We describe these questions, and give some answers, in Section 2. Most applications conjecturally extend to the flag variety (Sect. 2.11). The theorem is stated and proved in Section 3, and the applications are shown in Sections 4 and 5.

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1.1. Notation and philosophy. Fix a Grassmannian $G(k,n) = \mathbb{G}(k-1, n-1)$ over a base field (or ring) K. If α is a partition, let $\Omega_\alpha \in A^*(G(k,n))$ denote the corresponding Schubert class. Let $\Omega_\alpha(F_.)$ be the closed Schubert variety with respect to the flag $F_.$. Let $\mathbf{\Omega}_\alpha(F_.) \subset G(k,n) \times Fl(n)$ be the universal Schubert variety.

Let $\pi_i : G(k,n) \times Fl(n)^m \to G(k,n) \times Fl(n)$ $(1 \leq i \leq m)$ denote the projection, where the projection to $Fl(n)$ is from the ith $Fl(n)$ of the domain. We will make repeated use of the following diagram.

$$\begin{tikzcd} \pi_1^* \mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \pi_2^* \mathbf{\Omega}_{\alpha_2}(F_.^2) \cap \cdots \cap \pi_m^* \mathbf{\Omega}_{\alpha_m}(F_.^m) \arrow[r, hook] \arrow[d, "\mathbf{S}"] & G(k,n) \times Fl(n)^m \\ Fl(n)^m & \end{tikzcd} \tag{1}$$

Questions about Schubert problems often reduce to questions about the morphism $\mathbf{S}$, and in particular about $\mathbf{S}(p)$ where p is a general point of $Fl(n)^m$. Schubert induction involves studying this question by specializing p through a carefully chosen sequence of codimension 1 degenerations, ending with p being a totally degenerate point (where the m flags coincide). In each application, the property in question will behave well under these degenerations. The flavor of the statement is that if something is true for $m = 1$, then it is true for all m.

As an informal but illustrative example, consider the statement "there is a means of combinatorially computing the number of preimages of p". Then Schubert induction in this case amounts to the checker-tournament method of solving Schubert problems described in [V2, Sect. 3.12]. The base case is the trivial statement "there is one point in the zero-dimensional Schubert variety".

1.2. Acknowledgments. I am grateful to F. Sottile for advice and discussions. His philosophy is clearly present in this paper. In particular, the phrase "Schubert induction" appeared first in [S6, Sect. 1], although with a slightly different meaning. I thank A. Buch and A. Knutson for introducing me to this subject. I also thank B. Sturmfels for suggesting that [V2] might have numerical applications (see Sect. 2 Question 6), D. Allcock for discussions on the symmetric group, and B. Poonen for advice on the Chebotarev density theorem.

2. Questions and answers

Given a partition α, the condition (i.e. element of $A^*(G(k,n))$ corresponding to $\Omega_\alpha(F_.)$ is called a *Schubert condition*. A *Schubert problem* is the following:

2.1. Schubert problem. Given m Schubert conditions $\Omega_{\alpha_i}(F_.^i)$ with respect to general flags $F_.^i$ $(1 \leq i \leq m)$ whose total codimension is $\dim G(k,n)$, what is the cardinality of their intersection?

In other words, how many k-planes satisfy various linear algebraic conditions with respect to m general flags? (Or: what is the cardinality of $\mathbf{S}^{-1}(F_{\cdot}^{1}, \dots, F_{\cdot}^{m})$ for general $(F_{\cdot}^{1}, \dots, F_{\cdot}^{m}) \in Fl(n)^m$?) This is the natural generalization of the classical problem: how many lines in $\mathbb{P}^3$ meet four (fixed) general lines? The points of intersection are called the *solutions of the Schubert problem*. (For clarity's sake, we say that the number of solutions is the *answer* to the Schubert problem.) An immediate (if imprecise) follow-up is: What can one say about the solutions?

Suppose the base field is K, and $\alpha_1, \dots, \alpha_m$ are given such that $\dim(\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m}) = 0$. The corresponding Schubert problem is said to be *enumerative over* K if there are m flags $F_{\cdot}^{1}$, ..., $F_{\cdot}^{m}$ defined over K such that $\mathbf{S}^{-1}(F_{\cdot}^{1}, \dots, F_{\cdot}^{m})$ consists of $\deg(\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m})$ (distinct) K-points.

2.2. The answer to this problem over $\mathbb{C}$ is the prototype of the program in enumerative geometry. By the Kleiman-Bertini theorem [Kl], the Schubert conditions intersect transversely, i.e. at a finite number of reduced points. Hence the problem is reduced to one about the intersection theory of the Grassmannian. The intersection ring (the Schubert calculus) is known, using other interpretations of the Littlewood-Richardson co-efficients in combinatorics or representation theory.

Yet many natural questions remain:

2.3. Galois or monodromy groups of Schubert problems. The Galois or monodromy group of an enumerative problem measures three (related) things:

(a) (geometric) As the conditions are varied, how do the solutions permute?
(b) (arithmetic) What is the field of definition of the solutions, given the field of definition of the flags?
(c) (algebraic) What is the Galois group of the field extension of the "variety of solutions" over the "variety of conditions" (see (1))?

The modern study of such problems was initiated by J. Harris in [H]; the connection between (a) and (c) is made there. The connection to (b) is via the Hilbert irreducibility theorem, as the target of **S** is rational.

Question 1. *What is the Galois group of a Schubert problem?*

We partially answer this question. There is an explicit combinatorial criterion that implies that a Schubert problem has Galois group "at least alternating" (i.e. if there are d solutions, the group is A_d or S_d). This criterion holds over an arbitrary base ring. To prove it, we will discuss useful methods for analyzing Galois groups via degenerations. It seems to be quite strong: we are unaware of a single case to which it does not apply. For example:

2.4. *Theorem. — The Galois group of* any *Schubert problem on the Grassmannians $G(2,n)$ ($n \leq 16$) and $G(3,n)$ ($n \leq 9$) is either alternating or symmetric.*

A short `Maple` program applying the criterion to a general Schubert problem is available upon request from the author.

2.5. Enumerative geometry in positive characteristic. Enumerative geometry in positive characteristic is almost a stillborn field, because of the failure of the Kleiman-Bertini theorem. (Examples of the limits of our understanding are plane conics and cubics in characteristic 2 [Vn, Ber].) In particular, the Kleiman-Bertini Theorem fails in positive characteristic for all $G(k,n)$ that are not projective spaces (i.e. $1 < k < n-1$) — Kleiman's counterexample [Kl, ex. 9] for $G(2,4)$ easily generalizes.

Question 2. *Are Schubert problems enumerative over an algebraically closed field of positive characteristic?*

To answer this question, we first answer a logically prior one:

Question 3. *Is there any patch to the failure of the Kleiman-Bertini theorem on Grassmannians?*

A related natural question is:

Question 4. *Are Schubert problems enumerative over finite fields?*

We now answer all three questions. The appropriate replacement of Kleiman-Bertini is the following. We say a morphism $f : X \to Y$ is *generically smooth* if there is a dense open set V of Y and a dense open set U of $f^{-1}(V)$ such that f is smooth on U. If X and Y are varieties and f is dominant, this is equivalent to the condition that the function field of X is separably generated over the function field of Y.

2.6. *Generic smoothness theorem. — The morphism* $\mathbf{S}$ *is generically smooth. More generally, if* $Q \subset G(k,n)$ *is a subvariety such that* $Q \cap \mathbf{\Omega}_\alpha(F_\cdot) \to Fl(n)$ *is generically smooth for all* α*, then*

$$Q \cap \pi_1^* \mathbf{\Omega}_{\alpha_1}(F_\cdot^1) \cap \pi_2^* \mathbf{\Omega}_{\alpha_2}(F_\cdot^2) \cap \cdots \cap \pi_m^* \mathbf{\Omega}_{\alpha_m}(F_\cdot^m) \longrightarrow Fl(n)^m$$

is as well.

This begs the following question: is the only obstruction to the Kleiman-Bertini theorem for $G(k,n)$ that suggested by Kleiman, i.e. whether the variety in question intersects a general translate of all Schubert varieties transversely? More precisely, is it true that for all Q_1 and Q_2 such that $Q_i \cap \mathbf{\Omega}_\alpha(F_\cdot) \to Fl(n)$ is generically smooth for all α, and $i = 1, 2$, does it follow that

$$Q_1 \cap \sigma(Q_2) \longrightarrow PGL(n)$$

is also generically smooth, where $\sigma \in PGL(n)$?

Theorem 2.6 answers Question 3, and leads to answers to Questions 2 and 4:

2.7. *Corollary. —*

(a) *All Schubert problems are enumerative for algebraically closed fields.*

(b) *For any prime p, there is a positive density of points P defined over finite fields of characteristic p where $\mathbf{S}^{-1}(P)$ consists of* $\deg(\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m})$ *distinct points. Moreover, for a fixed m, there is a positive density of points that works for* all *choices of $\alpha_1 \ldots, \alpha_m$.*

Part (a) follows as usual (see Sect. 2.2). If $\dim(\Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_m}) = 0$, then Theorem 2.6 implies that $\mathbf{S}$ is generically separable (i.e. the extension of function fields is separable). Then (b) follows by applying the Chebotarev density theorem for function fields to

$$\coprod_{\alpha_1,\ldots,\alpha_m} \pi_1^* \mathbf{\Omega}_{\alpha_1}(F_\cdot^1) \cap \pi_2^* \mathbf{\Omega}_{\alpha_2}(F_\cdot^2) \cap \cdots \cap \pi_m^* \mathbf{\Omega}_{\alpha_m}(F_\cdot^m) \longrightarrow Fl(n)^m$$

(see for example [E, Lemma 1.2], although all that is needed is the curve case, e.g. [FJ, Sect. 5.4]).

Sottile has proved transversality for intersection of codimension 1 Schubert varieties [S7], and P. Belkale has recently proved transversality in general, using his proof of Horn's conjecture [Bel, Thm. 0.9].

2.8. Reality questions. The classical "reality problem" for Schubert problems [F1, p. 55], [F2, Ch. 13], [FP, Sect. 9.8] is:

Question 5. *Are all Schubert problems enumerative over $\mathbb{R}$?*

See [S1, S6] for this problem's history. The case $G(1,n)$ (and $G(n-1,n)$) is trivially linear algebra. Sottile proved the result for $G(2,n)$ (and $G(n-2,n)$) for all n, [S2], and for all problems involving only Pieri classes [S5]; see [S3] for further discussion. The case $G(2,n)$, as well as that of conics, also follows from [V1].

This question can be fully answered with Schubert induction.

2.9. *Proposition. — All Schubert problems for all Grassmannians are enumerative over $\mathbb{R}$. Moreover, for a fixed m, there is a set of m flags that works for* all *choices of $\alpha_1, \ldots, \alpha_m$.*

This argument carries through with $\mathbb{R}$ replaced by any field satisfying the implicit function theorem, such as $\mathbb{Q}_p$.

2.10. Effective numerical solutions (over $\mathbb{C}$) to all Schubert problems for all Grassmannians. Even over the complex numbers, questions remain.

Question 6. *Is there an effective numerical method for* solving *Schubert problems (i.e. calculating the solutions to any desired accuracy?)*

The case of intersections of "Pieri classes" was dealt with in [HSS]. For motivation in control theory, see for example [HV]. In theory, one could numerically solve Schubert problems using the Plucker embedding; however, this is unworkable in practice.

Schubert induction leads to an algorithm for effectively numerically finding all solutions to all Schubert problems over $\mathbb{C}$. The method will be described in [SVV], and the reasoning is sketched in Section 4.3.

2.11. Flag varieties. Conjecture 4.9 of [V2] would imply that the results of this paper except for those on Galois/monodromy groups apply to all Schubert problems on flag manifolds. In particular, as the conjecture is verified for $n \leq 5$ [V2, Prop. 4.10], the results all hold in this range. For example:

2.12. *Proposition. — All Schubert problems for $Fl(n)$ are enumerative over any algebraically closed field or any field with an implicit function theorem (e.g. $\mathbb{R}$) for $n \leq 5$. For a fixed m, there is a set of m flags that works for all choices of $\alpha_1, \ldots, \alpha_m$.*

(The generalizations of the other statements in the paper are equally straightforward.)

We note that in the case of triple intersections where the answer is 1, Knutson has shown that the solution to the problem can be obtained by using spans and intersections of the linear spaces in the three flags [K], see also Purbhoo's result [P].

3. THE MAIN THEOREM, AND ITS PROOF

3.1. Sketch of the Geometric Littlewood-Richardson rule [V2]. The key ingredient in the proof of the Schubert induction theorem 3.4 is the Geometric Littlewood-Richardson rule, which we sketch here.

The variety $Fl(n) \times Fl(n)$ is stratified by the locally closed subvarieties with fixed numerical data. For each $(a_{ij})_{i,j \leq n}$, the corresponding subvariety is $\{(F_\cdot, F'_\cdot) : \dim F_i \cap F'_j = a_{ij}\}$. We denote such numerical data by the *configuration* $\bullet$ (normally interpreted as a partition), and the corresponding locally closed subvariety by $X_\bullet$.

The variety $G(k,n) \times Fl(n) \times Fl(n)$ is the disjoint union of "two-flag Schubert varieties", locally closed subvarieties with specified numerical data. For each $(a_{ij}, b_{ij})_{i,j \leq n}$, the corresponding subvariety is $\{(F_\cdot, F'_\cdot, V) : \dim F_i \cap F'_j = a_{ij}, \dim F_i \cap F'_j \cap V = b_{ij}, \}$. We denote the data of the (b_{ij}) by $\circ$, so the locally closed subvarieties are indexed by the configuration $\circ\bullet$. Denote the corresponding two-flag Schubert variety by $X_{\circ\bullet}$. (Warning: the closure of a two-flag Schubert variety need not be a union of two-flag Schubert varieties [V2, Cor. 3.13(a)], so this is not a stratification.)

There is a *specialization order* $\bullet_{\text{init}}, \ldots \bullet_{\text{final}}$, corresponding to partial factorizations of the longest word in the Bruhat order [V2, Sect. 2.2]. If $\bullet \neq \bullet_{\text{final}}$ is in the specialization order, then let $\bullet_{\text{next}}$ be the next term in the order. We have $X_{\bullet_{\text{next}}} \subset \overline{X}_\bullet$, $\dim X_{\bullet_{\text{next}}} = \dim X_\bullet - 1$, $X_{\bullet_{\text{init}}}$ is dense in $Fl(n) \times Fl(n)$, and $X_{\bullet_{\text{final}}}$ is the diagonal in $Fl(n) \times Fl(n)$.

There is a subset of configurations $\circ\bullet$, called *mid-sort*, where $\bullet$ is in the specialization order.

3.2. *Geometric Littlewood-Richardson Rule, inexplicit form (cf.* [V2, Sect. 3]*). —*

(i) *For any two partitions* α_1, α_2, $\pi_1^*\mathbf{\Omega}_{\alpha_1}(F_{\cdot}^1) \cap \pi_2^*\mathbf{\Omega}_{\alpha_2}(F_{\cdot}^2) = \overline{X}_{\circ\bullet_{\text{init}}}$ *for some mid-sort* $\circ\bullet_{\text{init}}$, *or* α_2, $\pi_1^*\mathbf{\Omega}_{\alpha_1}(F_{\cdot}^1) \cap \pi_2^*\mathbf{\Omega}_{\alpha_2}(F_{\cdot}^2) = \emptyset$.

(ii) *For any mid-sort* $\circ\bullet_{\text{final}}$, $\overline{X}_{\circ\bullet_{\text{final}}} = \pi_1^*\mathbf{\Omega}_\alpha \cap \Delta \, (= \pi_2^*\mathbf{\Omega}_\alpha \cap \Delta)$ *for some* α, *where* Δ *is the pullback to* $G(k,n) \times Fl(n) \times Fl(n)$ *of the diagonal* $X_{\bullet_{\text{final}}}$ *of* $Fl(n) \times Fl(n)$.

(iii) *For any mid-sort* $\circ\bullet$ *with* $\bullet \neq \bullet_{\text{final}}$, *consider the diagram* [V2, equ. (1)]

(2)
$$\begin{tikzcd}
\overline{X}_{\circ\bullet} \arrow[r, hook, "open"] \arrow[d] & \overline{X}_{\circ\bullet} \arrow[d] & D_X \arrow[l, hook', "closed"'] \arrow[d] \\
X_\bullet \arrow[r, hook, "open"] & X_\bullet \cup X_{\bullet_{next}} & X_{\bullet_{next}}. \arrow[l, hook', "closed"']
\end{tikzcd}$$

The closures of $X_{\circ\bullet}$ *are taken in* $G(k,n) \times X_\bullet$ *and* $G(k,n) \times (X_\bullet \cup X_{\bullet_{next}})$ *respectively, and the Cartier divisor* D_X *is defined by fibered product. There are one or two mid-sort configurations (depending on* $\circ\bullet$*), denoted by* $\circ_{swap}\bullet_{next}$ *and/or* $\circ_{stay}\bullet_{next}$, *such that* $D_X = \overline{X}_{\circ_{swap}\bullet_{next}}$, $\overline{X}_{\circ_{stay}\bullet_{next}}$, *or* $\overline{X}_{\circ_{stay}\bullet_{next}} \cup \overline{X}_{\circ_{swap}\bullet_{next}}$.

There is a more precise version of this rule describing the mid-sort $\circ\bullet$, and $\circ_{\text{swap}}\bullet_{\text{next}}$ and $\circ_{\text{stay}}\bullet_{\text{next}}$ (see [V2, Sect. 3]). For almost all applications here this version will suffice, but the precise definition of mid-sort, $\circ_{\text{swap}}\bullet_{\text{next}}$, and $\circ_{\text{stay}}\bullet_{\text{next}}$ will be implicitly required for the Galois/monodromy results of Section 5.

3.3. Statement of Main theorem. Fix $Q \subset G(k,n)$, and define $S = S(\alpha_1, \dots, \alpha_{m-1}) \subset G(k,n) \times Fl(n)^{m-1}$ by

(3)
$$S := (Q \times Fl(n)^{m-1}) \cap \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_{\cdot}^1) \cap \cdots \cap \pi_{m-1}^*\mathbf{\Omega}_{\alpha_{m-1}}(F_{\cdot}^{m-1}).$$

Then S is irreducible, and the projection to $B := Fl(n)^{m-1}$ has relative dimension $\dim Q - \sum |\alpha_i|$. (This follows easily by constructing S as a fibration over Q.)

Let P be a property of morphisms depending only on dense open subsets of the target, i.e. if $U \subset Y$ is a dense open subset, then $f : X \to Y$ has P if and only if $f|_{f^{-1}(U)}$ has P (call this condition **(A)**). For such $S \to B$, and any mid-sort $\circ\bullet$, let ρ_1 and ρ_2 be the two projections from $B \times (Fl(n) \times Fl(n))$ onto its factors. Using (2), construct

(4)
$$\begin{tikzcd}
\rho_1^*S \cap \rho_2^*\overline{X}_{\circ\bullet} \arrow[r, hook, "\text{open}"] \arrow[d, "(\dagger)"] & \rho_1^*S \cap \rho_2^*\overline{X}_{\circ\bullet} \arrow[d] & \rho_1^*S \cap \rho_2^*D_X \arrow[l, hook', "\text{closed}"'] \arrow[d, "(\dagger\dagger)"'] & \begin{array}{c} \rho_1^*S\cap\rho_2^*\overline{X}_{\circ_{\text{swap}}\bullet_{\text{next}}} \\ \text{and/or} \\ \rho_1^*S\cap\rho_2^*\overline{X}_{\circ_{\text{stay}}\bullet_{\text{next}}} \end{array} \arrow[l] \arrow[dl, "(\dagger\dagger\dagger)"] \\
B \times X_\bullet \arrow[r, hook, "\text{open}"] & B \times (X_\bullet \cup X_{\bullet_{\text{next}}}) & B \times X_{\bullet_{\text{next}}}. \arrow[l, hook', "\text{closed}"'] &
\end{tikzcd}$$

As in (2), $\overline{X}_{\circ\bullet}$ is the closure of $X_{\circ\bullet}$ in the appropriate space; $\rho_2^*\overline{X}_{\circ\bullet}$ is the pullback of $\overline{X}_{\circ\bullet}$ from $X_\bullet$ or $X_\bullet \cup X_{\bullet_{\text{next}}}$, and similarly for the other terms of the top row. The upper right should be interpreted as

$$\rho_1^*S \cap \rho_2^*\overline{X}_{\circ_{\text{swap}}\bullet_{\text{next}}}, \quad \rho_1^*S \cap \rho_2^*\overline{X}_{\circ_{\text{stay}}\bullet_{\text{next}}}, \quad \text{or} \quad \rho_1^*S \cap \rho_2^*\overline{X}_{\circ_{\text{swap}}\bullet_{\text{next}}} \coprod \rho_1^*S \cap \rho_2^*\overline{X}_{\circ_{\text{stay}}\bullet_{\text{next}}},$$

as in the Geometric Littlewood-Richardson rule 3.2.

3.4. Schubert induction theorem. — *Suppose that for any such $S \to B$ and any mid-sort $\circ\bullet$,* **(B)** *if $(\dagger\dagger\dagger)$ has P, then $(\dagger\dagger)$ has P, and* **(C)** *if $(\dagger\dagger)$ has P, then $(\dagger)$ has P. If the projection*

$$(Q \times Fl(n)) \cap \mathbf{\Omega}_\alpha(F_.) \longrightarrow Fl(n) \tag{5}$$

has P for all partitions α (the "base case" of the Schubert induction), then the projection

$$(Q \times Fl(n)^m) \cap \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \cdots \cap \pi_m^*\mathbf{\Omega}_{\alpha_m}(F_.^m) \longrightarrow Fl(n)^m$$

has P for all m, $\alpha_1, \ldots, \alpha_m$.

In particular (taking $Q = G(k,n)$) if the projection

$$\mathbf{\Omega}_\alpha(F_.) \longrightarrow Fl(n)$$

has P for all α (condition **(D)**), then the projection

$$\mathbf{S} : \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \cdots \cap \pi_m^*\mathbf{\Omega}_{\alpha_m}(F_.^m) \longrightarrow Fl(n)^m$$

has P.

Proof. We show that

$$(Q \times Fl(n)^{m-1} \times X_\bullet) \cap \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \cdots \cap \pi_{m-1}^*\mathbf{\Omega}_{\alpha_{m-1}}(F_.^{m-1}) \cap \rho^*\overline{X}_{\circ\bullet} \to Fl(n)^{m-1} \times X_\bullet \tag{6}$$

(where ρ is the projection to $X_\bullet$) has P for all m and mid-sort $\circ\bullet$, by induction on $(m, \bullet)$, where $(m_1, \bullet_1)$ precedes $(m_2, \bullet_2)$ if $m_1 < m_2$, or $m_1 = m_2$ and $\bullet_1 < \bullet_2$ in the specialization order.

Base case $m = 1$, $\bullet = \bullet_{\text{final}}$. By the Geometric Littlewood-Richardson rule 3.2 (ii),

$$\begin{tikzcd}
(Q \times X_{\bullet_{\text{final}}}) \cap \rho^*\overline{X}_{\circ\bullet_{\text{final}}} \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & X_{\bullet_{\text{final}}} \arrow[d, equal] & \text{(i.e., (6))} \\
(Q \times X_{\bullet_{\text{final}}}) \cap \pi_1^*\mathbf{\Omega}_\alpha \cap \Delta \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & X_{\bullet_{\text{final}}} \arrow[d, leftrightarrow, "\cong"] \\
(Q \times Fl(n)) \cap \mathbf{\Omega}_\alpha(F_.) \arrow[r] & Fl(n)
\end{tikzcd}$$

has P by (5).

Inductive step, case $\bullet \neq \bullet_{\text{final}}$. By the inductive hypothesis,

$$(Q \times Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}) \cap \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \cdots \cap \pi_{m-1}^*\mathbf{\Omega}_{\alpha_{m-1}}(F_.^{m-1}) \cap \rho^*\overline{X}_{\circ_{\text{stay}}\bullet_{\text{next}}} \to Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}$$

and/or

$$(Q \times Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}) \cap \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \cdots \cap \pi_{m-1}^*\mathbf{\Omega}_{\alpha_{m-1}}(F_.^{m-1}) \cap \rho^*\overline{X}_{\circ_{\text{swap}}\bullet_{\text{next}}} \to Fl(n)^{m-1} \times X_{\bullet_{\text{next}}}$$

have P. Then an application of **(B)** and **(C)** shows that (6) has P as well.

Inductive step, case $\bullet = \bullet_{\text{final}}$, $m > 1$. Suppose $\overline{X}_{\circ\bullet_{\text{final}}} = \pi_1^*\mathbf{\Omega}_\alpha \cap \Delta$ and $\pi_1^*\mathbf{\Omega}_{\alpha_{m-1}}(F_.) \cap \pi_2^*\mathbf{\Omega}_\alpha(F'_.) = \overline{X}_{\circ'\bullet_{\text{init}}}$ (using the Geometric Littlewood-Richardson rule 3.2 (ii) and (i) respectively). Then

$$\begin{tikzcd}
(Q\times Fl(n)^{m-1}\times X_\bullet)\cap\pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1)\cap\cdots\cap\pi_{m-1}^*\mathbf{\Omega}_{\alpha_{m-1}}(F_.^{m-1})\cap\rho^*\overline{X}_{\circ\bullet} \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & Fl(n)^{m-1}\times X_\bullet \arrow[d, leftrightarrow, "\cong"] \\
(Q\times Fl(n)^{m-1}\times X_\bullet)\cap\pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1)\cap\cdots\cap\pi_{m-1}^*\mathbf{\Omega}_{\alpha_{m-1}}(F_.^{m-1})\cap\pi_m^*\mathbf{\Omega}_\alpha(F_.^m) \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & Fl(n)^m \arrow[d, equal] \\
(Q\times Fl(n)^{m-1}\times X_\bullet)\cap\pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1)\cap\cdots\cap\pi_{m-2}^*\mathbf{\Omega}_{\alpha_{m-2}}(F_.^{m-2})\cap\rho^*\overline{X}_{\circ'\bullet_{\text{init}}} \arrow[r] & Fl(n)^m
\end{tikzcd}$$

which has P as (by **(A)**)

$(Q \times Fl(n)^{m-1} \times X_\bullet) \cap \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \cdots \cap \pi_{m-2}^*\mathbf{\Omega}_{\alpha_{m-2}}(F_.^{m-2}) \cap \rho^*\overline{X}_{\circ'\bullet_{\text{init}}} \to Fl(n)^{m-2} \times X_{\bullet_{\text{init}}}$

has P by the inductive hypothesis. $\square$

For some applications, we will need a slight variation.

3.5. Schubert induction theorem, bis. — *Suppose P satisfies conditions* **(A–C)**. *Then if*

$$\coprod_{\alpha:\, \dim Q - |\alpha| = 0} (Q \times Fl(n)) \cap \mathbf{\Omega}_\alpha(F_.) \longrightarrow Fl(n)$$

has P then

$$\coprod_{\alpha_1,\ldots,\alpha_m:\, \dim Q - \sum|\alpha_i| = 0} (Q \times Fl(n)^m) \cap \pi_1^*\mathbf{\Omega}_{\alpha_1}(F_.^1) \cap \cdots \cap \pi_m^*\mathbf{\Omega}_{\alpha_m}(F_.^m) \longrightarrow Fl(n)^m$$

has P for all m.

The proof is identical to that of Theorem 3.4.

4. Applications

We now verify the conditions **(A–C)** for several P to prove the results claimed in Section 2.

4.1. Positive characteristic: Proof of Proposition 2.6. Let P be the property that the morphism f is generically smooth. Then P clearly satisfies **(A–C)** (note that the relative dimensions of († – † † †) are the same, and that $X_{\circ_{\text{stay}}\bullet_{\text{next}}}$ and $X_{\circ_{\text{swap}}\bullet_{\text{next}}}$ are disjoint), and the Schubert induction hypothesis **(D)**; apply Theorem 3.4. $\square$

4.2. Reality: Proof of Proposition 2.9. Let P be the property that there is a Zariski-dense subset U of real points of Y for which $\mathbf{S}^{-1}(p)$ consists of $\deg \mathbf{S}$ real points for all $p \in U$. Clearly P satisfies **(A–C)** and the Schubert induction hypothesis **(D)**. Apply Theorem 3.5. $\square$

As mentioned earlier, the same argument applies to any field satisfying the implicit function theorem, such as $\mathbb{Q}_p$.

4.3. Numerical solutions. Informally, this application corresponds to applying Theorem 3.5 to the property of generically finite morphisms $f : X \hookrightarrow G(k,n) \times Y \to Y$ "for each point of Y whose preimage is a set of points, there is an effective algorithm for numerically finding these points". Condition **(C)** corresponds to the fact that if $|f^{-1}(y)| = \deg f$ and the points of $f^{-1}(y)$ can be numerically calculated, then by the implicit function theorem, the points of $f^{-1}(y')$ can be numerically calculated for all y' such that $\dim f^{-1}(y') = 0$. This idea will be developed in [SVV].

5. Galois/monodromy groups of Schubert problems

5.1. We recall the "checker tournament" algorithm [V2, Sect. 3.12] for solving Schubert problems. We begin with m partitions, and we make a series of moves. Each move consists of one of the following.

(i) Take two partitions, and begin a checker game if possible, else end the tournament.
(ii) Translate a completed checker game back to a partition.
(iii) Make a move in an ongoing checkergame.

These parallel (i)–(iii) of Theorem 3.2. When one partition and no checker games are left, the tournament is complete. At step (iii), the checker tournament may bifurcate (if both a "stay" and a "swap" are possible), and both branches must be completed.

This answer to the Schubert problem can be interpreted as a creating a directed tree, where the vertices correspond to partially completed checker tournament. Each vertex has in-degree 1 (one immediate ancestor) except for the root (corresponding to the original Schubert problem), and out-degree (number of immediate descendants) between 0 and 2. The graph is constructed starting with the root, and for each vertex that is not a completed checkergame, a choice may be made (depending on (iii)) which may lead to a bifurcation. Vertices corresponding to a single partition and no checkergames are called *leaves*. (There may be other vertices with out-degree 0, arising from (i); these are not leaves.) The answer is the number of leaves of the tree.

The answer is of course independent of the choices made; in the description of [V2, Sect. 3.12], and in the proof of Theorems 3.4 and 3.5, each checker game was chosen to be completed before the next was begun.

5.2. *Theorem. — Suppose we are given a Schubert problem such that there is a directed tree as above, where each vertex with out-degree two satisfies either*

(a) *there are a different number of leaves on the 2 branches, or*
(b) *there is one leaf on each branch.*

Then the Galois group of the Schubert problem is at least alternating.

5.3. Specialization of monodromy. To prove the theorem, we will examine how Galois groups behave under specialization.

We say a generically finite morphism $f : X \to Y$ is *generically separable* if the corresponding extension of function fields is separable. Define the Galois group $\mathrm{Gal_f}$ of a generically finite and separable morphism to be the Galois group of the Galois closure of the corresponding extension of function fields.

5.4. *Remark: The complex case.* To motivate later statements over an arbitrary ground ring, we first consider the complex case. Suppose

$$\begin{tikzcd} W \arrow[d] & Y \arrow[l, hook', "\text{closed}"'] \arrow[d] \\ X & Z \arrow[l, hook', "\text{closed}"'] \end{tikzcd}$$

is a fiber diagram of complex schemes, where the vertical morphisms are proper generically finite degree d; W, X, and Z are irreducible varieties; Z is Cartier in X; X is regular in codimension 1 along Z; and Y is reduced. Then $\mathrm{Gal_{W\to X}}$ can be interpreted as an element of S_d by fixing a point of X with d preimages, and considering loops in the smooth locus of X based at that point, and their induced permutations of the preimages.

(a) If Y is irreducible, then by interpreting $\mathrm{Gal_{Y\to Z}}$ by choosing a general base point of Z and elements of the fundamental group of the smooth part of Z generating the Galois group, we have constructed an inclusion $\mathrm{Gal_{Y\to Z}} \hookrightarrow \mathrm{Gal_{W\to X}}$. In particular, if the first group is at least alternating, then so is the second.

(b) If Y has two components Y_1 and Y_2, which each map generically finitely onto Z with degrees d_1 and d_2 respectively (so $d_1 + d_2 = d$), then the same construction produces a subgroup H of $\mathrm{Gal_{Y_1\to Z}} \times \mathrm{Gal_{Y_2\to Z}}$ which surjects onto $\mathrm{Gal_{Y_i\to Z}}$ (for $i = 1, 2$), and an injection of H into $\mathrm{Gal_{W\to X}}$ (via the induced inclusion $S_{d_1} \times S_{d_2} \hookrightarrow S_d$).

Then a purely group-theoretical argument (Prop. 5.7) relying on Goursat's lemma will show that if $\mathrm{Gal_{Y_i\to Z}}$ is at least alternating ($i = 1, 2$), W is connected (so $\mathrm{Gal_{W\to X}}$ is transitive), and $d_1 \neq d_2$ or $d_1 = d_2 = 1$, then $\mathrm{Gal_{W\to X}}$ is at least alternating as well.

5.5. *The general case.* With this complex intuition in hand, we prove Remarks 5.4(a) and (b) over an arbitrary ring. Suppose $k_1 \subset k_2$ is a separable degree d field extension. Choose an ordering $x_1, \dots, x_d$ of the $\overline{k}_1$-valued points (over $\operatorname{Spec} k_1$) of $\operatorname{Spec} k_2$. If $g : X \to Y$ is a generically finite separable morphism, define the "Galois scheme" $\mathrm{GalSch_g}$ by

$$\overbrace{X \times_Y \cdots \times_Y X}^{\deg g} \setminus \Delta$$

where Δ is the "big diagonal". Recall that the Galois group of $k_1 \subset k_2$ can be interpreted as a subgroup of S_d as follows: σ is in the Galois group if and only if $(x_{\sigma(1)}, \dots, x_{\sigma(d)})$ is in the same component of $\mathrm{GalSch_{Spec\, k_2\to Spec\, k_1}}$ as $(x_1, \dots, x_d)$.

To understand how this behaves in families, let R be a discrete valuation ring with function field K and residue field k. Suppose the following is a fiber diagram

$$\begin{tikzcd}
X_K \arrow[r, hook, "\text{open}"] \arrow[d] & X_R \arrow[d] & X_k \arrow[l, hook', "\text{closed}"'] \arrow[d] \\
\operatorname{Spec} K \arrow[r, hook, "\text{open}"] & \operatorname{Spec} R & \operatorname{Spec} k \arrow[l, hook', "\text{closed}"']
\end{tikzcd}$$

where X_K is irreducible, X_k is reduced, and the vertical morphisms are finite and separable (and hence étale, using X_k reduced).

After choice of algebraic closures, there is a bijection from the $\overline{K}$-valued points of X_K with the $\overline{k}$-valued points of X_k.

By observing that each component of $\mathrm{GalSch}_{\mathrm{X_k \to Spec\, k}}$ lies in a unique irreducible component of $\mathrm{GalSch}_{\mathrm{X_R \to Spec\, R}}$ (as $\mathrm{GalSch}_{\mathrm{X_R \to Spec\, R}} \to \mathrm{Spec\, R}$ is étale along X_k), we see that Remarks 5.4 (a) and (b) hold in general (by applying these comments to $\mathcal{O}_{X,Z}$).

5.6. *Proof of Theorem 5.2.* Each vertex v corresponds to a diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f_v"'] & B \times G(k,n) \\
B &
\end{tikzcd}$$

where X is irreducible, and B is a product of flag varieties (one for each partition) and strata $X_{\circ\bullet}$ (one for each checkergame in process). The morphism f_v is generically finite and separable, and its degree is the answer to the corresponding enumerative problem.

We label each vertex v with the number of leaves on that branch (i.e. with $\deg f_v$), and with the Galois group $\mathrm{Gal_v}$ of that problem. We prove that $\mathrm{Gal_v} = \mathrm{A_{deg\, f_v}}$ or $S_{\deg f_v}$ for all v by induction on v. (This is a slight generalization of Schubert induction.)

If v is a leaf, the result is trivial.

Suppose next that v is a vertex with one descendant w, and $\mathrm{Gal_w}$ is at least alternating. If the move from v to w is of type (i) or (ii), then the morphism f_v is the same as f_w **[careful]**, so the result holds. If the move from vertex v is of type (iii) (so exactly one of $\{\text{stay}, \text{swap}\}$ is possible), then G_v is at least alternating by Remark 5.4(a).

Next, suppose v has two immediate descendants, so we are in case (iii) and both "stay" and "swap" are possible. If one branch has no leaves, then G_v is at least alternating by Remark 5.4(a), so assume otherwise. As X is irreducible, $\mathrm{Gal_v}$ is transitive. By Remark 5.4(b) and the group theoretic calculation of Proposition 5.7, $\mathrm{Gal_v}$ is at least alternating, and the inductive step is complete.

Thus by induction the root vertex has at least alternating Galois group, completing the proof of the Theorem 5.2. $\square$

5.7. *Proposition. — Suppose G is a transitive subgroup of S_{m+n} such that $G \cap (S_m \times S_n)$ contains a subgroup H such that the projection of H to S_m (resp. S_n) is either A_m ($m \geq 4$) or S_m (resp. A_n for $n \geq 4$, or S_n).*

(a) *If $m \neq n$, then $G = A_{m+n}$ ($m + n \geq 4$) or S_{m+n}.*
(b) *If $m = n = 1$ then $G = S_2$.*

Note that if $m = n$, then

$$\{e, (1, n+1)(2, n+2) \cdots (n, 2n)\} \rtimes (S_{\{1,\dots,n\}} \times S_{\{n+1,\dots,2n\}})$$

is a subgroup of S_{2n} whose intersection with $S_{\{1,\dots,n\}} \times S_{\{n+1,\dots,2n\}}$ surjects onto each of its factors.

Proof. Part (b) is trivial, so we prove (a). Assume without loss of generality that $n > m$.

Recall Goursat's lemma: if $H \subset G_1 \times G_2$, such that H surjects onto both factors, then there are normal subgroups $N_i \lhd G_i$ ($i = 1, 2$) and an isomorphism $\phi : G_1/N_1 \xrightarrow{\sim} G_2/N_2$ such that $(g_1, g_2) \in H$ if and only if $\phi(g_1 N_1) = g_2 N_2$.

We first show that if G is a transitive subgroup of S_{m+n} ($n > m \geq 3$) containing $A_m \times A_n$, then G contains any 3-cycle and hence A_{m+n}. Color the numbers 1 through m red and $m+1$ through $m+n$ green. Any monochromatic 3-cycle lies in $A_m \times A_n$ and hence G. Suppose τ is any element of G sending a green number to a red position. (i) If there are two numbers of each color in the positions of one color (say, red) then the conjugate of a 3-cycle in A_m by τ will be a 3-cycle α of 1 red and 2 green objects, and the conjugate of a different 3-cycle in A_m by τ will be a 3-cycle α of 1 green and 2 red objects. Similarly, (ii) if there is at least one number of each color in the positions of *both* colors, we can find a conjugate α of a 3-cycle in A_m or A_n by τ that is a 3-cycle α of 1 red and 2 green objects, and the conjugate of a 3-cycle in A_n or A_m by τ that is a 3-cycle α of 1 green and 2 red objects. By conjugating α and β further by elements of $A_m \times A_n$, we can obtain any non-monochromatic 3-cycle. Now τ falls into case (i) and/or (ii), or $n = m + 1$ and τ sends all red objects to green positions, and all but one green object to red positions. Suppose p is the green position containing the green object in τ, and σ is a 3-cycle in A_n moving p. Then $\tau^{-1}\sigma\tau$ is a permutation where exactly one red object is sent to a green position, and vice versa, and we are in case (ii). Thus in all cases G contains A_{m+n}, as desired.

We now deal with the case $m, n \geq 3$. By Goursat's lemma, G must contain $A_m \times A_n$. (For example, if the projections of H to S_m and S_n are surjective, then H arises from isomorphic quotients $S_n/N_m \cong S_n/N_n$. Then $(N_m, N_n) = (S_m, S_n)$ or $(N_m, N_n) = (A_m, A_n)$; in both cases $A_m \subset N_m$ and $A_n \subset N_n$.) Then apply the previous paragraph.

For the remaining cases, it is straightforward to see (using Goursat) that (i) if the image of H is A_n (resp. S_n) and $m = 1$, then $A_{m+n} \subset G$ (resp. $G = S_{m+n}$), and (ii) if H surjects onto A_n ($n \geq 4$) or S_n and $m = 2$, then $A_{m+n} \subset G$. □

5.8. Applying Theorem 5.2. Theorem 5.2 is quite strong, and can be checked with a naive computer program. For example, it implies that all Schubert problems for $G(2, n)$

for $n \leq 16$ are at least alternating. It also implies that all but a tiny handful of Schubert problems for Grassmannians of dimension less than 20 are at least alternating; we will describe these exceptions.

For $k > 1$, the criterion will fail for the Schubert problem $(1)^{k^2}$ on $G(k, 2k)$: the first degeneration (i.e. the first vertex with out-degree 2) will correspond to

$$(1)^{k^2} = (2)(1)^{k^2} + (1,1)(1)^{k^2}$$

and the two branches will have the same number of leaves by symmetry. More generally, if $1 \leq m < k$ and $(m, k) \neq (1, 2)$, the criterion will fail for the Schubert problem

$$(\overbrace{m, \ldots, m}^{m})(1)^{k^2-m^2}$$

on $G(k, 2k)$ for the same reason.

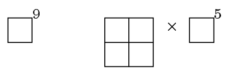

FIGURE 1. The two counterexamples of $G(3, 6)$

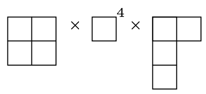

FIGURE 2. An induced counterexample in $G(3, 7)$

On $G(3, 6)$, the only counterexamples are of this sort, when $m = 1$ and 2, shown in Figure 1. By "embedding" these problems in larger problems, these trivially induce counterexamples in larger Grassmannians; for example, Figure 2 is a counterexample in $G(3, 7)$ that is really an avatar of the second example in $G(3, 6)$. We call counterexamples in $G(k, n)$ not arising in this way, i.e. involving only subpartitions not meeting the right column and bottom row of the rectangle, *primitive* counterexamples.

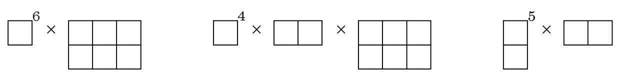

FIGURE 3. The primitive counterexamples in $G(3, 7)$

Then $G(3, 7)$ has only three counterexamples, shown in Figure 3, and the counterexamples in $G(4, 7)$ are given by the transposes of these. The Grassmannian $G(3, 8)$ has six counterexamples, shown in Figure 4, and $G(3, 9)$ has 13 counterexamples, shown in Figure 5.

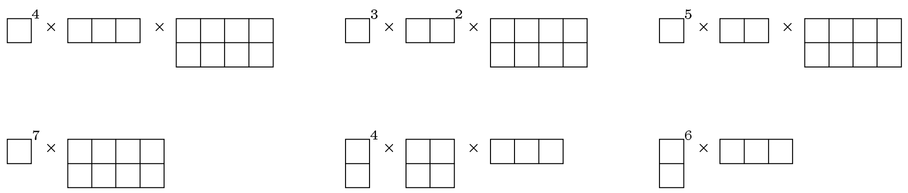

FIGURE 4. The primitive counterexamples in $G(3,8)$

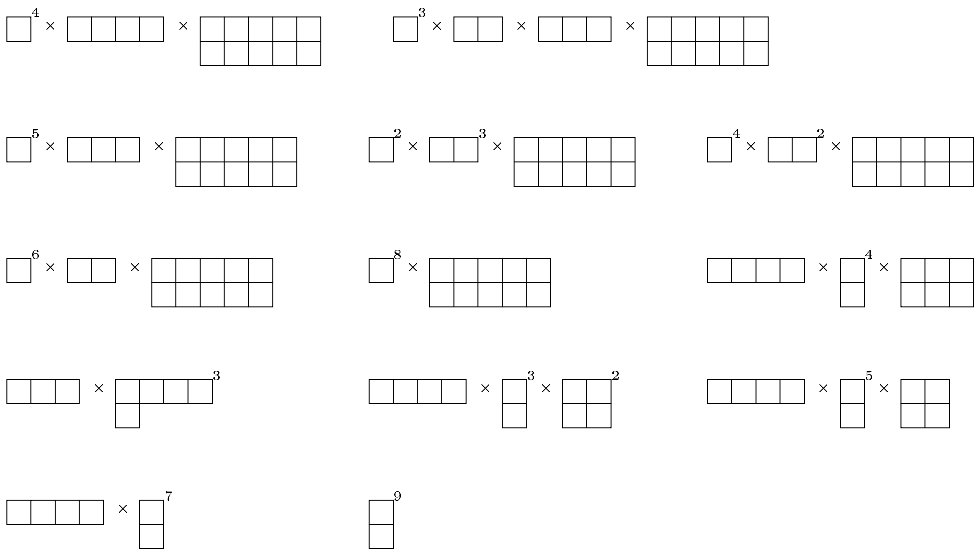

FIGURE 5. The primitive counterexamples in $G(3,9)$

All of these exceptions can be excluded with the following, slightly stronger criterion.

5.9. *Theorem. — Suppose we are given a Schubert problem such that there is a directed tree as above, where each vertex with two immediate descendants satisfies (a) or (b) of Theorem 5.2, or*

(c) *there are $m \neq 6$ leaves on each branch, and it is known that the corresponding Galois group is two-transitive.*

Then the Galois group of the Schubert problem is at least alternating.

In particular, to show that the Galois group is $(n-2)$-transitive, it often suffices to show that it is two-transitive.

As with Theorem 5.2, the proof reduces to the following variation of Proposition 5.7.

5.10. *Proposition. — Suppose G is a two-transitive subgroup of S_{2m} ($m \neq 6$) such that $G \cap (S_m \times S_m)$ contains a subgroup H such that the projection of H to both factors S_m is either A_m ($m \geq 4$) or S_m. Then $G = A_{2m}$ or S_{2m}.*

The proof is similar to that of Proposition 5.7, and is omitted.

If $m = n = 6$, D. Allcock has pointed out that the Mathieu group M_{12} can be expressed as a subgroup of S_{12} such that

$$M_{12} \cap (S_6 \times S_6) = \{(g, \sigma(g)) : g \in S_6\}$$

where σ is an outer automorphism of S_6. Thus Proposition 5.10 cannot be extended to $m = 6$.

We say two vertices v, w in a directed tree (as in Sect. 5.1) are *equivalent* if they are connected by a chain of edges $v_1 \to v_2 \to \cdots \to v_s$ ($(v_1, v_s) = (v, w)$ or (w, v)) and $\deg f_v = \deg f_w$ (and hence $= \deg f_{v_i}$ for all i). In each of the cases $G(3, n)$ ($6 \leq n \leq 9$) given above, it is possible to find such a tree satisfying Theorem 5.9 (a)–(c), where the vertices of type (c) are equivalent to vertices corresponding to Schubert problems (i.e. corresponding to a set of partitions, with no checkergames-in-progress), and to show by ad hoc means that these Schubert problems are two-transitive. (The details are omitted; this method should not be expected to be workable in general.) Hence all Schubert problems for these Grassmannians have Galois group at least alternating.

The Grassmannian $G(4, 8)$ has only 31 Schubert problems where the criterion of Theorem 5.2 does not apply (not shown here). Each of these cases may be reduced to checking that a certain Schubert problem is two-transitive. The author has not checked this two-transitivity.

In general, these ad hoc methods should not be expected to eliminate all counterexamples, although the author knows of no case where this approach fails.

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