\[ \pi_p, \text{ the value of } \pi \text{ in } \ell_p \]

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The two dimensional space \( \ell_p \) is the set of points in the plane, with the distance between two points \((x, y)\) and \((x', y')\) defined by \( (|x - x'|^p + |y - y'|^p)^{1/p}, 1 \leq p \leq \infty \). The distance of \((x, y)\) from the origin is then \( (|x|^p + |y|^p)^{1/p} \). The equation of the unit circle \( C_p \), i.e. the circle with its center at the origin and radius 1, is

\[
(|x|^p + |y|^p)^{1/p} = 1. \tag{1}
\]

Figure 1 shows \( C_p \) for \( p = 1, 3/2, 2, 3 \) and \( \infty \). Equation (1) is unchanged when \( x \) is replaced by \( -x \), when \( y \) is replaced by \( -y \), and when \( x \) and \( y \) are interchanged. Therefore \( C_p \) is symmetric about the \( y \)-axis, about the \( x \)-axis, and about the line \( x = y \).

We define \( \pi_p \) to be the ratio of the circumference of \( C_p \) to two times its radius, which is its diameter 2. The circumference is the integral of the element of arclength \( ds = \)
\[(|dx|^p + |dy|^p)^{1/p} \text{ around } C_p. \] Thus
\[
\pi_p = \frac{1}{2} \int_{C_p} (|dx|^p + |dy|^p)^{1/p} = \frac{1}{2} \int_{C_p} \left(1 + \left|\frac{dy}{dx}\right|^p\right)^{1/p} |dx|.
\tag{2}
\]

Because of the symmetry of \(C_p\), its circumference is equal to four times its arclength in the first quadrant, or eight times its arclength in the first quadrant between the lines \(x = 0\) and \(x = y\). When \(x = y\), (1) shows that \(x = 2^{-1/p}\), so the integral in (2) is 8 times the integral from 0 to \(2^{-1/p}\). By calculating \(dy/dx\) from (1), we can rewrite (2) as
\[
\pi_p = 4 \int_0^{2^{-1/p}} \left[1 + |x^{-p} - 1|^{1-p}\right]^{1/p} dx.
\tag{3}
\]

For \(p = 1\), (3) yields \(\pi_1 = 4(2^{1/p})(2^{-1/p}) = 4\). For \(p = \infty\), the integrand is 1 and the upper limit is 1, so \(\pi_\infty = 4\). At \(p = 2\), \(\pi_2 = \pi\). If geometry had been developed using the \(\ell_p\) distance instead of the \(\ell_2\) distance, \(\pi_p\) would have replaced \(\pi_2\), which is just the familiar \(\pi\).

Figure 2 shows a graph of \(\pi_p\) as a function of \(p\), obtained by numerical integration of (3). The graph shows that as \(p\) increases from \(p = 1\), \(\pi_p\) decreases monotonically from its maximum value \(\pi_1 = 4\) to its minimum value \(\pi_2 = \pi\), and then increases monotonically to \(\pi_\infty = 4\). Thus for each \(p\) in \(1 \leq p \leq 2\), there is a \(q\) in \(2 \leq q \leq \infty\) such that
\[
\pi_p = \pi_q.
\tag{4}
\]

To find \(q\) we recall that the Hölder inequality involves two exponents \(p\) and \(q\) related by
\[
\frac{1}{p} + \frac{1}{q} = 1.
\tag{5}
\]

We conjecture that (4) will hold when (5) does, and the numerical results shown in Figure 2 confirm this. In fact, when (5) holds, the domains bounded by \(C_p\) and \(C_q\) are polar to one another. Then a result of Schäffer [1] (see also Thompson [2]) shows that (4) holds.
We shall now give another proof that (4) holds when (5) does, by showing that then the integral (3) for \( \pi_p \) equals that for \( \pi_q \). We begin by writing the equation for the arc of \( C_p \) in the first quadrant in terms of a parameter \( t \in [0, \infty) \), setting \( x = f_1(t) \) and \( y = f_2(t) \). Then the length \( L_p \) of that arc can be written as

\[
L_p = \int_0^\infty \frac{f_2' \left( (f_1^{p(p-1)} + f_2^{p(p-1)})^{1/p} \right)}{f_1^{p-1}} \, dt = \int_0^\infty f_2' \left( 1 + \left( \frac{f_2}{f_1} \right)^{p(p-1)} \right)^{1/p} \, dt. \tag{6}
\]

The integrand in (6) is obtained from that on the right side of (2). We choose the parameter \( t \) such that \( t^{q/p} \) is the slope of the line from the origin to the point \( f_1(t), f_2(t) \) on \( C_p \), so that \( t^{q/p} = f_2(t)/f_1(t) \). From this equation and (1) we find that

\[
f_1(t) = (t^q + 1)^{-1/p}, \quad f_2(t) = (t^{-q} + 1)^{-1/p}. \tag{7}
\]

We parameterize \( C_q \) in the same way, setting \( x = g_1(t) \) and \( y = g_2(t) \), with

\[
g_1(t) = (t^p + 1)^{-1/q}, \quad g_2(t) = (t^{-p} + 1)^{-1/q}. \tag{8}
\]

Now we define the function \( F(t) = -f_1g_2 + f_2g_1 \). At the ends of the two arcs, \( t = 0 \) and \( t = \infty \), we have \( f_1 = g_1 \) and \( f_2 = g_2 \). Therefore \( F(0) = 0 \) and \( F(\infty) = 0 \), so \( \int_0^\infty F'(t) \, dt = 0 \). This equation can be rewritten as follows, by differentiating the definition of \( F(t) \) to get \( F'(t) \) and then transposing:

\[
\int_0^\infty (-f_1'g_2 + f_2'g_1) \, dt = \int_0^\infty (-g_1'f_2 + g_2'f_1) \, dt. \tag{9}
\]

The integrand on the left side of (9) can be rewritten as

\[
-f_1'g_2 + f_2'g_1 = f_2' \left( 1 + \left( \frac{f_2}{f_1} \right)^{p(p-1)} \right)^{1/p}. \tag{10}
\]
To prove (3) we first transform the left side as follows:

\[-f'_1g_2 + f'_2g_1 = -\left(-\frac{1}{p}\right) (t^q + 1)^{-\frac{p+1}{p}} q t^{q-1} (t^{-p} + 1)^{-1/q} + \left(-\frac{1}{p}\right) (t^{-q} + 1)^{-\frac{p+1}{p}} (-q) t^{q+1} (t^p + 1)^{-1/q} = \frac{q}{p} (t^q + 1)^{-\frac{p+1}{p}} (t^p + 1)^{-1/q} (t^{q-1+p/q} + t^{q(p+1)/p-(q+1)}) = \frac{q}{p} (t^q + 1)^{-\frac{p+1}{p}} (t^p + 1)^{1/p-1} t^{\frac{1}{p}-1} (t^p + 1) = \frac{q}{p} (t^q + 1)^{-\frac{p+1}{p}} (t^p + 1)^{1/p} t^{\frac{1}{p}-1} \]  

(11)

Then we transform the right side:

\[f'_2 \left(1 + \left( \frac{f_2}{f_1} \right)^{p(q-1)} \right)^{1/p} = f'_2 (1 + t^{(q/p)p(q-1)})^{1/p} = f'_2 (1 + t^{p})^{1/p} = -\frac{1}{p} (t^{-q} + 1)^{-\frac{p+1}{p}} (-q) t^{q-1} (1 + t^{p})^{1/p} = \frac{q}{p} (t^q + 1)^{-\frac{p+1}{p}} (t^p + 1)^{1/p} t^{\frac{2}{p}(p+1)-q-1} = \frac{q}{p} (t^q + 1)^{-\frac{p+1}{p}} (t^p + 1)^{1/p} t^{\frac{1}{p}-1} \]  

(12)

The last forms of (11) and (12) are the same, which proves (10).

The integral from 0 to \(\infty\) of the right side of (10) is just \(L_p\), as (6) shows. Therefore the integral of the left side, which is also the left side of (9), is also \(L_p\). A symmetrical argument shows that the right side of (9) is \(L_q\), so (9) shows that \(L_p = L_q\). This proves that (4) is true when \(p\) and \(q\) are related by (5).

The geometry behind this proof is rather interesting. Suppose \(q > p\). As \(t\) goes from 0 to 1, the point on \(C_p\) is "behind" the point on \(C_q\). At \(t = 1\) the \(p\)-point passes the \(q\)-point. The cumulative lengths at "time" \(t\) are not the same; the difference is twice the area of
the triangle spanned by the origin, the \( p \)-point and the \( q \)-point. This difference vanishes at \( t = \infty \), when the two points coincide.

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**References**


Figure 1. The unit circle $C_p$ in the first quadrant, defined by (1), for $p = 1, 3/2, 2, 3, \infty$.

Figure 2. $\pi_p$ as a function of $p$, obtained by numerical integration of (3).