Columbus April 5, 2003

Why the golden mean?

Ravi Vakil, Stanford University

http://math.stanford.edu/∼vakil/

The golden mean ϕ

$$\begin{aligned}\phi &= \frac{1+\sqrt{5}}{2}\\ &= 1.6180339...\end{aligned}$$

$$\phi^2 = \phi + 1$$

The other solution of this quadratic equation is ϕ's poor cousin

$$\begin{aligned}\sigma &= \frac{1-\sqrt{5}}{2}\\ &= 1-\phi\\ &= -1/\phi\\ &= -0.61803399...\end{aligned}$$

The Fibonacci numbers

$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n \geq 0).$

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
0	1	1	2	3	5	8	13	21	34	55

F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}
89	144	233	377	610	987	1597	2584

Useful fact:

$$\boxed{F_n = \frac{\phi^n - \sigma^n}{\sqrt{5}}}$$

(Remember $\sigma = 1 - \phi$.)

Fibonacci numbers hiding in Pascal's triangle

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7													
1																

Why?!!

The magic number 9899

$$1/9899 = 0.0001010203050813213 4...$$

Why?!!

Can you think of any more magic numbers? (Try 1/89 and 1/998.)

Miles and kilometers

1 mile is approximately 1.609 kilometers. (The conversion factor is roughly ϕ.)

1 mile is roughly 2 km.

2 miles is roughly 3 km.

3 miles is roughly 5 km.

5 miles is roughly 8 km.

8 miles is roughly 13 km. (8 mph is roughly 13 kmh)

$\vdots$

F_m miles is roughly F_{m+1} km.

Miles and kilometers, cont'd

What about other numbers?

$29 = 21 + 8$ miles is roughly $34 + 13 = 47$ km.

(What is the recipe in general? There is a hidden fact here.)

The many links between ϕ and Fibonacci numbers

ϕ is irrational. But you can approximate it by fractions. Recall that $\phi = 1.6180\ldots$. Notice:

$$
\begin{aligned}
1/1 &= 1 \\
2/1 &= 2 \\
3/2 &= 1.5 \\
5/3 &= 1.6666666\ldots \\
8/5 &= 1.6 \\
13/8 &= 1.625 \\
21/13 &= 1.6153846\ldots \\
34/21 &= 1.6190476\ldots \\
55/34 &= 1.6176470\ldots \\
89/55 &= 1.6181818\ldots \\
144/89 &= 1.6179775\ldots
\end{aligned}
$$

Ratios of successive Fibonacci numbers are *very good* approximations to ϕ. (In fact, it can be proved that they are the *best* approximations.)

Powers of ϕ over $\sqrt{5}$

n	$\phi^n/\sqrt{5}$
1	.723607
2	1.170820
3	1.894427
4	3.065248
5	4.959675
6	8.024922
7	12.984597
8	21.009519
9	33.994117
10	55.003636
$\vdots$	$\vdots$
20	6765.000030

Why?!!

Powers of ϕ

n	ϕ^n
1	1.618034
2	2.618034
3	4.236068
4	6.854101
5	11.090170
6	17.944272
7	29.034442
8	46.978714
9	76.013156
10	122.991869
$\vdots$	$\vdots$
20	15126.999934

Why?!!

Multiples of ϕ and ϕ^2

(recall $\phi^2 = \phi + 1$)

n	$n\phi$	$n\phi^2$
1	1.6180339	2.6180339
2	3.2360678	5.2360678
3	4.8541017	7.8541017
4	6.4721356	10.472136
5	8.0901695	13.090169
6	9.7082034	15.708203
7	11.326237	18.326237
8	12.944270	20.944270
9	14.562303	23.562303
10	16.180339	26.180339
$\vdots$	$\vdots$	$\vdots$
20	32.360678	52.360678

Why?!!

Geometry

The golden mean arises often in geometry. The moral reason for this relates to:

$$\sin 18^\circ = \frac{\phi - 1}{2}, \qquad \cos 36^\circ = \phi/2, \qquad \ldots$$

For example, it is present in the pentagram and decagon.

Platonic solids

- A cube of side length 1 has volume 1.
- A regular tetrahedron of side length 1 has volume $\frac{1}{6\sqrt{2}}$.
- A regular octahedron of side length 1 has volume $\sqrt{2}/3$.

What about the remaining two platonic solids, the dodecahedron and the icosahedron (with side length 1)?

Amazing fact! Both have volume of the form $a + b\phi$, where a and b are rational numbers.

Same for the buckyball (also known as a soccer ball or a truncated icosahedron)!

The Zometool volume theorem

More generally, anything one can build with Zometool has volume of the form $a + b\phi$ *where* a *and* b *are rational, where the length of a blue strut is* 1*.*

(*Translation:* the volume lies in the field $\mathbb{Q}(\phi)$.)

This has an easy proof!

How Zometool works: There are connectors that loosely look like blown-up dodechaedra. There are red struts that point in the directions of the pentagonal faces (from the center). There are yellow struts that point in the directions of the vertices. There are blue struts that point in the directions of the edges. The struts of each colors come in a variety of lengths differing by powers of ϕ.

Amazingly, the lengths are contrived so that you can build most interesting geometric models.

You really have to play with it to get an appreciation for how (well) it works.

Proof of the Zometool volume theorem

Pick three non-coplanar "blue" directions $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

Step 1.

We'll see that any point you can build starting at the origin and using various struts is of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

where $c_i \in \mathbb{Q}(\phi)$.

This can be done by demonstration.

(i) Given two blue directions, how show that you can get a third in the same plane.

(ii) Check that this lets you get all blue vectors.

(iii) Given *all* blue directions, show that you can get one (hence, by symmetry, all) yellow direction.

(iv) Given *all* blue and yellow directions, show that you can get *all* red directions.

Hence the volume of any polytope with vertices of this form is a multiple (by a magic number in $\mathbb{Q}(\phi)$) of the volume of the basic tetrahedron (spanned by the three original blue struts).

If we can easily show that the volume of the basic tetrahedron is rational, then we're done. However, I can't...

But if we can show that the volume of just one polytope built with Zometool is rational, then we win.

A second golden miracle: Wythoff's Game

Wythoff's game is a simple two-player game, with a simple winning strategy (to those in the know).

Start with two piles of coins. The players alternate taking away coins. When it is your turn, you may take

- any number of coins from the first pile, or
- any number of coins from the second pile, or
- the *same* number of coins from both piles.

The person to take the last coin wins.

Examples

If the number of coins is $(n, 0)$ $(n > 0)$:

If the number of coins is (n, n) $(n > 0)$:

If the number of coins is $(2, 1)$:

If the number of coins is $(3, 2)$:

Make a table:

10	w	w	w	w	w	w	**L**	w	w	w	w
9	w	w	w	w	w	w	w	w	w	w	w
8	w	w	w	w	w	w	w	w	w	w	w
7	w	w	w	w	**L**	w	w	w	w	w	w
6	w	w	w	w	w	w	w	w	w	w	**L**
5	w	w	w	**L**	w	w	w	w	w	w	w
4	w	w	w	w	w	w	w	**L**	w	w	w
3	w	w	w	w	w	**L**	w	w	w	w	w
2	w	**L**	w	w	w	w	w	w	w	w	w
1	w	w	**L**	w	w	w	w	w	w	w	w
0	**L**	w	w	w	w	w	w	w	w	w	w
	0	1	2	3	4	5	6	7	8	9	10

The slopes are ϕ and $1/\phi = -\sigma$! Irrational numbers are somehow produced by this discrete game!

How to win Wythoff's game

Remember the smallest "losing positions":

1	2
3	5
4	7
6	10
8	13
9	15
11	18
12	20

(and their opposites, e.g. $(2,1)$, $(5,3)$, $\ldots$).

Then when it is your turn, leave your opponent in a losing position.

Here is how to construct the "losing position" table from scratch.

We've seen these numbers before!

- (F_{2m}, F_{2m+1}) is a losing position.

But more so:

- The losing positions (a, b) with $a < b$ are precisely those numbers

$$(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor) = (\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n).$$

Another golden miracle: Why do Fibonacci numbers appear in nature?

Fibonacci numbers appear in pineapples, pine cones, sunflowers, etc. Why?

The same reason they turn up in abstract mathematics!

Greg Kuperberg, *A volume-preserving counterexample to the Seifert conjecture*, Comment. Math. Helv. **71** (1996), 70–97.

See also G. Kuperberg and K. Kuperberg, *Generalized counterexamples to the Seifert conjecture*, Annals of Math., **144** (1996), 239–268.

Greg Kuperberg's Elf Problem:

An elf dances around a circle of circumference ϕ. The elf takes n steps of length 1, and numbers his steps.

Conclusion

I hope you've enjoyed this brief tour through some ideas connected to the golden mean, and I hope that even experts have seen something new to whet their appetite for more.

Thank you!