A Beginner's Guide to Jet Bundles from the Point of View of Algebraic Geometry

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Although it may never be updated, this is a draft version, so please don't pass it on without the authour's permision. Suggestions and corrections would be appreciated.

1 Introduction

This short note is intended to provide a functional introduction to jet bundles from the point of view of enumerative algebraic geometry. These methods are certainly known, but as far as I know they have never been collected in one place. The title also admits another reading: the authour has little background in the field. This note is more a collection of my thoughts than a comprehensive introduction to the subject. I am most interested in explaining how jets give very quick solutions to a whole class of enumerative problems.

All references are to Hartshorne unless otherwise noted. Familiarity with some of the following will be useful: Chern classes (especially doing calculations and understanding Chern classes as degeneracy loci), vector bundles and coherent sheaves (II.5), and some differentials (II.8). You might want to skip the proofs the first time through. '$\sim$' denotes an equivalence relation between vector bundles; it means that their Chern polynomials are equal.

2 Jets

2.1 The Intuition

Let's think about the following problem. If you had a general pencil of degree d curves in $\mathbf{P}^2$, how many times would you expect to see a singular curve? One plan of attack is to consider an auxiliary bundle. We have our line bundle $L = \mathcal{O}(d)$ on $\mathbf{P}^2$. Consider a section $s \in H^0(\mathbf{P}^2, \mathcal{O}(d))$ near a point p ($(0,0)$ in local co-ordinates). s has a local defining equation near p

$$a_1 + a_x x + a_y y + a_{x^2} x^2 + \cdots .$$

Loosely speaking, we would like a (rank 3) bundle V that has for its stalk at p the information (a_1, a_x, a_y). Our global section s induces a section s' of V.

If s' is zero at p, then s is singular at p. For a web of general sections s', we expect this to happen $c_2(V)$ times over all of $\mathbf{P}^2$.

It turns out that the bundle we want is the *bundle of first-order jets* $J^1 L$, and that $J^1 L \sim L \otimes (\mathcal{O} \oplus \Omega)$. The Chern classes of this latter bundle are very easy to compute, and you can verify that

$$c_2(L \otimes (\mathcal{O} \oplus \Omega)) = 3(d-1)^2 \text{ points.}$$

2.2 The Method

In general, the bundle containing the information

$$(a_1, a_x, a_y, \ldots, a_{x^n}, \ldots, a_{xy^{n-1}}, a_{y^n})$$

of a section of L is the n^{th} *jet bundle* J^nL, and there is a natural map $H^0(\mathbf{P}^2, L) \to H^0(\mathbf{P}^2, J^nL)$ that "usually" maps general sections to general sections, allowing us to do Chern class calculations to work out degeneracy loci. Most important for computations,

$$J^nL \sim L \otimes (1 \oplus \Omega \oplus \ldots \oplus Sym^n\Omega)$$

(This will be proved in Section 2.4.)

2.3 An Example

If we have three general polynomials of degree d in $\mathbf{P}^1$, how many triple points do we expect to see in their net? We want a bundle to take into account (a_1, a_x, a_{x^2}); this is $J^2(\mathcal{O}(d))$. The answer will be $c_1(J^2(\mathcal{O}(d)))$.

$$\begin{aligned} J^2(\mathcal{O}(d)) &\sim \mathcal{O}(d) \otimes (\mathcal{O} \oplus \Omega \oplus \Omega^2) \\ &= \mathcal{O}(d) \otimes (\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-4)) \\ &= \mathcal{O}(d) \oplus \mathcal{O}(d-2) \oplus \mathcal{O}(d-4) \end{aligned}$$

$$\begin{aligned} \text{Thus } c_1(J^2(\mathcal{O}(d))) &= d + (d-2) + (d-4) \\ &= 3d - 6 \end{aligned}$$

Very quick!

2.4 Rigorously Defining Jet Bundles

Let Y be a Cohen-Macaulay variety. (So the results shown will hold for Y smooth, or more generally local complete intersection or even Gorenstein.) Consider $Y \times Y$ with diagonal Δ and projections π_1 and π_2.

$$\begin{array}{ccc} Y \times Y & \stackrel{\pi_2}{\to} & Y \\ \pi_1 \downarrow & & \\ Y & & \end{array}$$

Define $J^nL = \pi_{1*}(\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^{n+1} \otimes \pi_2^*L)$.

Quick fact (*): π_1 gives an isomorphism from the diagonal, so this gives us a lot of algebraic leeway when pushing forward a sheaf supported on (the reduced subscheme given by) the diagonal. In particular, the higher direct image sheaves will all vanish.

Note that $J^0L = L$.

Theorem 1 $0 \to L \otimes Sym^n\Omega \to J^nL \to J^{n-1}L \to 0$ *is exact.*

Proof.

$$\begin{aligned} Sym^n\Omega &= Sym^n(\pi_{1*}(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)) \\ &= \pi_{1*}Sym^n(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2) \\ &= \pi_{1*}(\mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1}) \end{aligned}$$

(Theorem II.8.21A p. 185; actually a result from Matsumura. Here we use the Cohen-Macaulay hypothesis.)

$$0 \to \mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1} \to \mathcal{O}/\mathcal{I}_\Delta^{n+1} \to \mathcal{O}/\mathcal{I}_\Delta^n \to 0 \text{ is exact}$$

$$\Rightarrow 0 \to (\mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1}) \otimes \pi_2^* L \to (\mathcal{O}/\mathcal{I}_\Delta^{n+1}) \otimes \pi_2^* L \to (\mathcal{O}/\mathcal{I}_\Delta^n) \otimes \pi_2^* L \to 0 \text{ is exact}$$

$$\Rightarrow 0 \to (\mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1}) \otimes \pi_1^* L \to (\mathcal{O}/\mathcal{I}_\Delta^{n+1}) \otimes \pi_2^* L \to (\mathcal{O}/\mathcal{I}_\Delta^n) \otimes \pi_2^* L \to 0 \text{ is exact}$$

as $\mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1}$ is supported on the reduced subscheme given by the diagonal. Pushing forward by π_1,

$$0 \to L \otimes Sym^n \Omega \to J^{n+1} L \to J^n L \to R^1 \pi_{1*}(\mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1}) \text{ is exact}$$

But by (*), $R^1 \pi_{1*}(\mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1}) = 0$. Ξ

Corollary 1 $J^n L \sim L \otimes (\mathcal{O} \oplus \Omega \oplus Sym^2 \Omega \oplus \ldots \oplus Sym^n \Omega) \sim L \otimes J^n \mathcal{O}$

Corollary 2 *If* $\dim Y = k$*, then* $J^n L$ *is a vector bundle of rank* $\binom{n+k-1}{k}$.

Proof. By induction using the exactness of

$$0 \to L \otimes Sym^n \Omega \to J^n L \to J^{n-1} L \to 0$$

Use the fact that if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is an exact sequence of quasi-coherent sheaves, with $\mathcal{F}$ and $\mathcal{H}$ vector bundles of rank f and h respectively, then $\mathcal{G}$ is a vector bundle of rank $f + h$. (Do this on affine open sets.) Ξ

There is a natural map $H^0(Y, L) \to H^0(Y, J^n L)$ that is the composition of the following maps:[1]

$$H^0(Y, L) \to H^0(Y \times Y, \pi_2^* L) \to H^0(Y \times Y, \mathcal{O}/\mathcal{I}_\Delta^n \otimes \pi_2^* L) \to H^0(Y, \pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^n \otimes \pi_2^* L)) = H^0(Y, J^n L)$$

These maps are easily checked to be compatible, by which I mean the following diagram commutes:

$$\begin{array}{ccc} H^0(Y, L) & \longrightarrow & H^0(Y, J^n L) \\ & \searrow \quad \nearrow & \\ & H^0(Y, J^{n+1} L) & \end{array}$$

In most cases, the map $H^0(Y, L) \to H^0(Y, J^n L)$ will "send general sections to sections that are general enough," so our Chern class calculations will give the correct information about the degeneracy locus, but this isn't always true. I haven't investigated criteria for when one would expect the existence of global sections of $J^n L$ coming from sections of L, that have degeneracy loci of expected dimension.

Theorem 2 *If* $s \in H^0(\mathbf{P}^2, L)$ *gives* $s' \in H^0(\mathbf{P}^n, J^{n-1} L)$*, then* s *has a zero of multiplicity* n *at a point* P *exactly when* s' *has a zero at* P*. (Thus* $J^n L$ *is really the bundle we want.)*

[1] The same method gives us maps $H^0(U, L) \to H^0(U, J^n L)$ for each open set $U \subset Y$. When $L = \mathcal{O}$, this is a map $\mathcal{O} \to J^n \mathcal{O}$ of vector bundles. The composition $\mathcal{O} \to J^n \mathcal{O} \to \mathcal{O}$ is the identity, so $\mathcal{O}$ is a summand of $J^n \mathcal{O}$. In particular, $J^1 \mathcal{O} = \mathcal{O} \oplus \Omega$. However, the composition of maps $J^n \mathcal{O} \to \mathcal{O} \to J^n \mathcal{O}$ is *not* the identity.

In general (when $L \neq \mathcal{O}$), the map $L \to J^n L$ is a map of sheaves, but *not* a map of $\mathcal{O}$-modules. As a result, $J^1 L$ won't usually split. The extension

$$0 \to \Omega \to L^* \otimes J^1 L \to \mathcal{O} \to 0$$

is likely given by the image of the class of L in

$$\text{Pic } Y = H^1(Y, \mathcal{O}_Y^*) \to H^1(Y, \mathcal{O}_Y) \to H^1(Y, \Omega) = Ext^1(\mathcal{O}, \Omega).$$

Proof. A section of L induces a section s' of $\pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^n \otimes \pi_2^* L) = J^{n-1}L$ and s'' of $\pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^n \otimes \pi_1^* L) = L \otimes J^{n-1}\mathcal{O}$. Although they are not (in general) the same vector bundle on Y, the induced sections have zeroes at the same points.

We compare the image of s at P with the induced image of s'' at P through the following sequences of isomorphisms:

$$\begin{aligned}
\pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^n \otimes \pi_1^* L) \otimes \mathcal{O}/\mathfrak{m} &= \pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^n) \otimes L \otimes (\mathcal{O}/\mathfrak{m}) \\
&= \pi_{1*}((\mathcal{O}/\mathcal{I}_\Delta^n) \otimes (\mathcal{O}/\pi_1^*\mathfrak{m})) \otimes L \\
&= \pi_{1*}(\mathcal{O}/(\mathcal{I}_\Delta^n \cap \pi_1^*\mathfrak{m})) \otimes L \\
&= (\mathcal{O}/\mathfrak{m}^n) \otimes L
\end{aligned}$$

The image of s'' in the stalk of $J^n L$ is precisely the image of s under the map

$$H^0(\mathbf{P}^2, L) \to H^0(\mathbf{P}^2, (\mathcal{O}/\mathfrak{m}^n) \otimes L).$$

Ξ

I found it worthwhile to actually see how this works by playing with simple examples. [Was that proof okay? Perhaps explain why we defined the jets as $\pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^n \otimes \pi_2^* L)$ instead of $\pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^n \otimes \pi_1^* L)$.]

2.5 Further Discussion

1. This method allows us to compute the Chern polynomial of $\pi_{1*}(\mathcal{O}/\mathcal{I}_\Delta^{n+1} \otimes \pi_2^* L)$ by way of a filtration whose successive quotients ($L \otimes Sym^n \Omega$) we understand well. The other common way of computing the Chern classes of a pushforward is the Grothendieck-Riemann-Roch formula, which would require proving that certain higher direct image sheaves vanish. (In this case, $R^i \pi_{1*} = 0$ ($i > 0$) for all sheaves supported on the diagonal, so G-R-R will work.) Finding filtrations seems to be a good way of dodging the unpleasantness of G-R-R.

2. We have an inverse system of vector bundles

$$\cdots \to J^n L \to J^{n-1} L \to \cdots \to J^1 L \to L \to 0$$

so we can construct the inverse limit $J^\infty L = \lim\limits_{\leftarrow} J^n L$. This vector bundle contains all the information of the power series expansion at each point. Likely

$$J^\infty L = \pi_{1*}((\lim_{\leftarrow} \mathcal{O}/\mathcal{I}_\Delta^n) \otimes \pi_2^* L).$$

$\lim\limits_{\leftarrow} \mathcal{O}/\mathcal{I}_\Delta^n$ might remind you of the formal completion of $Y \times Y$ along the diagonal Δ (see Section II.9 if you care).

3. We can define $J_n L = (J^n L)^*$. (This notation is my own.) This gives us a direct system:

$$0 \to L^* \to J_1 L \to J_2 L \to \cdots$$

which will give us $J_\infty L = \lim\limits_{\rightarrow} J_n L$. Presumably $J_\infty L = (J^\infty L)^*$. Locally, under reasonable conditions (such as smoothness), sections of J_∞ will look like (finite) polynomials in $\{\frac{d}{dx_1}, \frac{d}{dx_2}, \ldots\}$. (They will read off finite linear combinations of co-efficients in the power series.)

3 Splitting Jet Bundles

3.1 The Intuition and Method

The concept of jet bundles can be extended slightly to deal with a much wider class of problems. Here is a motivating problem: how many flexes will appear on a general degree d curve on $\mathbf{P}^2$? As before, we can evaluate this class by constructing the appropriate auxiliary bundle.

We construct a variety $\mathbf{P}T$ that is the projectivized tangent space of $\mathbf{P}^2$ (parametrizing lines, not quotients) with hyperplane class $\mathcal{M} = \mathcal{O}(1)$. ($\mathbf{P}T$ parametrizes those length 2 subschemes of $\mathbf{P}^2$ supported at one point.) Let ϕ be the projection $\phi : \mathbf{P}T \to \mathbf{P}^2$. We can easily compute its Chow ring.[2]

As before, each section s of $\mathcal{O}(d)$ looks locally like

$$a_1 + a_x x + a_y y + a_{x^2} x^2 + \cdots .$$

We would like a rank three vector bundle containing the information (a_1, a_x, a_{x^2}). A generic section of $L = \mathcal{O}(d)$ will induce a generic section of V, and our section of V will have $c_3(V)$ zeroes, which will correspond to flexes. It turns out that

$$V \sim L \otimes (\mathcal{O} \oplus \mathcal{M} \oplus \mathcal{M}^2)$$

- $L \otimes \mathcal{O}$ represents the a_1 information.
- $L \otimes \mathcal{M}$ represents the a_x information.
- $L \otimes \mathcal{M}^2$ represents the a_{x^2} information.

In general, if W keeps track of $a_{p_1(x,y)}$, $a_{p_2(x,y)}$, ... , $a_{p_n(x,y)}$ then $W \otimes \mathcal{M}$ keeps track of $a_{xp_1(x,y)}$, $a_{xp_2(x,y)}$, ... , $a_{xp_n(x,y)}$. For example, to hunt for cusps we would want to look at the bundle V' containing the information $a_1, a_x, a_y, a_{x^2}, a_{xy}$, so

$$V' \sim L \otimes (\mathcal{O} \oplus \Omega \oplus \Omega \otimes \mathcal{M})$$

- $L \otimes \mathcal{O}$ represents the a_1 information.
- $L \otimes \Omega$ represents the a_x, a_y information (as it did in $J^1 L$).
- $L \otimes (\Omega \otimes \mathcal{M})$ represents the a_{x^2}, a_{xy} information.

Counting flexes is now reduced to a computation.

$$A^*\mathbf{P}\Omega = \mathbf{Z}[h, m]/(h^3, m^2 + 3mh + 3h^2)$$

$$\begin{aligned} c_t(\mathcal{O}) &= 1 \\ c_t(\mathcal{M}) &= 1 + mt \\ c_t(\mathcal{M}^2) &= 1 + 2mt \end{aligned}$$

$$c_t(\mathcal{O} \oplus \mathcal{M} \oplus \mathcal{M}^2) = 1 + 3mt + 2m^2t^2$$

$$\begin{aligned} c_t(V) &= c_t(L \otimes (\mathcal{O} \oplus \mathcal{M} \oplus \mathcal{M}^2) \\ &= 1 + (3dh + 3m)t + (3d^2h^2 + 6mdh + 2m^2)t^2 + (d^3h^3 + 3md^2h^2 + 2m^2dh)t^3 \end{aligned}$$

$$\begin{aligned} \text{Thus } c_3(V) &= d^3h^3 + 3d^2mh^2 + 2dm^2h \\ &= 0 + 3d^2mh^2 - 6dmh^2 \\ &= 3d(d-2) \text{ points.} \end{aligned}$$

[2]Given a space Y with Chow ring R and a rank r vector bundle W with Chern classes c_1, ..., c_r, the Chow ring of $\mathbf{P}W$ is

$$A^*(\mathbf{P}W) = R[m]/(m^r + c_1 m^{r-1} + \ldots + c_{r-1}m + c_r)$$

where m is the class of $\mathcal{O}(1)$, the hyperplane class. (see Example 8.3.4, p. 141 of [1]). In this case, $m = \mathcal{M}$.

3.2 Analysis of the Method

For concreteness, we'll work on a smooth surface Y, but the analysis will carry through for any smooth variety.

Let S be a finite set of monomials of the form $x^m y^n$ with graded pieces denoted by S_n (and $S_{<n}$, $S_{\leq n}$ having their obvious meanings) such that

1. $S_n = \{x^n, x^{n-1}y, \ldots, x^{n-t(n)}y^{t(n)}\}$ for some integer $t(n) \in [-1, n]$. (If $t(n) = -1$, S_n is taken to be empty.)

2. If $x^m y^n \in S$ and $m > 0$ then $x^{m-1}y^n \in S$.

In other words, S looks like a basis for $k[x, y]$ modulo some monomial relations. The possible choices of S represent those subschemes (supported at one point) that can be parametrized by $\mathbf{P}T$ (or Y itself). Each S can be represented by a Ferrers diagram. For example, $\{1, x, y, x^2, x^3\}$ would be represented by:

The above two requirements correspond to characteristics of the diagram that are visually simple. Given a square in the diagram, Condition 1 states that (if possible) the square above and to the right is also in the diagram. Condition 2 states that (if possible) the square to the left is also in the diagram. Our construction will use the following commutative diagram, where each square is a fibre product.

$$
\begin{array}{ccccc}
\mathcal{X} & \rightarrow & \Delta & & \\
\downarrow & & \downarrow & & \\
\mathbf{P}T \times Y & \stackrel{(\phi,1)}{\rightarrow} & Y \times Y & \stackrel{\pi_2}{\rightarrow} & Y \\
\Pi_1 \downarrow & & \downarrow \pi_1 & & \downarrow \\
\mathbf{P}T & \stackrel{\phi}{\rightarrow} & Y & \rightarrow & \mathrm{pt}
\end{array}
$$

Π_1, Π_2 are the projections from $\mathbf{P}T \times Y$ to its factors. $\mathcal{X}$ is the incidence correspondence and is isomorphic to $\mathbf{P}T$ via Π_1. Note that

$$(\phi, 1)^{-1}\mathcal{I}_\Delta \cdot \mathcal{O}_{\mathbf{P}T \times Y} = \mathcal{I}_\mathcal{X}$$

As $\mathcal{M}$ is $\mathcal{O}(1)$ on $\mathbf{P}T$,

$$0 \to Q \to \phi^*\Omega \to \mathcal{M} \to 0 \tag{1}$$

is the exact sequence corresponding to the projective bundle, where Q is the dual of the universal quotient bundle. (Actually, $Q = \Omega_{\mathbf{P}_{T/P}} \otimes \mathcal{M}$, although we won't need this fact.) This induces a dual filtration on $\phi^* Sym^n \Omega$:

$$Sym^n\Omega \imath \mathcal{M} \otimes Sym^{n-1}\Omega \imath \mathcal{M}^2 \otimes Sym^{n-2}\Omega \imath \cdots \imath \mathcal{M}^n \imath 0. \tag{2}$$

We will define $J^S(L)$ with the following propoerties:

(A) $J^S(L) \sim \phi^*L \otimes J^S(\mathcal{O})$

(B) $J^{\{1,x,y,\ldots,x^n,x^{n-1}y,\ldots,y^n\}}L = \phi^* J^n(L)$

(C) $0 \to L \otimes (\mathcal{M}^{n-t(n)} \otimes Sym^{t(n)}\Omega) \to J^{S_{\leq n}}(L) \to J^{S_{<n}}(L) \to 0$ is exact. (Notice that $\mathcal{M}^{n-t(n)} \otimes Sym^{t(n)}$ is the $(n - t(n))^{\text{th}}$ term in the filtration (2).) This will justify the Chern class calculations described in the previous subsection. In general, if $S \subset T$, then there will be a natural surjective morphism $J^T \to J^S$.

(D) We have a natural map $H^0(Y, L) \to H^0(Y, J^S(L))$, and if $s \in H^0(Y, L)$ determines $s' \in H^0(Y, J^S(L))$, then s' is zero exactly when s is zero on the corresponding (length $\mid S_n \mid$) subscheme.

As we did with ordinary jets, we will take a sheaf of ideals $\mathcal{I}_S$ (to be defined soon) supported on $\mathcal{X}$, and

$$J^S(L) = \Pi_{1*}(\mathcal{O}/\mathcal{I}_S \otimes \Pi_2^* L).$$

We construct the $\mathcal{I}_S$ by way of two filtrations on $\mathcal{O}_{\mathbf{P}_{T\times Y}}$. The first comes from the filtration giving us the usual jet bundles:

$$\mathcal{O} \supset \mathcal{I}_\mathcal{X} \supset \mathcal{I}_\mathcal{X}^2 \supset \mathcal{I}_\mathcal{X}^3 \supset \cdots$$

As before, $\mathcal{I}_\mathcal{X}^k$ corresponds to $\{x^m y^n \mid m+n \geq k\}$. Diagramatically:

$\mathbf{P}T$ parametrizes the codimension 1 quotients of $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2$, which determines an ideal $\mathcal{J}$ such that $\mathcal{I}_\Delta^2 \subset \mathcal{J} \subset \mathcal{I}_\Delta$. Formally, $Q \hookrightarrow \mathcal{I}_\mathcal{X}/\mathcal{I}_\mathcal{X}^2$ (from eq. 1), so we get a fiber square

$$\begin{array}{ccc} \mathcal{J} & \rightarrow & \mathcal{I}_\mathcal{X} \\ \downarrow & & \downarrow \\ Q & \rightarrow & \mathcal{I}_\mathcal{X}/\mathcal{I}_\mathcal{X}^2 \end{array}$$

Notice that

$$\begin{aligned} \mathcal{M} &= (\mathcal{I}_\mathcal{X}/\mathcal{I}_\mathcal{X}^2)/Q \\ &= \mathcal{I}_\mathcal{X}/\mathcal{J} \end{aligned} \tag{3}$$

The second filtration is:

$$\mathcal{O} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \cdots.$$

$\mathcal{J}^k$ corresponds to $\{x^m y^n \mid n \geq k\}$. Diagramatically,

This allows us to define J^S in general. For example, to hunt for tacnodes, we must examine subschemes with basis $\{1, x, y, x^2, xy, x^3\}$, represented diagramatically by:

We think about what needs to be "cut out" from the J^∞ diagram:

to get: $\mathcal{I}_S = (\mathcal{J}^2, \mathcal{J} \cap \mathcal{I}_\mathcal{X}^3, \mathcal{I}_\mathcal{X}^4)$ ($= (\mathcal{J}^2, \mathcal{J} \cdot \mathcal{I}_\mathcal{X}^2, \mathcal{I}_\mathcal{X}^4)$ using the following lemma.)

[Do I need to explain the construction of $\mathcal{I}_S$ more explicitly? How can I better describe the construction of $\mathcal{J}$?]

Lemma 1 $\mathcal{I}_\mathcal{X}^m \cdot \mathcal{J}^n = \mathcal{I}_\mathcal{X}^{m+n} \cap \mathcal{J}^n$

Proof. Not difficult. Essentially, this is the same as saying that $(x,y)^m y^n = (x,y)^{m+n} \cap y^n$ in $k[x,y]$. Ξ

Lemma 2

$$\mathcal{M} \otimes \frac{\mathcal{I}_{\mathcal{X}}^t}{\mathcal{I}_{\mathcal{X}}^t \cap \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+1}} = \frac{\mathcal{I}_{\mathcal{X}}^{t+1}}{\mathcal{I}_{\mathcal{X}}^{t+1} \cap \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+2}}$$

Proof. By the previous lemma, the result is equivalent to:

$$\mathcal{M} \otimes \frac{\mathcal{I}_{\mathcal{X}}^t}{\mathcal{I}_{\mathcal{X}}^{t-s} \cdot \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+1}} = \frac{\mathcal{I}_{\mathcal{X}}^{t+1}}{\mathcal{I}_{\mathcal{X}}^{t+1-s} \cdot \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+2}} \tag{4}$$

Twisting the exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{X}}^{t-s} \cdot \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+1} \longrightarrow \mathcal{I}_{\mathcal{X}}^t \longrightarrow \frac{\mathcal{I}_{\mathcal{X}}^t}{\mathcal{I}_{\mathcal{X}}^{t-s} \cdot \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+1}} \longrightarrow 0$$

by $\mathcal{M} = \mathcal{I}_{\mathcal{X}}/\mathcal{J}$ (see eq. 3), we get:

$$0 \longrightarrow \frac{\mathcal{I}_{\mathcal{X}}^{t+1-s} \cdot \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+2}}{\mathcal{I}_{\mathcal{X}}^{t-s} \cdot \mathcal{J}^{s+1} + \mathcal{I}_{\mathcal{X}}^{t+1} J} \xrightarrow{\alpha} \frac{\mathcal{I}_{\mathcal{X}}^{t+1}}{\mathcal{I}_{\mathcal{X}}^t \mathcal{J}} \longrightarrow \mathcal{M} \otimes \frac{\mathcal{I}_{\mathcal{X}}^t}{\mathcal{I}_{\mathcal{X}}^{t-s} \cdot \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+1}} \longrightarrow 0$$

The cokernel of α is the cokernel of

$$\mathcal{I}_{\mathcal{X}}^{t+1-s} \cdot \mathcal{J}^s + \mathcal{I}_{\mathcal{X}}^{t+2} \longrightarrow \mathcal{I}_{\mathcal{X}}^{t+1}$$

which is what we sought to prove in eq. 4. Ξ

Theorem 3 *The following sequence is exact:*

$$0 \to \mathcal{M}^{n-t(n)} \otimes Sym^{t(n)}\Omega \to J^{S_{\leq n}}\mathcal{O} \to J^{S_{<n}}\mathcal{O} \to 0.$$

Proof.

$$\mathcal{I}_{S_{\leq n}} = (\ldots, \mathcal{J}^{t(n)+1} \cap \mathcal{I}_{\mathcal{X}}^n, \mathcal{I}_{\mathcal{X}}^{n+1})$$

$$\mathcal{I}_{S_{<n}} = (\mathcal{I}_{S_{\leq n}}, \mathcal{I}_{\mathcal{X}}^n)$$

$$0 \longrightarrow \Pi_{1*}(\mathcal{I}_{S_{<n}}/\mathcal{I}_{S_{\leq n}}) \longrightarrow \Pi_{1*}(\mathcal{O}_{\mathbf{P}_{T\times Y}}/\mathcal{I}_{S_{\leq n}}) \longrightarrow \Pi_{1*}(\mathcal{O}_{\mathbf{P}_{T\times Y}}/\mathcal{I}_{S_{<n}}) \longrightarrow 0$$

is exact, so

$$0 \longrightarrow \Pi_{1*}\frac{\mathcal{I}_{\mathcal{X}}^n}{\mathcal{J}^{t(n)+1} \cap \mathcal{I}_{\mathcal{X}}^n + \mathcal{I}_{\mathcal{X}}^{n+1}} \longrightarrow J^{S_{\leq n}}\mathcal{O} \longrightarrow J^{S_{<n}}\mathcal{O} \longrightarrow 0$$

is exact. But

$$\Pi_{1*}\frac{\mathcal{I}_{\mathcal{X}}^n}{\mathcal{J}^{t(n)+1} \cap \mathcal{I}_{\mathcal{X}}^n + \mathcal{I}_{\mathcal{X}}^{n+1}} = \mathcal{M}^{n-t(n)} \otimes Sym^{t(n)}\Omega$$

by the previous lemma (used repeatedly). Ξ

(C) will follow by a similar argument (cf. Theorem 1). (A) follows from (C). (B) follows from $\mathcal{I}_{\mathcal{X}}^{n+1} = \phi^*\mathcal{I}_{\Delta}^{n+1}$. (D) will likely follow from an argument similar to the proof of Theorem 2.

3.3 Further Discussion

1. *Another (Perhaps Better) Description of J^∞.* $\mathbf{P}\Omega = \text{Proj } R$ where R is a sheaf of graded algebras ($R = \bigoplus \mathcal{I}_\Delta^n/\mathcal{I}_\Delta^{n+1}$). Consider Spec R with the projection $\sigma : \text{Spec } R \to Y$. Then

$$\sigma_* \mathcal{O}_{\text{Spec } R} = J^\infty,$$

and the $\mathcal{I}_S$ are ideals in this sheaf of algebras.

2. Once again, finding a good filtration allows us to dodge G-R-R.

3. For Y of dimension greater than 2, instead of just splitting off a line bundle from Ω, we can split Ω completely. Instead of $\mathbf{P}T$, consider Fl T, a flag bundle over Y. Not co-incidentally, this construction is exactly the one used to prove the splitting principle (see [6] p. 270 or [1]).

4. As before, we can construct $\phi^* J_\infty$. We now have a double filtration of both $\phi^* J^\infty$ and $\phi^* J_\infty$. For Y of dimension greater than 2, we can get a multiple filtration, which should make fans of mixed Hodge structures jump for joy.

5. If we have a sheaf of ideals $\mathcal{I}$ in $\mathcal{J}^\infty$, $\mathcal{M} \otimes \mathcal{I}$ is $x\mathcal{I}$, so we can loosely think of $\mathcal{M}$ as "multiplication" or "twisting" by x.

4 Examples and Exercises

1. In a general pencil of degree d curves on $\mathbf{P}^2$, how many hyperflexes will appear?

2. In a general pencil of degree d hypersurfaces in $\mathbf{P}^n$, how many are singular?

3. In a general web of degree d curves in $\mathbf{P}^2$, how many tacnodes will appear?

4. Let $\mathbf{P}^N$ be the parameter space of degree d curves in $\mathbf{P}^2$. What is the degree of the (codimension 6) locus of curves with triple points?

5. Given a line bundle L on a surface (with known K and $c_2(T)$) and four general sections, how many nodes would you expect to see in a general pencil of sections? How many cusps in a general net? How many tacnodes in a general web?

6. Show that a general cubic surface in $\mathbf{P}^3$ has 27 lines.

7. In a general pencil of cubic surfaces in $\mathbf{P}^3$, show that three lines will come together at a single point 100 times. (Strictly speaking, a slightly different construction is necessary here. But the same formalism works.)

8. Prove part of the classical Plücker formulas for curves (see [2] p. 288):

   Let C be an irreducible curve of degree d in $\mathbf{P}^2$ with only δ nodes and κ cusps as singularities, and such that its dual curve is of degree d^*. Show that

$$d^* = d(d-1) - 2\delta - 3\kappa.$$

   (The rest of the Plücker formulas can also be proved in this fashion.)

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References

[1] Fulton, Intersection Theory.

[2] Griffiths and Harris, Principles of Algebraic Geometry.

[3] Hartshorne, Algebraic Geometry.

[4] Matsumura, Commutative Algebra or Commutative Ring Theory.

[5] D.J. Saunders, The Geometry of Jet Bundles, Cambridge University Press, 1989, Cambridge (LMS 142). Although this book is intended as an introduction 'for the reader who is in mathematicial physics, and who has a knowledge of modern differential geometry,' the book looks quite readable.

[6] Shafarevich, Algebraic Geometry I.