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The Mathematics of Doodling

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Thanks. Everyone doodles in their own particular way. I'd like to tell you about the way I doodle, and some of the mathematics that I realized later that it connected to.

I've used these ideas as questions for talented students, at high school, undergraduate, and even the graduate level. There is a lot of room for exploration, and some of the answers are quite sophisticated.

Drawing circles around something.

First question: Are things getting more circular? Why?

Prior question: What precisely (mathematically) are we doing?

Given a plane set X, if $r \geq 0$, define the rth neighborhood of X:

$$N_r(X) = \{y : |y - x| \leq r \text{ for some } x \in X\}.$$

Reworded question: In some sense does

$$N_r(N_r(\cdots(N_r(N_r(X)))\cdots))$$

become more and more circular?

Does size matter?

How does our choice of r affect the question?

How does $N_1(N_1(X))$ compare to $N_2(X)$?

Answer: $N_{a+b}(X) = N_a(N_b(X))$.

The triangle inequality!

Reworded question: As $r \to \infty$, does $N_r(X)$ get more circular?

Answer: Yes! Suppose p is a point of X. Let C_t be the circle of radius t around p. Pick an R such that $X \subset C_R$.

Then

$$\{p\} \subset X \subset C_R,$$

so

$$C_r = N_r(\{p\}) \subset N_r(X) \subset N_r(C_R) = C_{r+R}.$$

Thus as $r \to \infty$, the border of $N_r(X)$ is squeezed between two circles whose ratio of radii goes to 1. We've answered the question!

Are we done with this doodle? No, we're just getting started!

For simplicity, let X be a convex polygon. Let P denote perimeter. What is $P(N_r(X))$?

$$P(N_r(X)) = P(X) + 2\pi r.$$

Let A denote area. What is $A(N_r(X))$?

$$A(N_r(X)) = A(X) + rP(X) + \pi r^2$$

Question: $\frac{d}{dr}A = P$. Why?!

Subtle observation: $A(N_r(X))$ is a quadratic polynomial in r, and its coefficients have good geometric meaning.

Let's generalize. Suppose X is convex, but with differentiable boundary. (Experts can consider piecewise differentiable boundaries.)

It will be more convenient to define N_r in terms of the normal vector.

Fact:

$$P(N_r(X)) = P(X) + 2\pi r$$

$$A(N_r(X)) = A(X) + rP(X) + \pi r^2$$

still hold! (Proof?)

This is one of my two favorite (hard) problems in multivariable calculus involving Green's theorem.

Let's give up convexity. What happens? How should we change the problem in order to have a good solution?

New concept: Area with multiplicity. But nothing changes: our formulas for perimeter and area still hold.

What about a figure eight? (We have to make sense of area!)

$$P(N_r(X)) = P(X).$$

$$A(N_r(X)) = A(X) + P(X)r.$$

We've lost the πr^2!? What happened?

To answer this question, consider a more complicated shape. This has winding number 2.

$$P(N_r(X)) = P(X) + 4\pi r,$$

$$A(N_r(X)) = A(X) + P(X)r + 2\pi r^2.$$

This is the winding number!

Let's go back to the winding number 1 case from now on.

What happens if we add a hole? First, let's consider just small r.

$$P(N_r(X)) = P(X)$$

$$A(N_r(X)) = A(X) + rP(X).$$

We've lost the πr^2 (again)! How best to make sense of this?!

But first, what happens when r gets large?

Answer: *Area with multiplicity 2 again*.

Many holes, many pieces?

Answer:

$$A(N_r(X)) = A(X) + rP(X) + \chi(X)\pi r^2$$

where $\chi(X)$ is the *Euler characteristic*: the number of connected components minus the number of holes.

More generalizations:

What happens in three dimensions?

What happens in one dimension?

What happens in n dimensions?

Let's do this with a box, of height h, length ℓ, and width w. Now let V denote volume and A denote surface area.

$$
\begin{aligned}
&V(N_r(X)) \\
&\quad = V(X) + A(X)r + (h + \ell + w)\pi r^2 + \frac{4}{3}\pi r^3
\end{aligned}
$$

This works for a convex body in general. We "know" the meaning of three of the coefficients. We have a fourth! (Exercise: Calculate it for a sphere.)

A beautiful Russian problem.

A Russian train company has a rule: you are not allowed packages (in a box) whose sum of dimensions (length plus width plus height) is more than $1m$. Is it possible to cheat by taking an illegal box, putting it in a larger box that is legal?

Answer: No!

Suppose X and Y are boxes, and $X \subset Y$.

Then $N_r(X) \subset N_r(Y)$.

$$V(X) + A(X)r + (h_X + \ell_X + w_X)\pi r^2 + \frac{4}{3}\pi r^3$$

$$\leq V(Y) + A(Y)r + (h_Y + \ell_Y + w_Y)\pi r^2 + \frac{4}{3}\pi r^3$$

Now let $r \to \infty$.

$$h_X + \ell_X + w_X \leq h_Y + \ell_Y + w_Y$$

as desired.

Problem solved!

In dimension n,

$$\begin{aligned} V(N_r(X)) &= V(X) + A(X)r + \cdots \\ &\quad +\chi(X)(\text{vol of } n\text{-sphere} \\ &\quad \text{of radius } r) \end{aligned}$$

Another definition of these invariants (from Gian-Carlo Rota's beautiful lectures "Introduction to Geometric Probability", AMS video, given as the AMS Colloquium lecture in 1998) is as follows.

$n = 2$, *one-dimensional invariant (aka perimeter).*

Consider the average length of the shadow of a convex body. Multiply it by π. You get the perimeter!

Example: A circle of radius R *always* has a shadow of area $2R$, so its perimeter is $2\pi R$.

Another example: What is the average length of the shadow of a line segment of length 1?

If the answer is x, then πx is the "perimeter", i.e. 2, so $x = 2/\pi$. (Check it!)

This connects to the famous experimental method of determining the value of π by dropping needles of length 1 on a table marked with parallel lines 1 unit apart, and counting how often a needle meets a line. (Buffon's Needle Problem)

$n = 3$, two-dimensional invariant (aka surface area).

Consider the average area of the shadow of a convex body. Multiply it by 4. You get the surface area!

Example: a sphere of radius R *always* has shadow of area πR^2, so its surface area is $4\pi R^2$!

Hard question: prove this for a box.

$n = 3$, one-dimensional invariant ($4h + 4w + 4\ell$ for a box):

Take the average length of its shadow on a 1-dimensional screen. Multiply by a certain constant. *Exercise:* find this constant by trying this out for X a line segment.

A related Hilbert problem.

Can you dissect a cube and rearrange the pieces to obtain a regular pyramid?

(In dimension two, the answer is yes: you can dissect any polygon and rearrange the pieces to make any desired other polygon with the same area. In other words, the only invariant up to dissection is *area*.)

Answer: No.

The proof uses a pumped-up version of this one-dimensional invariant. In dimension three, there are *two* invariants: volume, and this new one-dimensional invariant!

Another doodle.

Slide a chord of length 2 around a corner.

A first question: what is the shape (equation) of the envelope?

Answer: $(x/2)^{2/3} + (y/2)^{2/3} = 1$.

A second question: Slide a chord of length 2 around the inside of a shape, such as a square.

Mark the locus of its midpoint.

What happens?

Going around a square corner, you get a quarter circle! (Prove it!)

A connection to my washing machine.

This series of questions came to me while doing laundry. Here's why.

What about other shapes? Slide a chord of length 2 around the inside of a shape.

Around another angle?

How about a circle of radius R?

You get a circle of radius $\sqrt{R^2 - 1}$. Nothing pretty here, right?

Wrong! The *area* between the two circles is *always* π, *regardless* of R.

Let's go back to the other examples.

Fact. If you slide a chord around a (large enough convex) X, and the locus of its midpoint sweeps out a shape Y, then the area between X and Y is *always* π.

Why?!

(This is my other favorite (hard) problem in multivariable calculus involving Green's theorem.)

Follow-ups: What happens if you give up convexity? Etc. etc.

This fact also relates to the *previous* doodle. Start with a convex X. Travel around it counterclockwise. Construct a new shape Y as follows. Instead of taking a step of length 1 in the *normal* direction, take a step of length 1 in the *tangent* direction. What happens with a circle? With a square? An equilateral triangle? In general?

Why is this true in general? (More Green's theorem fun!)

What happens if you take a step halfway between the normal and tangent directions?

Conclusion:

Lurking behind even the most trivial-looking doodles can be mathematics of surprising beauty and power.

Thank you!