AN EXAMPLE OF A NICE VARIETY WHOSE RING OF GLOBAL SECTIONS IS NOT FINITELY GENERATED

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1. Introduction

When learning algebraic geometry, one might naively think that any variety has finitely generated ring of global sections. I certainly thought this. Brian Osserman asked about this in the introductory course I taught in fall 1999 (as part of a larger question he was considering, and eventually solved).

What follows is an example of a variety whose ring of global sections is not finitely generated. The variety is as nice as can be: it is a nonsingular quasiprojective threefold, that can be defined over basically any nice (infinite) field. The example was worked out with Johan de Jong, and a suggestion of Brian Conrad's sparked the idea. (So thanks are due to Brian O., Johan, and Brian C.)

If anyone knows of other examples in the literature (or even this one), I'd be curious to hear about it. They certainly exist (as lots of people know that the naive hope enunciated in the first paragraph is not true), but no one we've chatted with seems to know of any easy concrete examples. (I certainly haven't done any sort of literature search.)

I should mention that Brian Conrad has a sketch of an example by Raynaud, but there are some mysterious steps. I have a copy of the sketch in case anyone is interested.

2. The example

Let E be an elliptic curve over some ground field k, N a degree 0 non-torsion invertible sheaf on E, and P a positive-degree invertible sheaf on E. Then $H^0(E, N^m \otimes P^n)$ is nonzero if and only if either (i) $n > 0$, or (ii) $m = n = 0$ (in which case the sections are elements of k). Thus the ring $R = \oplus_{m,n \geq 0} H^0(E, N^m \otimes P^n)$ is not finitely generated.

Now let X be the total space of the vector bundle $N \oplus P$ over E. Then the ring of global sections of X is R.

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3. Interesting variations

One variant is a nonsingular quasiaffine threefold. Let Y be the total space of $N \oplus P^*$ (where N is the line bundle, and P^* is a k^*-bundle), and suppose P has degree at least 3. Once again the ring of global sections $\oplus_{m \geq 0, n \in \mathbb{Z}} H^0(E, N^m \otimes P^n)$ is not finitely generated.

Here's why Y is quasiaffine. P gives an embedding of E into $\mathbb{P}^{\deg P - 1}$. Let A be affine cone of the image, and B the complement of the origin in A. There is a natural morphism $\pi : B \to E$, expressing B as the total space of P^* over E. Extend $\pi^* P^*$ (which is an invertible sheaf on B) over A (by taking the reflexive hull; note that $A \setminus B$ is codimension 2 in A) to a coherent sheaf M, and then let Z be the spectrum of the symmetric algebra of M (so Z is affine). Then note that Y is an open subset of Z.

Another interesting fact: without doing any calculations, we see that $Z \setminus Y$ is codimension 1 in Z. Otherwise, (i) the normalization $\tilde{Z}$ of Z is affine and thus has finitely generated ring of global sections, (ii) Y is an open subset of $\tilde{Z}$, with complement of codimension 2, and (iii) any global section of Y would extend to a global section of $\tilde{Z}$ (and any global section of $\tilde{Z}$ would restrict to a global section of Y) giving us a contradiction.

Exercise: Find a nonsingular threefold whose ring of global sections requires at least N generators, where N is some huge number.

Interesting question: Find an example of a nonsingular variety over a finite field whose ring of global sections is not finitely generated. I don't have any idea how to approach this.

Other things to idly wonder about: Are there any examples in dimension 2? What does $Z \setminus Y$ look like in the quasiaffine example? What happens in the quasiaffine example if P has degree 1 or 2? (A should still exist: take the total space of P^*, and contract the zero-section — at least in the category of algebraic spaces — which is possible by a theorem of M. Artin, as it has negative self-intersection.)

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