Abstracts

Murphy's Law for the Hilbert scheme (and the Chow variety, and moduli spaces of surfaces of general type, and stable maps, and nodal and cuspidal plane curves, and ...)

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Define an equivalence relation on singularities generated by: If $(X, p) \to (Y, q)$ is a smooth morphism, then $(X, p) \sim (Y, q)$. We say that *Murphy's Law* holds for a moduli space if every singularity type of finite type over $\mathbb{Z}$ appears on that moduli space.

Theorem 1. *The following moduli spaces satisfy Murphy's Law.*

- **1a.** *the Hilbert scheme of nonsingular curves in projective space*
- **1b.** *the moduli space of maps of smooth curves to projective space* $\mathcal{M}_g(\mathbb{P})$
- **1c.** *the Chow variety of nonsingular curves in projective space (where only seminormal singularities are allowed)*
- **2a.** *the (coarse or fine) moduli space of smooth surfaces (with ample canonical bundle)*
- **2b.** *the Hilbert scheme of nonsingular surfaces in* $\mathbb{P}^5$*, and the Hilbert scheme of surfaces in* $\mathbb{P}^4$
- **3.** *more generally, the moduli space of smooth n-folds* ($n > 1$) *(with ample canonical bundle)*
- **4a.** *branched covers of* $\mathbb{P}^2$ *with only simple branching (nodes and cusps), in characteristic not* 2 *or* 3
- **4b.** *the "Severi variety" of plane curves with a fixed numbers of nodes and cusps, in characteristic not* 2 *or* 3

In the lecture, more spaces were shown to satisfy Murphy's Law; for the sake of brevity we have kept the list short. A weaker equivalence relation may also be used.

We sketch some philosophy and history, and then outline the proof. I am grateful to the organizers and participants in the Oberwolfach workshop on Classical Algebraic Geometry for many comments, in particular for pointing out that Theorem 2 was first proved by Mnëv. I thank F. Catanese for sharing his expertise, and D. Abramovich for sharing his book (5). The results stated here will appear in (10).

The moral of Theorem 1 is as follows: in algebraic geometry, we know that some moduli spaces of interest are "well-behaved" (e.g. equidimensional, having at worst finite quotient singularities, etc.), often because they are constructed as Geometric Invariant Theory quotients of smooth spaces: e.g. the moduli space of curves, the moduli space of vector bundles on a curve, the moduli space of branched covers of $\mathbb{P}^1$ (the Hurwitz scheme, or space of admissible or twisted covers), the Hilbert scheme of divisors on projective space. In other cases, there has been some effort to try to bound how "bad" the singularities can get. Theorem 1 in essence states that these spaces can be arbitrarily singular, and gives a means of constructing an example where any given behavior happens. To make this quite explicit, one can construct a smooth curve in projective space whose deformation space has any given number of components, each with a given singularity type, with any given non-reduced behavior along various associated subschemes. Similarly, one can give a smooth surface of general type in characteristic 17 that lifts to 17^7 but not to 17^8.

There is a folklore belief that the Hilbert scheme satisfies Murphy's Law, which was first explicitly stated in (4) p. 18. (The MathReview for this book MR1631825 shows the mathematical community's discomfort with the informal nature of the traditional statement of

Murphy's Law.) I am not sure of the origin of Murphy's Law, but it seems reasonable to ascribe it to Mumford (see his famous "pathologies" paper (9)) and Hartshorne.

On the other hand, other moduli spaces were believed (or hoped) to be better-behaved. For example, Severi stated that the space of plane curves with given numbers of nodes and cusps was unobstructed (see the MathReview MR0897672 to (6)). J. Wahl gave the first counterexample in (12), and (6) gives another. Theorem 1 **4b** shows that Severi was "maximally wrong".

The proof is by drawing connections among various moduli spaces. We begin with a remarkable result of Mnëv. Define the *incidence scheme of points and lines in* $\mathbb{P}^2$, a locally closed subscheme of $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n = \{p_1, \dots, p_m, l_1, \dots, l_n\}$.

- We are given some specified incidences: For each pair (p_j, l_i), either p_j is required to lie on l_i, or p_j is required not to lie on l_i.
- The points are required to be distinct, and the lines are required to be distinct.
- Given any two lines, there a point required to be on both of them.
- Each line contains at least three points.

Theorem 2. *(Special case of Mnëv's Universality Theorem) Every singularity type appears on some incidence scheme.*

The original statement was in (7), (8); for a later exposition see (5) p. 13.

Outline of Theorem 1. Given any singularity type, we begin with an incidence scheme, and a point on it with that singularity type. Then consider the surface S that is the blow-up of $\mathbb{P}^2$ at the points p_j, along with the marked divisor D that is the proper transform of the union of the l_i. Deformations of (S, D) correspond to deformations of the l_i and p_j preserving the incidences, so the deformation space of (S, D) has the same singularity type. Choose two other divisor classes on S that are equivalent to D modulo 2, and sufficiently ample, and choose effective divisors D', D'' in these classes. (This is a smooth choice; the resulting deformation space hence has the same singularity type.) Then use Catanese's construction (1) to produce a bidouble ($\mathbb{Z}/2 \times \mathbb{Z}/2$) cover with branch divisor given by these three divisors, whose deformations are precisely those of (S, D, D', D''); this surface has ample canonical bundle. Then (2) Proposition 4.3 ensures that this surface has no extra automorphisms (other than $\mathbb{Z}/2 \times \mathbb{Z}/2$), ensuring that the coarse moduli space has the same singularity type as well. (In characteristic 2, we take tritriple covers instead, using Pardini's work on abelian covers.)

We have thus shown **2a**. By taking four or five sections of a sufficiently ample bundle, we obtain **2b**. By taking three sections, and using Wahl's results (showing that deformations of a generic branched cover of $\mathbb{P}^2$ are the same as deformations of the branch divisor preserving the nodes and cusps), we obtain **4**. By taking the product of the surface with high-genus curves, we obtain **3** (as deformations of the product are products of the deformations, by van Opstall's thesis (11)). By embedding the surface by a complete linear system corresponding to a sufficiently ample divisor, and slicing the surface with a sufficiently high degree hypersurface, we obtain **1**, using Fantechi and Pardini's key result of (3).

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