Contents

RESEARCH SUMMARY AND PROPOSAL

RAVI VAKIL

1. Introduction

My area of research is algebraic geometry. My main interest to date has been intersection theory on moduli spaces, especially of curves and stable maps.

Stable maps provide a powerful new technique to try to understand the geometry of curves in varieties, using the geometry of (the moduli space of) curves. For example, many classical questions in enumerative geometry are now tractable. Starting with the WDVV-equations, this has been a recurring theme in Gromov-Witten theory. Conversely, and perhaps more fundamentally, one can study the moduli space of curves by studying maps from curves to varieties.

I am also interested in the interactions of algebraic geometry with nearby fields, especially mathematical physics, number theory, symplectic and differential geometry, and combinatorics.

Perhaps the best way to give an idea of my research interests to date and a proposal for future research is to give a quick summary of my articles and work in progress. All of my articles are available on my preprints webpage, http://www-math.mit.edu/~vakil/preprints.html.

Date: October 10, 2000.

2. Algebraic geometry

Enumerative geometry of curves via degeneration methods, [V2]. To appear (in more concise form) as **The enumerative geometry of rational and elliptic curves in projective space**, [V4]. The striking enumerative content of Kontsevich's *and Manin's?* First Reconstruction Theorem was that one could effectively recursively compute the enumerative geometry of rational curves (of any degree) in $\mathbb{P}^n$. More precisely, one could compute how many degree d rational curves are incident to various generally chosen linear spaces of various dimensions. The method is by degenerating conditions to lie in a fixed hyperplane; the intermediate numbers are now known as relative Gromov-Witten invariants of projective space.

This work should be extendable to give an algorithm for computing all characteristic numbers of genus (0 and) 1 curves in $\mathbb{P}^n$ (see also [V5]). I am saving this project for a future graduate student.

The method used in [V4] was degeneration of incidence conditions. A different version of the same idea also gives a method of **Counting curves on rational surfaces**, [V8]: Suppose one wishes to count the number of curves of a given genus in a given divisor class through a certain number of points. One can move the points to a fixed $\mathbb{P}^1$ one at a time, and determine what degenerations turn up (and with one multiplicities). The two main applications: (i) the enumerative geometry of curves of any genus in any divisor class on any Hirzebruch is determined, and (ii) all Gromov-Witten invariants of all but 2 of the Fano surfaces are determined. As far as I know, this is the only workable means of computing arbitrary-genus Gromov-Witten invariants of a surface. Also, the invariants coming up in (i) are relative Gromov-Witten invariants.

Recursions for characteristic numbers of genus one plane curves, [V5], gives a fast, short algorithm to compute characteristic numbers of genus one plane curves of all degrees, using a well-known relation "in the Picard group of the j-stack", the identity $\omega = \Delta/12$. In [V5], some foundational facts about characteristic numbers of maps are also proved, to set up [V6] and [V7].

Thanks to stable maps, the ... **Characteristic numbers of quartic plane curves**, [V6], I completed a verification of a computation of Zeuthen's. Thanks to stable maps (and knowledge of the geometry of genus 3 curves), this is surprisingly easy: identify the "relevant" boundary divisors on the appropriate component of the space of maps, work out their characteristic numbers, and use equalities in the Picard group. As a side benefit, this gives a fast calculation of the characteristic number of the cubics. Remarkably, and presumably not coincidentally, many steps parallel (in a vague sense) those of the nineteenth century algebraic geometer Zeuthen.

The coincidences that came up in [V6] led to many more coincidences that appeared in **Twelve points on the projective line, branched covers, and rational elliptic fibrations**, [V9]. **Put brief summary here!** This relates many fun and classical facts, and ties together (and reproves) various facts due to Coble, Zariski, Mumford, and Zeuthen.

A useful lemma, relating degenerations in the Hilbert scheme and in the space of stable maps, will appear in **A tool for stable reduction of curves on surfaces**, [V13].

Gromov-Witten theory and Mirror symmetry [PV15], with R. Pandharipande, consists of notes meant to accompany Pandharipande's lectures at the Clay Institute summer school of 2000.

2.1. Maps to $\mathbb{P}^1$, and the geometry of the moduli space of curves.

2.1.1. *Motivation: The dream of Faber and Pandharipande.* The moduli space of pointed curves, $\mathcal{M}_{g,n}$, is a central object in algebraic geometry. Mumford defined a subring of its cohomology ring, called the *tautological ring* $R^*(\mathcal{M}_{g,n})$, generated by natural classes, including certain natural (algebraic) codimension 1 ψ-classes (one for each marked point, denoted ψ_1 through ψ_n) and λ-classes (Chern classes of the Hodge bundle, denoted λ_0 through λ_g). This concept generalizes readily to $\overline{\mathcal{M}}_{g,n}$ and other partial compactifications of $\mathcal{M}_{g,n}$.

Recent interest in the tautological ring has been largely motivated by work of C. Faber. Of particular note is his celebrated conjecture that $R^*(\mathcal{M}_g)$ "behaves like" the algebraic cohomology ring of a smooth projective variety of dimension $g-2$; in particular it should be Gorenstein with socle in degree $g-2$.

Faber and Pandharipande have also speculated that similar results should hold for $\overline{\mathcal{M}}_{g,n}$, as well as the partial compactification of $\mathcal{M}_{g,n}$

corresponding to curves of compact type. Moreover, they speculate that these results hold in the Chow ring, not just in the cohomology ring. (The Chow ring is a much more refined algebraic version of the cohomology ring.) Although they are reluctant to dignify these speculations with the name "conjecture", behind these ideas is a grand dream: that the tautological ring (considered as a subring of the Chow ring) of $\overline{\mathcal{M}}_{g,n}$ (and of the other partial compactifications of $\mathcal{M}_{g,n}$ mentioned above) is fundamentally a combinatorial object.

2.1.2. *Witten's conjecture.* In the early 1990's, Witten made a remarkable conjecture about all top intersections of ψ-classes in $H^*(\overline{\mathcal{M}}_{g,n})$, proved by Kontsevich, [K]. He conjectured that the generating function of such top intersections (Witten's "free energy of a point" F) satisfies a certain differential equation (KdV). Later, this was shown to imply F is also annihilated by a sequence of differential operators related to the Virasoro algebra. Using Witten's conjecture, one can compute all top intersections of ψ-classes in $H^*(\overline{\mathcal{M}}_{g,n})$. By work of Faber, [F2], this allows one to compute *all* top intersections in the tautological ring (for any g, n). Remarkably, despite the huge amount of previous work on the moduli space, this was not possible before. The difficulty of getting a hold of these top intersections has made progress in understanding the intersection theory of $\overline{\mathcal{M}}_{g,n}$ very slow.

2.1.3. *My work in this area.* One thrust of my recent work has been to try to study the geometry of $\overline{\mathcal{M}}_{g,n}$ through maps to $\mathbb{P}^1$ (and vice versa).

The key idea is this. Given a genus g and a partition α of d, one can define the *Hurwitz number* H^g_α counting genus g covers of $\mathbb{P}^1$, branched over ∞ with monodromy type α, and simply branched at an appropriate number of other fixed points.

Some naive exploration leads one to realize that there is a rich structure in the numbers H^g_α for fixed g; this was first explored by combinatorialists in the 1990's as a response to questions from physicists studying field theory. One aspect of this structure is that Hurwitz numbers have some sort of "polynomiality". This can be expressed in two ways. First, for fixed g and m, $H^g_{\alpha_1,\dots,\alpha_m}$ is a symmetric polynomial in the α of degrees between $2g-3+m$ and $3g-3+m$. Second, there is an ansatz for writing the Hurwitz generating function that also is polynomial in nature; this was a conjecture of Goulden and Jackson.

An earlier version of Goulden and Jackson's conjecture dealt with the genus 1 case. I proved their conjecture (i.e. polynomiality in this case) in **Genus 0 and 1 Hurwitz numbers: Recursions, formulas, and graph-theoretic interpretations**, [V7]. Part of my goal was to express Hurwitz numbers in terms of graphs, although I haven't made this work in higher genus. Since then, a couple of other proofs have appeared.

A few years ago, Ekedahl et al announced a wonderful formula for Hurwitz numbers explaining the first sort of polynomiality, [ELSV1]: the coefficients of the polynomial were precisely monomials in the tautological ring!

Based on their formula, Goulden, Jackson and I wrote **The Gromov-Witten potential of a point, Hurwitz numbers, and Hodge integrals**, which shows several things, including: (i) After a highly non-trivial change of variables, the Hurwitz generating function is a natural generalization of the generating function of ψ-integrals on $\overline{\mathcal{M}}_{g,n}$ (i.e. Witten's free energy of a point). (ii) It proves Goulden and Jackson's general polynomiality conjecture. In the course of a proof, it gives a faster (and more general) proof of an ansatz of the physicists Itzykson and Zuber. (iii) This proof gives a means of proving all (true) recursions for Hurwitz numbers in given genus, shutting down the cottage industry of Hurwitz recursions. (iv) The polynomiality also gives a closed form for simple Hurwitz numbers of any given genus.

[GJV10] relied on the announcement [ELSV1], and it wasn't completely clear that a proof was forthcoming, so T. Graber and I proved it in **Hodge integrals, Hurwitz numbers, and virtual localization**, [GV11]. We used virtual localization ([GP], with a twist), and we also see it as the first step toward virtual localization on a space of "relative stable maps". Later, Ekedahl et al indeed put out a proof in [ELSV2], which is quite different from ours, and very enlightening.

The set-up of [GV11] gives a short, low-tech proof that **The socle of the tautological ring of $\overline{\mathcal{M}}_{g,n}$ is one-dimensional** (with Graber, [GV12]); this is the first surprising prediction of Faber and Pandharipande's "dream". (In short, it states that the image of the 0-cycles of $\overline{\mathcal{M}}_{g,n}$ in the tautological ring is one-dimensional. This is obvious in cohomology, but surprising in Chow — the 0-dimensional Chow group of $\overline{\mathcal{M}}_{g,n}$ is expected to be uncountably generated in general.)

2.1.4. *Future work.* (a) I intend to use the combinatorial structure of the Hurwitz numbers to further understand the combinatorial structure of ψ-integrals. In particular, the constraints of the Virasoro algebra and Witten's conjecture should have combinatorial meaning.

(b) A key step in Kontsevich's (non-algebraic) proof of Witten's conjecture was the expression of ψ-integrals as sums over ribbon graphs. Ribbon graphs also have a strong connection to Hurwitz numbers, via Arnol'd's "edge-ordered graph" construction, [A] (discovered independently by A. Okounkov and Pandharipande).

I hope to understand how to express integrals involving ψ-classes and one λ-class as sums over ribbon graphs. If one can then conjecture such a formula for a general (top) intersection in the tautological ring, [F2] would give a means of proving it. Then much of Faber's conjecture would be reduced to combinatorial statements about ribbon graphs. (This programme will be difficult, but even partial results would shed light on the tautological ring.)

(c) One can similarly define *double Hurwitz numbers* $H^g_{\alpha,\beta}$, counting branched covers, simply branched away from two points, above which branching is given by partitions α and β. I am studying these further (with Goulden and Jackson), with the goal of proving an ELSV-type formula for double Hurwitz numbers in terms of the tautological ring. There are four reasons why this study may lead to insight into the tautological ring. *(i)* Double Hurwitz numbers count branched covers of $\mathbb{C}^*$ with given monodromy over the two "missing points"; as $\mathbb{C}^*$ is Calabi-Yau, structure is expected in these invariants. *(ii)* The generating function for double Hurwitz numbers is a τ-function for the Toda hierarchy, [O], which can be used to prove relations in the tautological ring. *(iii)* Double Hurwitz numbers have a good ribbon graph interpretation. *(iv)* Using (iii) and an argument counting points in polytopes, I can show that double Hurwitz numbers satisfy a homogeneity condition, which is a slightly weaker version of the polynomiality condition satisfied by usual Hurwitz numbers.

I am also overseeing J. Song's work extending these ideas to maps to higher-genus curves. One possible goal is a verification of predictions of the Virasoro conjecture where the target is a curve; another is a version of Göttsche's conjecture, [Gö] (where the target once again is one-dimensional). Song is a Ph.D. student in physics at MIT; I am member of Song's thesis committee. Some of Song's results are already very promising, including an unexpected link between Hodge integrals on $\overline{\mathcal{M}}_{g,2}$ and hypergeometric functions.

2.2. **Relative Gromov-Witten invariants.** There has been a great deal of work in extending Gromov-Witten theory to a relative setting (informally dealing with maps from a curve to a smooth projective variety, with prescribed tangencies with a smooth hypersurface). In the symplectic category, a great deal has been done by many people, including Y. Ruan, A.-M. Li, Ionel, Parker, and many more. Recent work of J. Li has connected this theory to the algebraic category; there is now a great deal of productive work to be done.

After I understand the "virtual fundamental class" in this setting, I would like to prove a virtual localization formula for relative invariants (with Graber); a first step toward these ideas was given in [GV11].

One motivation for this generalizes the Hurwitz picture. In order to study the invariants of X, consider instead relative invariants of $(X \times \mathbb{P}^1, X \times \{\infty\})$, with the obvious torus action. One may hope to relate these relative invariants to the (non-relative) invariants of X itself by localization; if X is a point, this is precisely the proof [GV11] of the ELSV formula. By choosing different linearizations, one finds combinatorial relations among the invariants of X.

If the Hurwitz approach yields a new, algebraic, proof of Witten's conjecture (which looks likely, thanks to ideas of Okounkov and Pandharipande), then this could lead to an approach to the general Virasoro conjecture. This is pure speculation, not a programme. However, it does motivate studying relative invariants further.

3. Articles outside of algebraic geometry

I have also pursued interests outside of algebraic geometry.

On Conway's recursive sequence (with T. Kubo), [KV1], explains and extends the remarkable structure behind a seemingly chaotic sequence of Conway's (that had been much-studied before). It is the most cited of my articles.

On the Steenrod length of real projective spaces: Finding longest chains in certain directed graphs, [V3], was used in J. D. Christensen's M.I.T. Ph.D. thesis (in topology, under Haynes Miller).

The ideas in [V9], and an attempt to disprove a geometric conjecture of Donagi's, have led to an ongoing project in number theory with M. Bhargava, the first paper of which will be [BV14], **On the 3-part of the class numbers of quartic fields**. There isn't much known about class numbers of number fields as a whole, but the 2-part of class numbers of quadratic number fields and the 3-part of quadratic and cubic number fields is understood by work of Gauss and Davenport-Heilbronn. For example, it is known that a positive density of cubic number fields have 3-part in their class group, and a positive density don't (and similarly for quadratic number fields). We have an approach to do the same for the 3-part of class numbers of quartic fields. There are several other applications of this point of view as well; we plan on finishing the first paper in this sequence in the coming year.

References

[KV1] T. Kubo and R. Vakil, *On Conway's recursive sequence*, Discrete Math. **152** (1–3) (1996), 225–252.

[V2] R. Vakil, *Enumerative geometry of curves via degeneration methods*, Harvard Ph.D. thesis, 1997.

[V3] R. Vakil, *On the Steenrod length of real projective spaces: Finding longest chains in certain directed graphs*, Gould Anniversary Volume of Discrete Math **204** (1–3) (1999), 415–425.

[V4] R. Vakil, *The enumerative geometry of rational and elliptic curves in projective space*, J. fuer die Riene und Ang. Math. (Crelle's journal), to appear.

[V5] R. Vakil, *Recursions for characteristic numbers of genus one plane curves*, Arkiv för Matematik, to appear.

[V6] R. Vakil, *Characteristic numbers of quartic plane curves*, Can. J. Math. **51** no. 5 (1999), 1089–1120.

[V7] R. Vakil, *Genus 0 and 1 Hurwitz numbers: Recursions, formulas, and graph-theoretic interpretations*, Trans. A.M.S., to appear.

[V8] R. Vakil, *Counting curves on rational surfaces*, manuscripta mathematica **102** (2000), 53–84.

[V9] R. Vakil, *Twelve points on the projective line, branched covers, and rational elliptic fibrations*, Math. Ann., to appear.

[GJV10] I.P. Goulden, D.M. Jackson, and R. Vakil, *The Gromov-Witten potential of a point, Hurwitz numbers, and Hodge integrals*, submitted.

[GV11] T. Graber and R. Vakil, *Hodge integrals, Hurwitz numbers, and virtual localization*, submitted.

[GV12] T. Graber and R. Vakil, *The socle of the tautological ring of* $\overline{\mathcal{M}}_{g,n}$ *is one-dimensional*, manuscript in preparation.

[V13] R. Vakil, *A tool for stable reduction of curves on surfaces*, preprint, submitted.

[BV14] M. Bhargava and R. Vakil, *On the 3-part of the class numbers of quartic number fields*, work in progress and manuscript in preparation.

[PV15] R. Pandharipande and R. Vakil, *Gromov-Witten theory and Mirror symmetry*, 98 pp., to appear in the Clay Institute textbook coming out of the summer school of 2000.

[V16] R. Vakil, *A Mathematical Mosaic: Patterns and Problem-Solving*, Brendan Kelly Publ., 1997.

[KPV17] K. Kedlaya, B. Poonen, R. Vakil, *The William Lowell Putnam Mathematical Competition: Problems, Solutions, and Commentary 1985–2000*, book in preparation, expected to appear Summer 2001, published by Math. Ass. Amer.

[CH] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[ELSV1] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, C. R. Acad. Sci. Paris **328** (1999), Série I, 1171–1180.

[ELSV2] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, math.AG/0004096v2.

[GP] T. Graber and R. Pandharipande, *Virtual localization*, Invent. Math. **135** (1999), no. 2, 487–518.

[A] V. Arnol'd, *Topological classification of trigonometric polynomials and combinatorics of graphs with an equal number of vertices and edges*, Functional analysis and its applications **30** (1996), no. 1, 1–17.

[F1] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, in *Moduli of curves and abelian varieties*, 109–129, Aspects Math., **E33**, Vieweg, Braunschweig, 1999.

[F2] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians*, in *New trends in algebraic geometry (Warwick, 1996)*, 93–109, London Math. Soc. Lecture Note Ser. **264**, Cambridge Univ. Press, Cambridge, 1999.

[Gö] L. Göttsche, *A conjectural generating function for numbers of curves on surfaces*, Comm. Math. Phys. **196** (1998), no. 3, 523–533.

[K] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[O] A. Okounkov, *Toda equations for Hurwitz numbers*, math.AG/0004128.

[AV] D. Abramovich and A. Vistoli, *Compactifying the space of stable maps*, math.AG/9908167. *Probably kill.*

[CrTa] M. Crescimanno and W. Taylor, *Large N phases of chiral QCD_2*, Nuclear Phys. B **437** (1995), no. 1, 3–24. *Probably kill.*

[LR] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I*, math.AG/9803036.

Dept. of Mathematics, MIT, Cambridge MA 02139

E-mail address: `vakil@math.mit.edu`