THE MODULI SPACE OF n POINTS ON THE LINE IS CUT OUT BY SIMPLE QUADRICS WHEN n IS NOT SIX

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ABSTRACT. A central question in invariant theory is that of determining the relations among invariants. Geometric invariant theory quotients come with a natural ample line bundle, and hence often a natural projective embedding. This question translates to determining the equations of the moduli space under this embedding. This note deals with one of the most classical quotients, the space of ordered points on the projective line. We show that under any linearization, this quotient is cut out (scheme-theoretically) by a particularly simple set of quadric relations, with the single exception of the Segre cubic threefold (the space of six points with equal weight). Unlike many facts in geometric invariant theory, these results (at least for the stable locus) are field-independent, and indeed work over the integers.

CONTENTS

1. INTRODUCTION

We consider the space of n (ordered) points on the projective line, up to automorphisms of the line. In characteristic 0, the best description of this is as the Geometric Invariant Theory quotient $(\mathbb{P}^1)^n//SL(2)$. This is one of the most classical examples of a GIT quotient, and is one of the first examples given in any course (see [MS, §2], [MFK, §3], [N, §4.5], [D, Ch. 11], [DO, Ch. I], ...). We find it remarkable that well over a century after this fundamental invariant theory problem first arose, we have so little understanding of its defining equations, when they should be expected to be quite simple.

The construction of the quotient depends on a linearization, in the form of a choice of weights of the points $\mathbf{w} = (w_1, \ldots, w_n)$ (the *weight vector*). We denote the resulting

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quotient $M_{\mathbf{w}}$. A particularly interesting case is when all points are treated equally, when $\mathbf{w} = (1, \dots, 1) = 1^n$. We call this the *equilateral* case because of its interpretation as the moduli space of equilateral space polygons in symplectic geometry (e.g. [HMSV, §2.3]). At risk of confusion, we denote this important case M_{1^n} by M_n for simplicity. We say that n points of $\mathbb{P}^1$ are $\mathbf{w}$-*stable* (resp. $\mathbf{w}$-*semistable*) if the sum of the weights of points that coincide is less than (resp. no more than) half the total weight. The dependence on $\mathbf{w}$ will be clear from the context, so the prefix $\mathbf{w}$- will be omitted. The n points are *strictly semistable* if they are semistable but not stable. Then $M_{\mathbf{w}}$ is a projective variety, and GIT gives a natural projective embedding. The stable locus of $M_{\mathbf{w}}$ is a fine moduli space for the stable points of $(\mathbb{P}^1)^n$. The strictly semistable locus of $M_{\mathbf{w}}$ is a finite set of points, which are the only singular points of $M_{\mathbf{w}}$. The question we wish to address is: what are the equations of $M_{\mathbf{w}}$?

We prefer to work as generally as possible, over the integers, so we now define the moduli problem of stable n-tuples of points in $\mathbb{P}^1$. For any scheme B, *families of stable n-tuples of points in $\mathbb{P}^1$ over B* are defined to be n morphisms to $\mathbb{P}^1$ such that no more than half the weight is concentrated at a single point of $\mathbb{P}^1$. More precisely, a family is a morphism $(\phi_1, \dots, \phi_n) : B \times \{1, \dots, n\} \to \mathbb{P}^1$ such that for any $I \subset \{1, \dots, n\}$ such that $\sum_{i \in I} w_i \geq \sum_{i=1}^n w_i/2$, we have $\cap_{i \in I} \phi_i^{-1}(p) = \emptyset$ for all p. Then there is a fine moduli space for this moduli problem, quasiprojective over $\mathbb{Z}$, which indeed has a natural ample line bundle, the one suggested by GIT. (This is well-known, but in any case will fall out of our analysis.) In parallel with GIT, we define stable points. Hence this space has extrinsic projective geometry. The question we will address in this context is: what are its equations?

The main moral of this note is taken from Chevalley's construction of Chevalley groups: when understanding a vector space defined geometrically, choosing a basis may obscure its structure. Instead, it is better to work with an equivariant generating set, and equivariant linear relations. A prototypical example is the standard representation of $\mathfrak{S}_n$, which is best understood as the permutation representation on the vector space generated by $e_1, \dots, e_n$ subject to the relation $e_1 + \cdots + e_n = 0$. As an example, we give yet another short proof of Kempe's theorem (Theorem 2.3), which has been called the "deepest result" of classical invariant theory [Ho, p. 156] (in the sense that it is the only result that Howe could not prove from standard constructions in representation theory). Another application is the computation in §2.11 of the degree of all $M_{\mathbf{w}}$.

We now state our main theorem. We will describe a natural (equivariant) set of "graphical" generators of the algebra of invariants (in Section 2). The algebraic structure of the invariants is particularly transparent in this language, and as an example we give a short proof of Kempe's Theorem 2.3, and give an easy basis of the group of invariants (by "non-crossing variables", Theorem 2.5). We then describe some geometrically (or combinatorially) obvious relations, the (linear) *sign relations*, the (linear) *Plücker relations*, and the (quadratic) *simple binomial relations*.

1.1. *Main Theorem. — With the single exception of* $\mathbf{w} = (1, 1, 1, 1, 1, 1)$, *the following holds.*

(a) *Over a field of characteristic* 0*, the GIT quotient* $(\mathbb{P}^1)^n//SL_2$ *(with its natural projective embedding) is cut out scheme-theoretically by the sign, Plücker, and simple binomial relations.*
(b) *Over* $\mathbb{Z}$ *(or any base scheme), the fine moduli space of stable* n*-tuples of points on* $\mathbb{P}^1$ *is quasiprojective over* $\mathbb{Z}$*. Under its natural embedding, its closure is cut out by the sign, Plücker, and simple binomial relations.*

The exceptional case $\mathbf{w} = (1,1,1,1,1,1)$ is the Segre cubic threefold.

Note that Geometric Invariant Theory does not apply to $SL(2)$-quotients in positive characteristic, as $SL(2)$ is not a reductive group in that case. Thus for (b) we must construct the moduli space by other means.

The idea of the proof is as follows. We first reduce the question to the equilateral case, where n is even. We do this by showing a stronger result, which reduces such questions about the *ideal of relations* of invariants to the equilateral case.

1.2. *Theorem (reduction to equilateral case, informal statement). — For any weight* $\mathbf{w}$*, there is a natural map from the graded ring of projective invariants for* $1^{|\mathbf{w}|}$ *to those for* $\mathbf{w}$*. Under this map, each of our generators for* $1^{|\mathbf{w}|}$ *is sent to either a generator for* $\mathbf{w}$*, or to zero. Moreover, the relations for* $1^{|\mathbf{w}|}$ *generate the relations for* $\mathbf{w}$*.*

In Section 2.12, we will state this result precisely, and prove it, once we have introduced some terminology. This is a stronger result than we will need, as it refers to the full ring of projective invariants, not just to the quotient variety. It is also a stronger statement than simply saying that $M_{\mathbf{w}}$ is naturally a linear section of $M_{|\mathbf{w}|}$.

We then verify Theorem 1.1 "by hand" in the cases $n = 2m \leq 8$ (§2.14). The cases $n = 6$ and $n = 8$ are the base cases for our later argument (which is ironic, as in the $n = 6$ case the result does not hold!). In §4, we show that the result holds set-theoretically, and that the projective variety is a fine moduli space away from the strictly semistable points. The strictly semistable points are more delicate, as the quotient is not naturally a fine moduli space there; we instead give an explicit description of a neighborhood of a strictly semistable point, as the affine variety corresponding to rank one $(n/2-1) \times (n/2-1)$ matrices with entries distinct from 1, using the Gel'fand-MacPherson correspondence. We prove the result in this neighborhood in §3.

1.3. Related questions. Our question is related to another central problem in invariant theory: the invariants of binary forms, or equivalently n unordered points on $\mathbb{P}^1$, or equivalently equations for $M_n/\mathfrak{S}_n$. These generators and relations are more difficult, and the relations are certainly not just quadratic. Mumford describes Shioda's solution for $n = 8$ [Sh] as "an extraordinary tour de force" [MFK, p. 77]. One might dream that the case $n = 10$ might be tractable by computer, given the explicit relations for M_{10} described here.

We remark on the relation of this paper to our previous paper [HMSV]. That paper dealt with the *ideal* of relations among the invariants, and a central result was that this ideal was cut out by equations of degree at most 4. The present article deals with the moduli space as a projective variety; it is not at all clear that our quadrics generate the ideal of invariants.

Question. Do the simple binomial relations generate the ideal of relations among invariants (if $n \neq 6$)?

By Theorem 1.2, it suffices to consider the equilateral case. [HMSV] suggests one approach: one could hope to show that the explicit generators given there lie in the ideal given by these simple binomial quadrics.

Even special cases are striking, and are simple to state but computationally too complex to verify even by computer. For example, we will describe particularly attractive relations for M_{10} of degree degrees 3, 5, 7 (§2.10); do these lie in the ideal of our simple quadrics? We will describe a quadric relation for M_{12} (§2.6). Is this a linear combination of our simple binomial quadrics?

These questions lead to more speculation. The shape of the proof of the main theorem suggests an explanation for the existence of an exception: when the number of points gets "large enough", the relations are all inherited from "smaller" moduli spaces, so at some point (in our case, when $n = 8$) the relations "stabilize".

Speculative question. We know that M_n satisfies Green's property N_0 (projectively normality — Kempe's Theorem 2.3), and the questions above suggest that M_n might satisfy property N_1 (projectively normal and cut out by quadrics) for $n > 6$. Might it be true that each p, M_n satisfies N_p for $p \gg 0$? (Property N_2 means that the scheme satisfies N_1, and the syzygies among the quadrics are linear. These properties measure the "niceness" of the ideal of relations.)

Speculative question. Might similar results hold true for other moduli spaces of a similar flavor, e.g. $\overline{\mathcal{M}}_{0,n}$ or even $\overline{\mathcal{M}}_{g,n}$? Are equations for moduli spaces inherited from equations of smaller moduli spaces for $n \gg 0$? For further motivation, see the striking work of [KT] on equations cutting out $\overline{\mathcal{M}}_{0,n}$. Also, an earlier example of quadratic equations inherited from smaller moduli spaces appeared in [BP], in the case of Cox rings of del Pezzo surfaces. I thank B. Hassett for pointing out this reference.

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2. The invariants of n points on $\mathbb{P}^1$ as a graphical algebra

We give a convenient alternate description of the generators (as a group) of the ring of invariants of n ordered points on $\mathbb{P}^1$. By *graph* we will mean a *directed* graph on n vertices labeled 1 through n. Graphs may have multiple edges, but may have no loops. The *multidegree* of a graph Γ is the n-tuple of valences of the graph, denoted $\mathbf{deg}\,\Gamma$. The bold font is a reminder that this is a vector. We consider each graph as a set of edges. For each edge e of Γ, let $h(e)$ be the head vertex of e and $t(e)$ be the tail. We use multiplicative

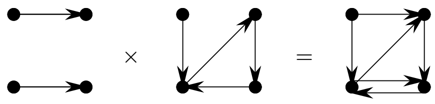

FIGURE 1. Multiplying (directed) graphs

notation for the "union" of two graphs: if Γ and Δ are two graphs on the same set of vertices, the union is denoted by $\Gamma \cdot \Delta$ (so for example $\mathbf{deg}\,\Gamma + \mathbf{deg}\,\Delta = \mathbf{deg}\,\Gamma \cdot \Delta$), see Figure 1. (We will occasionally use additive and subtractive notation when we wish to "subtract" graphs. We apologize for this awkwardness.)

For each graph Γ, define $X_\Gamma \in H^0((\mathbb{P}^1)^n, \mathcal{O}_{(\mathbb{P}^1)^n}(\mathbf{deg}\,\Gamma))$ by

$$X_\Gamma = \prod_{\text{edge } e \text{ of } \Gamma} (p_{h(e)} - p_{t(e)}) = \prod_{\text{edge } e \text{ of } \Gamma} (u_{h(e)}v_{t(e)} - u_{t(e)}v_{h(e)}). \tag{1}$$

If S is a non-empty set of graphs of the same degree, $[X_\Gamma]_{\Gamma \in S}$ denotes a point in projective space $\mathbb{P}^{|S|-1}$ (assuming some such X_Γ is nonzero of course). For any such S, the map $(\mathbb{P}^1)^n \dashrightarrow [X_\Gamma]_{\Gamma \in S}$ is easily seen to be invariant under $SL(2)$: replacing p_i by $p_i + a$ preserves X_Γ; replacing p_i by ap_i changes each X_Γ by the same factor; and replacing p_i by $1/p_i$ also changes each X_Γ by the same factor.

The First Fundamental Theorem of Invariant Theory [D, Thm. 2.1] states that, given a weight $\mathbf{w}$, the ring of invariants of $(\mathbb{P}^1)^n//SL_2$ is generated (as a group) by the X_Γ where $\mathbf{deg}\,\Gamma$ is a multiple of $\mathbf{w}$. The translation to the tableaux is as follows. Choose any ordering of the edges $e_1, \ldots, e_{|\Gamma|}$ of Γ. Then X_Γ corresponds to any $2 \times |\Gamma|$ tableau where the top row of the ith column is $h(e_i)$ and the bottom row is $t(e_i)$. We will soon see advantages of this graphical description as compared to the tableaux description.

We now describe several types of relations among the X_Γ, which will all be straightforward: the sign relations, the Plücker (or straightening) relations, the simple binomial relations, and the Segre cubic relation.

2.1. *The sign (linear) relations.* The *sign relation* $X_{\Gamma \cdot \vec{xy}} = -X_{\Gamma \cdot \vec{yx}}$ (Figure 2) is immediate, given the definition (1). Because of the sign relation, we may omit arrowheads in identities where it is clear how to consistently insert them (see for example Figures 7 and 9, where even the vertices are implicit).

2.2. *The Plücker (linear) relations.* The identity of Figure 3 may be verified by direct calculation. If Γ is any graph on n vertices, and Δ_1, Δ_2, Δ_3 are three graphs on the same vertices given by identifying the four vertices of Figure 3 with the some four of the n vertices of Γ, then

$$X_{\Gamma \cdot \Delta_1} + X_{\Gamma \cdot \Delta_2} + X_{\Gamma \cdot \Delta_3} = 0. \tag{2}$$

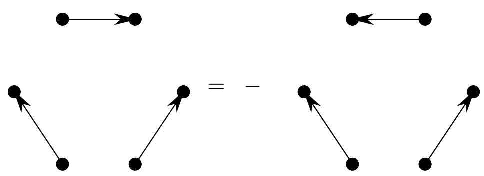

FIGURE 2. An example of the sign relation

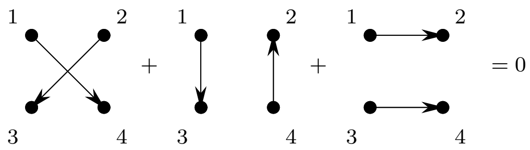

FIGURE 3. The Plücker relation for $n = 4$ (and $\mathbf{w} = (1, 1, 1, 1)$)

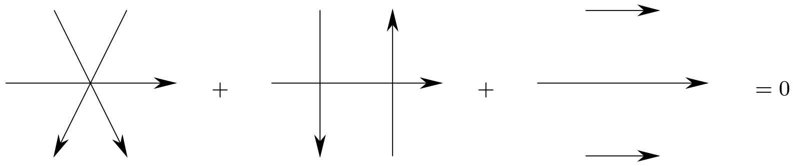

FIGURE 4. An example of a Plücker relation

These relations are called *Plücker relations* (or *straightening rules*). See Figure 4 for an example. We will sometimes refer to this relation as the Plücker relation for $\Gamma \cdot \Delta_1$ with respect to the vertices of Δ_1.

Using the Plücker relations, one can reduce the number of generators to a smaller set, which we will do shortly (Proposition 2.4). However, a central thesis of this article is that this is the wrong thing to do too soon; not only does it obscure the $\mathfrak{S}_n$ symmetry of this generating set, it also makes certain facts opaque. As an example, we give a new proof of Kempe's theorem. The proof will also serve as preparation for the proof of the main theorem, Theorem 1.1.

2.3. *Kempe's Theorem* [HMSV, Thm. 4.6]. — *The lowest degree invariants generate the ring of invariants.*

Note that the lowest-degree invariants are of weight $\epsilon_{\mathbf{w}}\mathbf{w}$, where $\epsilon_{\mathbf{w}} = 1$ if $|\mathbf{w}|$ is even, and 2 if $|\mathbf{w}|$ is odd.

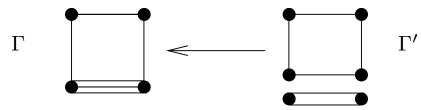

FIGURE 5. Constructing Γ' from Γ (example with $\mathbf{w} = (1,1,2,2)$, $d = 2$)

Proof. We begin in the case when $\mathbf{w} = (1, \dots, 1)$ where n is even. Recall Hall's Marriage Theorem: given a finite set of men M and women W, and some men and women are compatible (a subset of $M \times W$), and it is desired to pair the women and men compatibly, then it is necessary and sufficient that for each subset S of women, the number of men compatible with at least one of them is at least $|S|$.

Given a graph Γ of multidegree $(d, \dots, d)$, we show that we can find an expression $\Gamma = \sum \pm \Delta_i \cdot \Xi_i$ where $\mathbf{deg}\, \Delta_i = (1, \dots, 1)$. Divide the vertices into two equal-sized sets, one called the "positive" vertices and one called the "negative" vertices. This creates three types of edges: positive edges (both vertices positive), negative edges (both vertices negative), and neutral edges (one vertex of each sort). When one applies the Plücker relation to a positive edge and a negative edge, all resulting edges are neutral (see Figure 3, and take two of the vertices to be of each type). Also, each regular graph must have the same number of positive and negative edges. Working inductively on the number of positive edges, we can use the Plücker relations so that all resulting graphs have only neutral edges. We thus have an expression $\Gamma = \sum \pm \Gamma_i$ where each Γ_i has only neutral edges and is hence a bipartite graph. Each vertex of Γ_i has the same valence d, so any set of p positive vertices must connect to at least p negative edges. By Hall's Marriage Theorem, we can find a matching Δ_i that is a subgraph of Γ_i, with "residual graph" Ξ_i (i.e. $\Gamma_i = \Delta_i \cdot \Xi_i$). Thus the result holds in the equilateral case.

We next treat the general case. If $|\mathbf{w}|$ is odd, it suffices to consider the case $2\mathbf{w}$, so by replacing $\mathbf{w}$ by $2\mathbf{w}$ if necessary, we may assume $\epsilon_{\mathbf{w}} = 1$. The key idea is that $M_{\mathbf{w}}$ is a linear section of $M_{|\mathbf{w}|}$. Suppose $\deg \Gamma = d\mathbf{w}$. Construct an auxiliary graph Γ' on $|\mathbf{w}|$ vertices, and a map of graphs $\pi : \Gamma' \to \Gamma$ such that (i) the preimage of vertex i of Γ consists of w_i vertices of Γ', (ii) π gives a bijection of edges, and (iii) each vertex of Γ' has valence d, i.e. Γ' is d-regular. (See Figure 5 for an illustrative example. There may be choice in defining Γ'). Then apply the algorithm of the previous paragraph to Γ'. By taking the image under π, we have our desired result for Γ. $\square$

Choosing a planar representation makes termination of certain algorithms straightforward as well, as illustrated by the following argument. Consider the vertices of the graph to be the vertices of a regular n-gon, numbered (clockwise) 1 through n. A graph is said to be *non-crossing* if no two edges cross. Two edges sharing one or two vertices are considered not to cross. A variable X_Γ is said to be non-crossing if Γ is.

2.4. *Proposition (graphical version of "straightening algorithm"). — For each* $\mathbf{w}$*, the non-crossing variables of degree* $\mathbf{w}$ *generate* $\langle X_\Gamma \rangle_{\mathbf{deg}\,\Gamma = \mathbf{w}}$ *(as an abelian group).*

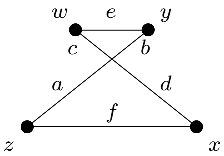

FIGURE 6. The triangle inequality implies termination of straightening: $b + c > e$, $a + d > f$

This is essentially the straightening algorithm (e.g. [D, §2.4]) in this situation.

Proof. We explain how to express X_Γ in terms of non-crossing variables. If Γ has a crossing, choose one crossing $wx \cdot yz$ (say $\Gamma = wx \cdot yz \cdot \Gamma'$), and use the Plücker relation (2) involving $wxyz$ to express Γ in terms of two other graphs $wy \cdot xz \cdot \Gamma'$ and $wz \cdot xy \cdot \Gamma'$. Repeat this if possible. We now show that this process terminates, i.e. that this algorithm will express X_Γ in terms of non-crossing variables. Both of these graphs have lower sum of edge-lengths than Γ (see Figure 6, using the triangle inequality on the two triangles with side lengths a, d, f and b, c, e). As there are finite number of graphs of weight $\mathbf{w}$, and hence a finite number of possible sums of edge-lengths, the process must terminate. $\square$

2.5. *Theorem (non-crossing basis of invariants). — For each* $\mathbf{w}$*, the non-crossing variables of degree* $\mathbf{w}$ *form a basis for* $\langle X_\Gamma \rangle_{\mathbf{deg}\,\Gamma = \mathbf{w}}$.

Proof. Proposition 2.4 shows that the non-crossing variables span, so it remains to show that they are linearly independent. Assume otherwise that they are not always linearly independent, and thus there s a simplest nontrivial relation R for some smallest $\mathbf{w}$ (where the $\mathbf{w}$ are partially ordered by $|\mathbf{w}|$ and $\#\mathbf{w}$). The relation R states that some linear combination of these graphs is 0. Suppose this relation involves $\#\mathbf{w} = n$ vertices. Then not all of the graphs in R contain edge $(n-1)n$ (or else we could remove one copy of the edge from each term in R and get a smaller relation).

Then identify vertices $(n-1)$ and n, throwing out the graphs containing edge $(n-1)n$. This "contraction" gives a natural bijection between non-crossing graphs on n vertices with multidegree $\mathbf{w}$, not containing edge $(n-1)n$, and non-crossing graphs on $n-1$ vertices with multidegree $\mathbf{w}'$ (where $w'_{n-1} = w_{n-1} + w_n$). The resulting relation still holds true (in terms of invariants, we insert the relation $p_{n-1} = p_n$ into the older relation). We have contradicted the minimality of R, so no such counterexample R exists. $\square$

2.6. *Binomial (quadratic) relations.* We next describe some obvious binomial relations. If $\mathbf{deg}\,\Gamma_1 = \mathbf{deg}\,\Gamma_2$ and $\mathbf{deg}\,\Delta_1 = \mathbf{deg}\,\Delta_2$, then clearly $X_{\Gamma_1 \cdot \Delta_1} X_{\Gamma_2 \cdot \Delta_2} = X_{\Gamma_1 \cdot \Delta_2} X_{\Gamma_2 \cdot \Delta_1}$. We call these the *binomial relations*. A special case are the *simple binomial relations* when $\mathbf{deg}\,\Delta_i = (1,1,1,1,0,\dots,0) = 1^4 0^{n-4}$, or some permutation thereof. Examples are shown in Figures 7 and 9.

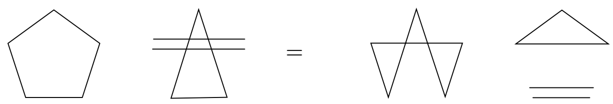

FIGURE 7. A simple binomial relation for $n = 5$

FIGURE 8. The building blocks of Figure 7

(We have now defined all the relations relevant to the Main Theorem 1.1, so the reader is encouraged to reread its statement.)

In the even democratic case, the smallest quadratic binomial relations that are not simple binomial relations appear for $n = 12$, and

$$\mathbf{deg}\,\Gamma_i = \mathbf{deg}\,\Delta_i = (1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0).$$

In the introduction, we asked if these quadratics are linear combinations of the simple binomial relations.

By the Plücker relations, the binomial relations are generated by those where the Γ_i and Δ_i are non-crossing, and similarly for the simple binomial relations. This restriction can be useful to reduce the number of equations, but as always, symmetry-breaking obscures other algebraic structures. (Note that even though we may restrict to the case where Γ_i and Δ_j are non-crossing, we may not restrict to the case where $\Gamma_i \cdot \Delta_j$ are non-crossing, as the following examples with $n = 5$ and $n = 8$ show.)

As an example, consider the case $n = 5$ (with the smallest democratic linearization $(2, 2, 2, 2, 2)$). One of the simple binomial relations is shown in Figure 7. The building blocks Γ_i and Δ_j are shown Figure 8. In fact, these quadric relations cut out M_5 in $\mathbb{P}^5$, as can be checked directly, or as follows from Theorem 1.1. The $\mathfrak{S}_5$-representation on the quadrics is visible.

2.7. As a second example, consider $n = 8$ (and the democratic linearization $(1, \ldots, 1)$). Because there are $\binom{8}{4}/2 = 35$ ways of partitioning the 8 vertices into two subsets of size 4, and each such partition gives one simple binomial relation (where the Γ_i and Δ_j are non-crossing, see comments two paragraphs previous), we have 35 quadric relations on M_8, shown in Figure 9.

The space of quadric relations forms an irreducible 14-dimensional $\mathfrak{S}_8$-representation, which we show by representation theory. If V is the vector space of quadratic relations,

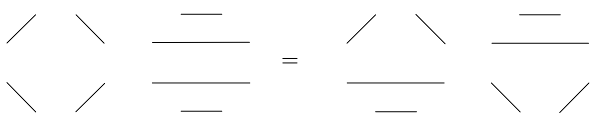

FIGURE 9. One of the simple binomial relations for $n = 8$ points

we have the exact sequence

$$0 \to V \to \operatorname{Sym}^2 H^0(M_8, \mathcal{O}(1)) \to H^0(M_8, \mathcal{O}(2)) \to 0$$

of representations. By counting non-crossing graphs, we can calculate $h^0(M_8, \mathcal{O}(2)) = 91$ (the 8th Riordan number or Motzkin sum) and $h^0(M_8, \mathcal{O}(1)) = C_4 = 14$ (the 4th Catalan number), from which $\dim V = 14$. As the representation $H^0(M_8, \mathcal{O}(1))$ is identified, we can calculate the representation $\operatorname{Sym}^2 H^0(M_8, \mathcal{O}(1))$, and observe that the only 14-dimensional subrepresentations it contains are irreducible. (Simpler still is to compute the character of the 196-dimensional representation $H^0(M_8, \mathcal{O}(1))^{\otimes 2}$, and decompose it into irreducible representations, using Maple for example, finding that it decomposes into representations of dimension $1 + 14 + 14 + 20 + 35 + 56 + 56$; $\operatorname{Sym}^2 H^0(M_8, \mathcal{O}(1))$ is of course a subrepresentation of this.)

As our quadric relations are nontrivial, and form an $\mathfrak{S}_8$-representation, we have given generators of the quadric relations. Necessarily they span the same vector space of the 14 relations given in [HMSV, §8.3.4]. Our relations have the advantage that the $\mathfrak{S}_8$-action is clear, but the major disadvantage that it is not a priori clear that the vector space they span has dimension 14. We suspect there is an $\mathfrak{S}_8$-equivariant description of the linear relations among the generators, but we have been unable to find one.

In [HMSV, §8.3.4] it was shown that the ideal of relations of M_8 was generated by these fourteen quadrics, and hence by our 35 simple binomial quadrics. We will use this as the base case of our induction later.

2.8. *The Segre cubic relation.* Other relations are also clear from this graphical perspective. For example, Figure 10 shows an obvious relation for M_6, which is well-known to be a cubic hypersurface (the Segre cubic hypersurface, see for example [DO, p. 17] or [D, Example 11.6]). As this is a nontrivial cubic relation (this can be verified by writing it in terms of a non-crossing basis), it must be the Segre cubic relation. Interestingly, although the relation is not $\mathfrak{S}_6$-invariant, it becomes so modulo the Plücker relations (2). Note that there are no (nontrivial) binomial relations for M_6 (which is cut out by this cubic), so the Segre relation cannot be in the ideal generated by the binomial relations.

2.9. *Remark: Segre cubic relations for $n \geq 8$.* There are analogous cubic relations for $n \geq 8$, by simply adding other vertices. The $n = 8$ case is given in Figure 11. For $n \geq 8$, these Segre cubic relations lie in the ideal generated by the simple binomial relations. We will use this in the proof of Theorem 1.1. This follows from the case $n = 8$, which can be verified in a couple of ways. As stated above, [HMSV] shows that the ideal cutting out

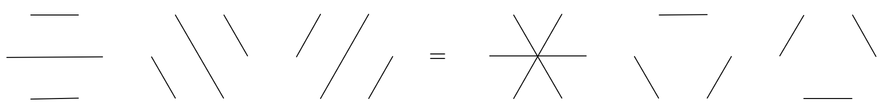

FIGURE 10. The Segre cubic relation (graphical version)

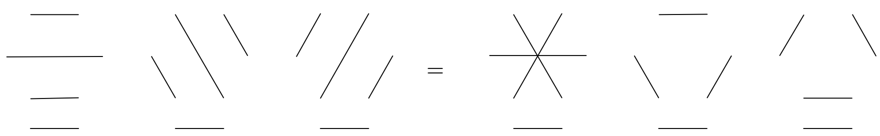

FIGURE 11. The Segre cubic relation for $n = 8$

M_8 is generated by the fourteen quadrics of [HMSV, §8.3.4], which by §2.7 is the ideal generated by the simple binomial relations, and the cubic lies in this ideal. One may also verify that the Segre relation lies in the ideal generated by the fourteen quadrics by explicit calculation (omitted here).

2.10. *Other relations.* There are other relations, that we will not discuss further. For example, consider the democratic case for n even. Then $\mathfrak{S}_n$ acts on the set of graphs. Choose any graph Γ. Then

$$\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) X^i_{\sigma(\Gamma)} = 0$$

is a relation for i odd and $1 < i < n-1$. Reason: substituting for X's in terms of p's (or more correctly the u's and v's) using (1) to obtain an expression E, and observing that $\mathfrak{S}_n$ acts oddly on E, we see that we must obtain a multiple of the Vandermonde, which has degree $(n-1, \dots, n-1) > \deg E$. Hence $E = 0$. It is not clear that this is a nontrivial relation, but it appears to be so in small cases. In particular, the case $n = 6$, $i = 3$ is the Segre cubic relation. In the introduction, we asked if these relations for $n = 10$ lie in the ideal generated by the simple binomial quadric relations.

2.11. Degree of the GIT quotient. As an application of these coordinates, we compute the degree of all $M_{\mathbf{w}}$. For example, we will use this to verify that the degree is 1 when $|\mathbf{w}| = 6$ and $\mathbf{w} \neq (1, \dots, 1)$, although this can also be done directly.

We would like to intersect the moduli space $M_{\mathbf{w}}$ with $n-3$ coordinate hyperplanes of the form $X_\Gamma = 0$ and count the number of points, but these hyperplanes will essentially never intersect properly. Instead, we note that each hyperplane $X_\Gamma = 0$ is reducible, and consists of a finite number of components of the form $M_{\mathbf{w}'}$ where the number of points $\#\mathbf{w}'$ is $n-1$. We can compute the multiplicity with which each of these components appears. The algorithm is then complete, given the base case $n = 4$. Here, more precisely, is the algorithm.

(a) (trivial case) If $n = 3$, the moduli space is a point, so the degree is 1.

(b) (base case) If $\mathbf{w} = (d, d, d, d)$, then $\deg M_{\mathbf{w}} = d$, as the moduli space is isomorphic to $\mathbb{P}^1$, embedded by the d-uple Veronese. (This may be seen by direct calculation, or by noting that a base-point-free subset of those variables of degree (d, d, d, d) are "dth powers" of variables of degree $(1, 1, 1, 1)$.)

(c) If $n > 4$ and $\mathbf{w}$ satisfies $w_j + w_k \leq \sum w_i/2$ for all j, k, we choose any Γ of weight $\mathbf{w}$. We can understand the components of $X_\Gamma = 0$ by considering the morphism $\pi : (\mathbb{P}^1)^n - U_{\mathbf{w}} \to [X_\Gamma]_{\deg \Gamma = \mathbf{w}}$, where $U_{\mathbf{w}}$ is the unstable locus. Directly from the formula for X_Γ, we see that for each pair of vertices j, k with an edge joining them, such that $w_j + w_k < \sum w_i/2$, there is a component that can be interpreted as $M_{\mathbf{w}'}$, where $\mathbf{w}'$ is the same as $\mathbf{w}$ except that w_j and w_k are removed, and $w_j + w_k$ is added (call this w_0 for convenience). We interpret this as removing vertices j and k, and replacing them with vertex 0. This component corresponds to the divisor

$$(u_j v_k - u_k v_j)^{m_{jk}} = 0 \tag{3}$$

on the source of π, where m_{jk} is the number of edges joining vertices j and k. If Δ is the reduced version of this divisor, $u_j v_k - u_k v_j = 0$, then the correspondence between between $\Delta \to M_{\mathbf{w}}$ and $(\mathbb{P}^1)^{n-1} - U_{\mathbf{w}'} \to M_{\mathbf{w}'}$ is as follows. For each Γ' of degree $\mathbf{w}'$, we lift $X_{\Gamma'}$ to any X_Γ where Γ is a graph on $\{1, \dots, n\}$ of degree $\mathbf{w}$ whose "image" in $\{1, \dots, n\} \cup \{0\} \setminus \{j, k\}$ is Γ'. (In other words, to w_j of the w_0 edges meeting vertex 0 in Γ', we associate edges meeting vertex j in Γ, and similarly with j replaced by k.) If Γ'' is any other lift, then $X_\Gamma = \pm X_{\Gamma''}$ on Δ, because using the Plücker relations, $X_\Gamma \pm X_{\Gamma''}$ can be expressed as a combination of variables containing edge jk, which all vanish on Δ.

From (3), the multiplicity with which this component appears is m_{ij}, the number of edges joining vertices j and k.

If $w_j + w_k = \sum w_i/2$, then $M_{\mathbf{w}'}$ is a strictly semistable point, and of dimension 0 smaller than $\dim M_{\mathbf{w}} - 1$, and hence is not a component. (Our base case is $n = 4$, not 3, for this reason.)

(d) If $n \geq 4$ and there are j and k such that $w_j + w_k > \sum w_i/2$, then the rational map $(\mathbb{P}^1)^4 \dashrightarrow M_{\mathbf{w}}$ has a base locus. Any graph X_Γ of degree $\mathbf{w}$ necessarily contains a copy of edge jk, so $(u_j v_k - u_k v_j)$ is a factor of any of the X_Γ. Hence $M_{\mathbf{w}}$ is naturally isomorphic to $M_{\mathbf{w} - e_j - e_k}$, so we replace $\mathbf{w}$ by $\mathbf{w} - e_j - e_k$, and repeat the process. Note that if $n = 4$, then the final resulting quadruple must be of the form (d, d, d, d).

For example, $\deg M_4 = 1$, $\deg M_6 = 3$, $\deg M_8 = 40$, and $\deg M_{10} = 1225$ were computed by hand. (This appears to be sequence A012250 on Sloane's *On-line encyclopedia of integer sequences* [Sl].) The calculations $\deg M_6 = 3$ and $\deg M_{2,2,2,2,2} = 5$ are shown in Figure 12. and 13 respectively. At each stage, $\mathbf{w}$ is shown, as well as the Γ used to calculate the next stage. In these examples, there is essentially only one such $\mathbf{w}'$ at each stage, but in general there will be many. The vertical arrows correspond to identifying components of X_Γ (step (c)). The first arrow in Figure 12 is labeled $\times 3$ to point out the reader that the next stage can be obtained in three ways. The degrees are obtained inductively from the bottom up.

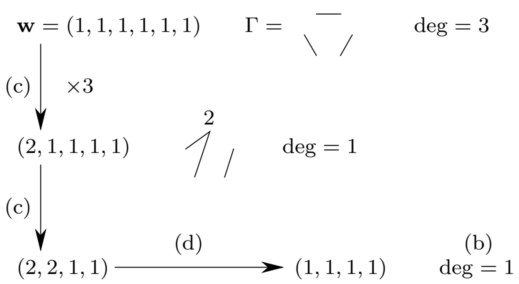

FIGURE 12. Computing $\deg M_6 = 3$ (recall that M_6 is the Segre cubic threefold $\mathcal{S}_3$)

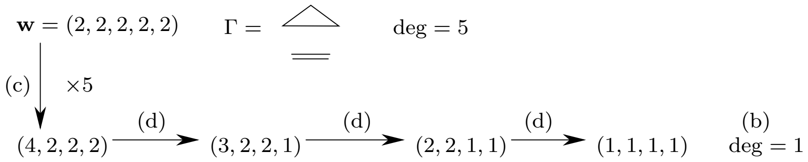

FIGURE 13. Computing $\deg M_{(2,2,2,2,2)} = 5$ using an inconvenient choice of Γ (recall that $M_{(2,2,2,2,2)}$ is a degree 5 del Pezzo surface)

(The reader is encouraged to show that $\deg M_8 = 40$, and that this algorithm indeed gives $\deg M_{d\mathbf{w}} = d^{n-3} \deg M_{\mathbf{w}}$.)

2.12. Reduction of the Main Theorem 1.1 to the equilateral case. We next show that the Main Theorem 1.1 in the equilateral case (when $\mathbf{w} = (1, \ldots, 1)$) implies the Main Theorem 1.1 in general. The argument is similar in spirit to our proof of Kempe's Theorem 2.3.

Consider the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & I_n \arrow[r] \arrow[d, "\alpha"] & \oplus_\Gamma \mathbb{Z} X_\Gamma \arrow[r, "\phi"] \arrow[d, "\beta"] & R_n \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & I_{\mathbf{w}} \arrow[r] & \oplus_\Omega \mathbb{Z} X_\Omega \arrow[r, "\psi"] & R_{\mathbf{w}} \arrow[r] & 0, \end{tikzcd}$$

where R_n (resp. $R_{\mathbf{w}}$) is the ring of invariants for 1^n (resp. $\mathbf{w}$), the Γ's range over matchings of n vertices, $n = |\mathbf{w}|$ is even, and the Ω's range over multi-valence $\mathbf{w}$ graphs. The map β takes X_Γ to X_Ω where Ω is given by identifying vertices of Γ within the same clump; if a loop is introduced then it maps to zero. (We take the clumps to be subsets of adjacent vertices in the n-gon.)

2.13. *Theorem. — The map $\alpha : I_n \to I_{\mathbf{w}}$ is surjective.*

Proof. It is clear that β is surjective (and hence γ too). Thus by the five lemma, it suffices to prove that $\ker \beta$ surjects onto $\ker \gamma$.

We have that $R_n = \oplus_G \mathbb{Z} \cdot \phi(X_G)$, where G ranges over regular non-crossing graphs on $\{1, \ldots, n\}$. (In other words, such $\phi(X_G)$ form a $\mathbb{Z}$-basis of R_n.) Similarly $R_{\mathbf{w}} = \oplus_H \mathbb{Z} \cdot \psi(X_H)$ as H ranges over non-crossing graphs of multi-valence a multiple of $\mathbf{w}$. For each H, there is exactly one non-crossing G for which $\gamma(\phi(X_G)) = \psi(X_H)$. Thus $\ker(\gamma) = \oplus_G \mathbb{Z} \cdot \phi(X_G)$ where the sum is over those non-crossing G which contain least one edge which connects two vertices in a single clump. Fix such a G, with an edge $a \to b$ where a and b are in the same clump. We will show that $X_G \in \ker \beta$.

Partition $\{1, \ldots, n\}$ into two equal sized subsets A and B ("positive" and "negative") such that $a \in A$, and $b \in B$. As in the proof of Kempe's theorem 2.3, we can write $X_G = \sum_i \pm X_{G_i}$, where the G_i are (possibly crossing) graphs each containing the edge $a \to b$. (The process described in the proof of Theorem 2.3 involves trading a pair of edges, one positive and one negative, for two neutral edges. No neutral edges such as $a \to b$ are affected by this process.)

By applying Hall's marriage theorem (repeatedly) to each G_i, we can write $X_{G_i} = \prod_{j=1}^k \phi(X_{\Gamma_{i,j}})$, where the $\Gamma_{i,j}$ are matchings, and $k = \deg(X_G)$. (At least) one $\Gamma_{i,j}$ contains the edge $a \to b$, so, $\beta(\prod_{j=1}^k X_{\Gamma_{i,j}}) = 0$. Hence $\beta(X_G) = \beta(\sum \pm X_{G_i}) = 0$ as desired. $\square$

2.14. Verification of the Main Theorem 1.1 in small cases. The cases $|\mathbf{w}| = 2$ and $|\mathbf{w}| = 4$ are trivial.

If $|\mathbf{w}| = 6$ and $\mathbf{w} \neq (1, \ldots, 1)$, then $\mathbf{w} = (3,2,1)$, $(2,2,2)$, $(2,2,1,1)$, or $(2,1,1,1,1)$. The first two cases are points, and the next two cases were verified to have degree 1 in §2.11 (see Figure 12).

The case $\mathbf{w} = (1,1,1,1,1,1,1,1)$ was verified in §2.7, so by §2.12, the case $|\mathbf{w}| = 8$ follows.

Thus the cases $|\mathbf{w}| \geq 10$ remain.

3. An analysis of a neighborhood of a strictly semistable point

We now show the result in a neighborhood of a strictly semistable point, in the equilateral case $w = 1^{n=2m}$, in characteristic 0, by explicitly describing an affine neighborhood of such a point. This affine neighborhood has a simple description: it is the space of $(m-1) \times (m-1)$ matrices of rank at most 1, where no entry is 1 (Lemma 3.3). The strictly semistable point corresponds to the zero matrix.

3.1. The Gel'fand-MacPherson correspondence: the moduli space as a quotient of the Grassmannian. We begin by recalling the Gel'fand-MacPherson correspondence, an alternate description of the moduli space. The Plücker embedding of the Grassmannian $G(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ is via the line bundle $\mathcal{O}(1)$ that is the positive generator for $\operatorname{Pic} G(2,n)$.

This generator may be described explicitly as follows. Over $G(2,n)$, we have a tautological exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow Q \longrightarrow 0 \tag{4}$$

where S is the tautological rank 2 subbundle (over $[\Lambda] \in G(2,n)$, it corresponds to $\Lambda \subset \mathbb{C}^n$), and Q is the tautological rank $n-2$ quotient bundle. Then $\wedge^2 S = \mathcal{O}(1)$ is a line bundle, and is the dual to $\mathcal{O}(1)$. Dualizing (4) we get a map $\wedge^2\mathcal{O}^{\oplus n} \to \wedge^2 S^*$. Then $\wedge^2 S^*$ can be easily checked to be be generated by these global sections. We call these sections s_{ij}, and note that they satisfy the following relations: the sign relations $s_{ij} = -s_{ji}$ (so $s_{ii} = 0$) inherited from $\wedge^2\mathcal{O}^{\oplus n}$, and the Plücker relations

$$s_{ij}s_{kl} - s_{ik}s_{jl} - s_{jk}s_{il} = 0.$$

These equations cut out the Grassmannian in $\mathbb{P}^{\binom{n}{2}-1}$.

The connection to n points on $\mathbb{P}^1$ is as follows. Given a general point of the Grassmannian corresponding to the subspace Λ of n-space, we obtain n points on $\mathbb{P}^1$, by considering the intersection of Λ with the n coordinate hyperplanes, and projectivizing. This breaks down if Λ is contained in a coordinate hyperplane. (The point $[\Lambda]$ is GIT-stable if the resulting n points in $\mathbb{P}^1$ are GIT-stable, and similarly for semistable. We recover the cross-ratio of four points via $s_{ij}s_{kl}/s_{il}s_{jk}$.)

Let $D(s_{1n})$ be the distinguished open set where $s_{1n} \neq 0$. (In the correspondence with marked points, this corresponds to the locus where the first point is distinct from the last point.) Then $D(s_{1n})$ is isomorphic to $\mathbb{A}^{2(n-2)}$, with good coordinates as follows. Given $\Lambda \notin D(s_{1n})$, choose a basis for Λ, written as a $2 \times n$ matrix. As $\Lambda \notin D(s_{1n})$, the first and last columns are linearly independent, so up to left-multiplication by $GL(2)$, there is a unique way to choose a basis where the first column is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the last column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We choose the "anti-identity" matrix rather than the identity matrix, because we will think of the first column as $[0;1] \in \mathbb{P}^1$, and the last column as $[1;0]$. (Another interpretation is as follows. If Λ is interpreted as a line in $\mathbb{P}^{n-1}$, and $H_1, \dots, H_n$ are the coordinate hyperplanes, then if Λ does not meet $H_1 \cap H_n$, then it meets H_1 at one point of $H_1 - H_1 \cap H_n \cong \mathbb{C}^{n-2}$ and H_n at one point of $H_n - H_1 \cap H_n \cong \mathbb{C}^{n-2}$, and Λ is determined by these two points. The coordinates on the first space are the x's, and the coordinates on the second are the y's.)

Thus if the $2 \times n$ matrix is written

$$\begin{bmatrix} 0 & x_2 & x_3 & \cdots & x_{n-1} & 1 \\ 1 & y_2 & y_3 & \cdots & y_{n-1} & 0 \end{bmatrix}$$

then we have coordinates $x_2, \dots, x_{n-1}, y_2, \dots, y_{n-1}$ on our affine chart. For convenience, we define $x_1 = 0$, $y_1 = 1$, $x_n = 1$, $y_n = 0$.

Under the trivialization $(\mathcal{O}(1), s_{1n})|_{D(s_{1n})} \cong (\mathcal{O}, 1)|_{D(s_{1n})}$, in these coordinates, the section s_{ij} may be interpreted as

$$s_{ij} = x_j y_i - x_i y_j.$$

We can use this to immediately verify the Plücker relations. We also recover the x_i and y_j from the sections via

$$x_i = s_{1i}/s_{1n} \qquad y_j = s_{jn}/s_{1n}. \tag{5}$$

The Grassmannian has dimension $2(n-2) = 2n-4$. To obtain our moduli space, we take the quotient of $G(2,n)$ by the maximal torus $T \subset SL(2,n)$, which has dimension $n-1$. (Thus as expected the quotient has dimension $n-3$.) We will write elements of this maximal torus as $\lambda = (\lambda_1, ..., \lambda_n)$. To describe the linearization, we must describe how λ acts on each s_{ij}: λ_i acts on s_{ij} with weight 1, and on the rest of the s_{ij}'s by weight 0. This action certainly preserves our relations.

Then we can see how to construct the quotient as a Proj: the terms that have weight $(d, d, \dots, d)$ correspond precisely to d-regular graphs on our n vertices. Hence we conclude that this projective scheme is precisely the GIT quotient of n points on the projective line, as the graded rings are the same. This is the Gel'fand-MacPherson correspondence. The relations we have described on our X_Γ clearly come from the relations on the Grassmannian. (That is of course no guarantee that we have them all!)

3.2. A neighborhood of a strictly semistable point. Let $\pi : G(2,2m)^{ss} \to M_{\mathbf{w}}$ be the quotient map. Let p be a strictly semistable point of the moduli space $M_{\mathbf{w}}$, without loss of generality the image of $(0, \dots, 0, \infty, \dots, \infty)$. We say an edge ij on vertices $\{1, \dots, 2m\}$ is *good* is $i \leq m < j$ (if it "doesn't connect two 0's or two ∞'s"). We say a graph on $\{1, \dots, 2m\}$ is *good* all of its edges are good. We say an edge or graph is *bad* if it is not good. Let P be the set of good matchings of $\{1, \dots, 2m\}$. Let

$$U_P = \{q \in M_{\mathbf{w}} : X_\Gamma(q) \neq 0 \text{ for all } \Gamma \in P\}.$$

(In the dictionary to n points on $\mathbb{P}^1$, this corresponds to the set where none of the first m points is allowed to be the same as any of the last m points.) Note that $p \in P$, and $\pi^{-1}(U_P) \subset D(s_{1,2m})$.

3.3. *Lemma. — U_P is an affine variety, with coordinate ring generated by W_{ij} and Z_{ij} ($1 < i \leq m < j < 2m$) with relations*

$$W_{ij}W_{kl} = W_{il}W_{kj} \tag{6}$$

(i.e. the matrix $[W_{ij}]$ has rank 1) and

$$Z_{ij}(W_{ij} - 1) = 1 \tag{7}$$

(i.e. the matrix $[W_{ij}]$ has no entry 1).

This has a simple interpretation: U_P is isomorphic to the space of $(m-1) \times (m-1)$ matrices of rank at most 1, where each entry differs from 1, and p is the unique singular point, corresponding to the zero matrix. (We remark that this is the cone over the Segre embedding of $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$.) Hence we have described a neighborhood of the singular point rather explicitly.

Proof. Let $V_P = \{[\Lambda] \in G(2,2m) : s_{ij}(\Lambda) \neq 0 \text{ for all } i \leq m < j\}$, so $V_P = \pi^{-1}(U_P)$. Then V_P is an open subset of $D(s_{1,2m})$. In terms of the coordinates on $D(s_{1,2m}) \cong \mathbb{C}^{4m-4}$ described above, V_P is described by

$$x_j y_i - x_i y_j \neq 0, \qquad x_j \neq 0, \qquad y_i \neq 0 \tag{8}$$

for $i \leq m < j$. Let T be the maximal torus $T \subset SL(2m)$. By our preceding discussion, using (5), λ_k ($1 < k < 2m$) acts on x_k and y_k with weight 1, and on the other x_l and y_l's with weight 0. The torus λ_1 acts on x_l with weight 0, and on y_l with weight -1. The torus λ_{2m} acts on x_l with weight -1, and on y_l with weight 0.

We analyze the quotient V_P/T by writing T as a product, $T = T'T''$, and restricting to invariants of T', then of T''. Let T' be the subtorus of T such that $\lambda_1 = \lambda_{2m}$, and T'' be the subtorus $(\lambda, 1, \dots, 1, \lambda^{-1})$. We take the quotient first by T'. It is clear that the invariants are given by $u_i = x_i/y_i$ for $1 < i \leq m$ and $v_j = y_j/x_j$ for $m+1 \leq j < 2m$. From (8), the quotient of V_P is cut out by the inequality $u_i v_j - 1 \neq 0$.

T'' acts on this quotient as follows: λ acts on u_i by weight -2, and v_j by weight 2. The invariants of this quotient by T'' are therefore generated by $u_i v_j$ and $1/(u_i v_j - 1)$.

Hence if we take $W_{ij} = u_i v_j$, the invariants are generated by the W_{ij}, subject to the relations that the matrix $[W_{ij}]$ has rank 1, and also $W_{ij} \neq 1$. □

Now let $I \subset \mathbb{C}[\{X_\Gamma\}]$ be the ideal of relations of the invariants of $M_{\mathbf{w}}$, and let I_V be the ideal generated by the linear Plücker relations and the simple binomial relations. We have already shown that $I_V \subset I$.

Let S be the multiplicative system of monomials in X_Γ generated by those X_Γ where $\Gamma \in P$.

3.4. *Theorem. — If $n = 2m \geq 8$, then $S^{-1} I_V = S^{-1} I$. In other words, the sign, Plücker, and simple binomial relations cut out the moduli space on this open subset.*

As the Main Theorem 1.1 is true for $n = 2m = 8$ (§2.7), this theorem holds in that "base" case.

Proof. By Γ we will mean a general matching, and by Δ, we will mean a matching in P. We have a surjective map

$$\mathbb{C}[\{X_\Gamma/X_\Delta\}]/S^{-1}I_V \to \mathbb{C}[\{X_\Gamma/X_\Delta\}]/S^{-1}I$$

(that we wish to show is an isomorphism), and Lemma 3.3 provides an isomorphism

$$\mathbb{C}[\{X_\Gamma/X_\Delta\}]/S^{-1}I \cong \mathcal{O}(U_P) \cong \mathbb{C}[\{W_{ij}, Z_{ij}\}]/J_{WZ},$$

where $J_{WZ} \subset \mathbb{C}[\{W_{ij}\}, \{Z_{ij}\}]$ is the ideal generated by the relations (6) and (7).

By comparing the moduli maps, we see that this isomorphism is given by

$$W_{ij} \mapsto \frac{X_{1i \cdot j(2m) \cdot \Gamma}}{X_{1j \cdot i(2m) \cdot \Gamma}}, \qquad Z_{ij} \mapsto \frac{X_{1j \cdot i(2m) \cdot \Gamma}}{X_{1(2m) \cdot ij \cdot \Gamma}} \tag{9}$$

where Γ is any matching on $\{1, \dots, 2m\} - \{1, i, j, 2m\}$ such that $1j \cdot i(2m) \cdot \Gamma \in P$. (By the simple binomial relations, this is independent of Γ.) The description of the isomorphism in the reverse direction is not so pleasant, and we will spend much of the proof avoiding describing it explicitly.

We thus have a surjective morphism

$$\psi : \mathbb{C}[\{X_\Gamma/X_\Delta\}] \to \mathbb{C}[\{W_{ij}, Z_{ij}\}]/J_{WZ}$$

whose kernel is $S^{-1}I$, which contains $S^{-1}I_V$. We wish to show that the kernel is $S^{-1}I_V$. We do this as follows. For each $1 < i \le m < j < 2m$, fix a matching $\Gamma_{i,j}$ on $\{1,\dots,2m\} - \{1,i,j,2m\}$ so that $1j \cdot i(2m) \cdot \Gamma_{i,j} \in P$. Consider the subring of $\mathbb{C}[\{X_\Gamma/X_\Delta\}]$ generated by

$$w_{ij} = \frac{X_{1i \cdot j(2m) \cdot \Gamma_{i,j}}}{X_{1j \cdot i(2m) \cdot \Gamma_{i,j}}}, \qquad z_{ij} = \frac{X_{1j \cdot i(2m) \cdot \Gamma_{i,j}}}{X_{1(2m) \cdot ij \cdot \Gamma_{i,j}}}.$$

(Compare this to (9).) Call this subring $\mathbb{C}[\{w_{ij}, z_{ij}\}]/J_{wz}$.

The proof consists of two steps. *Step 1.* We show that *any* element of $\mathbb{C}[\{X_\Gamma/X_\Delta\}]$ differs from an element of $\mathbb{C}[\{w_{ij}, z_{ij}\}]/J_{wz}$ by an element of $S^{-1}I_V$. We do this in several smaller steps. *Step 1a.* We show that any X_Γ/X_Δ can be written as a linear combination of $X_{\Delta'}/X_\Delta$ (where Δ' is also good). *Step 1b.* We show that any such $X_{\Delta'}/X_\Delta$ may be expressed (modulo $S^{-1}I_V$) in terms of $X_{ik\cdot jl \cdot \Gamma}/X_{il \cdot jk \cdot \Gamma}$, where $i,j \le m < k,l$, and Γ is good. *Step 1c.* We show that any such expression can be written (modulo $S^{-1}I_V$) in terms of w_{ij} and z_{ij}, i.e. modulo $S^{-1}I_V$, any such expression lies in $\mathbb{C}[\{w_{ij}, z_{ij}\}]/J_{wz}$.

Step 2. The kernel of the map $\psi : \mathbb{C}[\{w_{ij}, z_{ij}\}]/J_{wz} \to \mathbb{C}[\{W_{ij}, Z_{ij}\}]/J_{WZ}$ (given by $w_{ij} \mapsto W_{ij}$, $z_{ij} \mapsto Z_{ij}$) lies in $S^{-1}I_V$.

We now execute this strategy.

Step 1a. We first claim that X_Γ/X_Δ ($\Delta \in P$) is a linear combination of units $X_{\Delta'}/X_\Delta$ (i.e. $\Delta' \in P$) modulo the Plücker relations (the linear relations, which are in $S^{-1}I_V$). We prove the result by induction on the number of bad edges. The base case (if all edges of Γ are good, i.e. $\Gamma \in P$) is immediate. Otherwise, Γ has at least two bad edges, say ij and kl, where $i,j \le m < k,l$. Then $X_\Gamma = \pm X_{\Gamma - \{ij,kl\} + \{ik,jl\}} \pm X_{\Gamma - \{ij,kl\}+\{il,jk\}}$ is a Plücker relation, and the latter two terms have two fewer bad edges, completing the induction.

Step 1b. We show that any element $X_{\Delta'}/X_\Delta$ of $\mathbb{C}[\{X_\Gamma/X_\Delta\}]$ (Δ' good) is congruent (modulo $S^{-1}I_V$) to an element of the form $X_{ik\cdot jl\cdot\Gamma}/X_{il\cdot jk\cdot\Gamma}$, where $i,j \le m < k,l$, and Γ is good. We prove this by induction on m. If $m=4$, the result is true (§2.7). Assume now that $m > 4$. If Δ' and Δ share an edge e, then let $\overline{\Delta'}$ and $\overline{\Delta}$ be the graphs on $2m-2$ vertices obtained by removing this edge e. Then by the inductive hypothesis, the result holds for $X_{\overline{\Delta'}}/X_{\overline{\Delta}}$. By taking the resulting expression, and "adding edge e to the subscript of each term", we get an expression for $X_{\Delta'}/X_\Delta$. Finally, if Δ' and Δ share no edge, suppose in Δ', 1 is connected to $(m+1)$; in Δ, 1 is connected to $(m+2)$; and in Δ', $(m+2)$ is connected to 2. This is true after suitable reordering. Say $\Delta' = 1(m+1)\cdot 2(m+2)\cdot\Gamma'$ and $\Delta = 1(m+2)\cdot\Gamma$. Then

$$\frac{X_{\Delta'}}{X_\Delta} = \frac{X_{1(m+2)\cdot 2(m+1)\cdot\Gamma'}}{X_{1(m+2)\cdot\Gamma}} \cdot \frac{X_{1(m+1)\cdot 2(m+2)\cdot\Gamma'}}{X_{1(m+2)\cdot 2(m+1)\cdot\Gamma'}}.$$

For each factor of the right side, the numerator and the denominator "share an edge", so we are done.

Step 1c. We next show that any such can be written (modulo $S^{-1}I_V$) in terms of w_{ij} and z_{ij}, i.e. modulo $S^{-1}I_V$ lies in $\mathbb{C}[\{w_{ij}, z_{ij}\}]/J_{wz}$. If $2m = 8$, the result again holds (§2.7). Assume now that $2m > 8$. Given any $X_{ik\cdot jl\cdot\Gamma}/X_{ij\cdot kl\cdot\Gamma}$ as in Step 1b, we will express it (modulo $S^{-1}I_V$) in terms of w_{ij} and z_{ij}. By the simple binomial relation (i.e. modulo $S^{-1}I_V$), we may assume that Γ is any good matching on $\{1, \dots, 2m\} - \{i, j, k, l\}$, and in particular that there are edges $ab, cd \in \Gamma$ such that $\{1, 2m\} \subset \{a, b, c, d, i, j, k, l\}$. Then the result for $m = 4$ implies that we can write $X_{ik\cdot jl\cdot ab\cdot cd}/X_{ij\cdot kl\cdot ab\cdot cd}$ can be written in terms of w_{ij} and z_{ij} (in terms of the "$m = 4$ variables"). By taking this expression, and "adding in the remaining edges of Γ", we get the desired result for our case.

Step 2. We will show that the kernel of the map $\psi : \mathbb{C}[\{w_{ij}, z_{ij}\}]/J_{wz} \to \mathbb{C}[\{W_{ij}, Z_{ij}\}]/J_{WZ}$ (given by $w_{ij} \mapsto W_{ij}$, $z_{ij} \mapsto Z_{ij}$) lies in $S^{-1}I_V$.

In order to do this, we need only verify that the relations (6) and (7) are consequences of the relations in $S^{-1}I_V$.

We first verify (6). By the simple binomial relation, we may write

$$W_{ij} = \frac{X_{1i\cdot j(2m)\cdot kl\cdot\Gamma}}{X_{1j\cdot i(2m)\cdot kl\cdot\Gamma}}, \qquad W_{kl} = \frac{X_{1k\cdot l(2m)\cdot ij\cdot\Gamma}}{X_{1l\cdot k(2m)\cdot ij\cdot\Gamma}}, \tag{10}$$

$$W_{il} = \frac{X_{1i\cdot l(2m)\cdot jk\cdot\Gamma}}{X_{1l\cdot i(2m)\cdot jk\cdot\Gamma}}, \qquad W_{jk} = \frac{X_{1j\cdot k(2m)\cdot il\cdot\Gamma}}{X_{1k\cdot j(2m)\cdot il\cdot\Gamma}}. \tag{11}$$

We thus wish to show that modulo $S^{-1}I_V$, the product of the terms in (10) equals the product of the terms in (11). Choose any edge $e \in \Gamma$. The analogous question with $m = 4$, with $\Gamma - e$ "removed from the subscripts", is true (§2.7). Hence, by "adding $\Gamma - e$ back in to the subscripts", we get the analogous result here.

We next verify (7):

$$1 - W_{i,j} = \frac{X_{1j\cdot i(2m)\cdot\Gamma_{i,j}}}{X_{1j\cdot i(2m)\cdot\Gamma_{i,j}}} - \frac{X_{1i\cdot j(2m)\cdot\Gamma_{i,j}}}{X_{1j\cdot i(2m)\cdot\Gamma_{i,j}}} \equiv \frac{X_{1(2m)\cdot ij\cdot\Gamma_{i,j}}}{X_{1j\cdot i(2m)\cdot\Gamma_{i,j}}} = 1/Z_{i,j} \pmod{S^{-1}I_V}$$

where the equivalence uses a linear Plücker relation. $\square$

4. Proof of Main Theorem

We have reduced to the equilateral case $\mathbf{w} = 1^n$, $n = 2m$, where $n \geq 10$.

The reader will notice that we will use the simple binomial relations very little. In fact we just use the inductive structure of the moduli space: given a matching Δ on $n - k$ of n vertices ($4 \leq k < n$), and a point $[X_\Gamma]_\Gamma$ of V_n, then either these X_Γ with $\Delta \subset \Gamma$ are all zero, or $[X_\Gamma]_{\Delta\subset\Gamma}$ satisfies the Plücker and simple binomial relations for k, and hence is a point of V_k if $k \neq 6$. (The reader should think of this rational map $[X_\Gamma] \dashrightarrow [X_\Gamma]_{\Delta\subset\Gamma}$ as a forgetful map, remembering only the moduli of the k points.) In fact, even if $k = 6$ (and $n \geq 8$), the point must lie in M_6, as the simple binomial relations for $n > 6$ induce the Segre cubic relation (§2.9). The central idea of our proof is, ironically, to use the case $n = 6$, where the Main Theorem 1.1 doesn't apply.

We will call such Δ, where the X_Γ with $\Delta \subset \Gamma$ are not all zero and the corresponding point of M_6 is stable, a *stable* $(n-6)$*-matching*. One motivation for this definition is that given a stable configuration of n points on $\mathbb{P}^1$, there always exists a stable $(n-6)$-matching. (Hint: Construct Δ inductively as follows. We say two of the n points are in the same *clump* if they have the same image on $\mathbb{P}^1$. Choose any y in the largest clump, and any z in the second-largest clump; yz is our first edge of Δ. Then repeat this with the remaining vertices, stopping when there are six vertices left.) Caution: This is false with 6 replaced by 4!

Main Theorem 1.1 is a consequence of the following two statements, and Theorem 3.4. Indeed, **(I)** and **(II)** show Theorem 1.1 set-theoretically, and scheme-theoretically away from the strictly semistable points (in characteristic 0), and Theorem 3.4 deals with (a neighborhood of) the strictly semistable points.

(I) There is a natural bijection between points of V_n with no stable $(n-6)$-matching, and strictly semistable points of M_n.

(II) If B is any scheme, there is a bijection between morphisms $B \to V_N$ missing the "no stable $(n-6)$-matching" locus (i.e. missing the strictly semistable points of M_n, by (I)) and stable families of n points $B \times \{1, \dots, n\} \to \mathbb{P}^1$. (In other words, we are exhibiting an isomorphism of functors.)

One direction of the bijection of (I) is immediate. The next result shows the other direction.

4.1. *Claim. — If* $[X_\Gamma]_\Gamma$ *is a point of* V_n *(*$n \geq 10$*) having no stable* $(n-6)$*-matching, then* $[X_\Gamma]_\Gamma$ *is a strictly semistable stable point of* M_n*.*

Several of the steps will be used in the proof of (II). We give them names so they can be referred to later.

Proof. Our goal is to produce a partition of n into two subsets of size $n/2$, such that the point of M_n given by this partition is our point of V_n. Throughout this proof, partitions will be assumed to mean into two equal-sized subsets.

We work by induction. We will use the fact that the result is also true for $n = 6$ (tautologically) and $n = 8$ (as $V_8 = M_8$, §2.7).

Fix a matching Δ such that $X_\Delta \neq 0$. By the inductive hypothesis, each edge xy yields a strictly semistable point of M_{n-2}, and hence a partition of $\{1, \dots, n\} - \{x, y\}$, by considering all matchings containing xy. Thus for each $xy \in \Delta$, we get a partition of $\{1, \dots, n\} - \{x, y\}$. If wx, yz are two edges of Δ, then we get the same induced partition of $\{1, \dots, n\} - \{w, x, y, z\}$ (from the inductive hypothesis for $n-4$), so all of these partitions arise from a single partition $\{1, \dots, n\} = S_0 \coprod S_1$.

4.2. Δ *two-overlap argument.* As this partition is determined using any two edges of Δ, we would get the same partition if we began with any Δ' sharing two edges with Δ, such that $X_{\Delta'} \neq 0$.

Defining the map to $\mathbb{P}^1$. Define $\phi : S_0 \coprod S_1 = \{1, \ldots, n\} \to \mathbb{P}^1$ by $S_0 \to 0$ and $S_1 \to 1$. For each matching Γ, define X'_Γ using these points of $\mathbb{P}^1$. Rescale (or normalize) all the X_Γ so $X'_\Delta = X_\Delta$. We will show that $X'_\Gamma = X_\Gamma$ for all Γ, which will prove Claim 4.1.

4.3. *One-overlap argument.* For any Γ sharing an edge xy with Δ, $X'_\Gamma = X_\Gamma$, for the following reason: $[X_\Xi]_{xy \in \Xi}$ lies in M_{n-2} by the inductive hypothesis, and this point of M_{n-2} corresponds to the map ϕ (as the partition $S_0 \coprod S_1$ was determined using this point of M_{n-2}), so $[X_\Xi]_{xy \in \Xi} = [X'_\Xi]_{xy \in \Xi}$, and the normalization $X'_\Delta = X_\Delta \neq 0$ ensures that $X'_\Xi = X_\Xi$ for all Ξ containing xy.

4.4. *Reduction to* Γ *with* $X'_\Gamma \neq 0$. It suffices to prove the result for those graphs Γ, all of whose edges connect S_0 and S_1 (i.e. no edge is contained in S_0 or S_1; equivalently, $X'_\Gamma \neq 0$). We show this by showing that *any* X_Γ is a linear combination of such graphs, by induction on the number i of edges of Γ contained in S_0 (= the number contained in S_1). The base case $i = 0$ is tautological. For the inductive step, choose an edge $wx \in \Gamma$ contained in S_0 and an edge yz contained in S_1. Then the Plücker relation using Γ and $wxyz$ (with appropriate signs depending on the directions of edges) is

$$\pm X_\Gamma \pm X_{\Gamma - wx - yz + wy + xz} \pm X_{\Gamma - wx - yz + wz + xy} = 0,$$

and both $\Gamma - wx - yz + wy + xz$ and $\Gamma - wx - yz + wz + xy$ have $i - 1$ edges contained in S_0, and the result follows.

4.5. $pqrs$ *argument, first version.* Finally, assume that $X'_\Gamma \neq 0$ and that Γ shares no edge with Δ. See Figure 14. Let qr be an edge of Γ (so $\phi(q) \neq \phi(r)$), and let pq and rs be edges of Δ containing q and r respectively (so $\phi(p) \neq \phi(q)$ and $\phi(r) \neq \phi(s)$). Then $\phi(p) \neq \phi(s)$, as ϕ takes on only two values. Let $\Delta' = \Delta - pq - rs + qr + ps$, so $X'_{\Delta'} \neq 0$ as $\phi(q) \neq \phi(r)$ and $\phi(p) \neq \phi(s)$. Then $X_{\Delta'} = X'_{\Delta'}$ by the one-overlap argument 4.3, as Δ' shares an edge with Δ (indeed all but two edges), so $X_{\Delta'} \neq 0$. Hence by the Δ two-overlap argument 4.2, Δ' defines the same partition $S_0 \coprod S_1$, and hence the same map $\phi : \{1, \ldots, n\} \to \mathbb{P}^1$. Finally, Γ shares an edge with Δ', so $X'_\Gamma = X_\Gamma$ by the one-overlap argument 4.3.

We have thus completed the proof of Claim 4.1. □

Proof of (II). The result boils down to the following desideratum: Given any $(n-6)$-matching Δ on some $\{1, \ldots, n\} - \{a, b, c, d, e, f\}$, there should be a bijection between:

(a) morphisms $\pi : B \to V_n$ contained in the open subset where Δ is a stable $(n-6)$-matching, and
(b) stable families of points $\phi : B \times \{1, \ldots, n\} \to \mathbb{P}^1$ where $\phi|_{B \times \{a, \ldots, f\}}$ is also a stable family, and for any edge xy of Δ, $\phi|_{B \times \{x\}}$ does not intersect $\phi|_{B \times \{y\}}$.

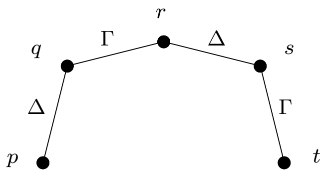

FIGURE 14. The $pqrs$ argument (vertex t is used in §4.10)

We have already described the map (b) $\Rightarrow$ (a) in §2. We now describe the map (a) $\Rightarrow$ (b), and verify that (a) $\Rightarrow$ (b) $\Rightarrow$ (a) is the identity. (It will then be clear that (b) $\Rightarrow$ (a) $\Rightarrow$ (b) is the identity: given a stable family of points parameterized by B, we get a map from B to an open subset of M_n, which is a fine moduli space, hence (b) $\Rightarrow$ (a) is an injection. The result then follows from the fact that (a) $\Rightarrow$ (b) $\Rightarrow$ (a) is the identity.)

We work by induction on n. The case $n = 8$ was checked earlier (§2.7).

The map to $\mathbb{P}^1$. Given an element of (a), define a family of n points of $\mathbb{P}^1$ (an element of (b)) as follows. (i) $\phi : B \times \{a, \dots, f\} \to \mathbb{P}^1$ is given by the corresponding map $B \to M_6$. (ii) If yz is an edge of Δ, we define $B \times (\{1, \dots, n\} - \{y, z\}) \to \mathbb{P}^1$ extending (i) by considering the matchings containing yz, which by the inductive hypothesis give a point of M_{n-2}. (iii) The morphisms of (ii) agree "on the overlap", as given two edges wx and yz of Δ, we get $B \times (\{1, \dots, n\} - \{w, x, y, z\}) \to \mathbb{P}^1$ by considering the matchings containing $wx \cdot yz$, which by the inductive hypothesis give a map to M_{n-4}. Here we are using that $n \geq 10$; and if $n = 10$, we need the fact that the Segre cubic relation cutting out M_6 is induced by the quadrics cutting out M_n for $n \geq 8$ (Remark 2.9). Thus we get a well-defined morphism $\phi : B \times \{1, \dots, n\} \to \mathbb{P}^1$.

4.6. Δ *two-overlap argument, cf.* §*4.2.* If Δ' is another matching on $\{1, \dots, n\} - \{a, \dots, f\}$ sharing at least 2 edges with Δ, with $X_{\Delta' \cdot \Xi} \neq 0$ for some matching Ξ of $\{a, \dots, f\}$, we obtain the same ϕ, as ϕ can be recovered by considering only two edges of Δ when using (ii).

Defining X'. Define X'_Γ for all matchings Γ using ϕ and the moduli morphism of eqn. (1). The coordinates X_Γ are projective (i.e. the set of X_Γ is defined only up to scalars); scale them so that $X_{\Delta \cdot \Xi} = X'_{\Delta \cdot \Xi}$ for all matchings Ξ of $\{a, \dots, f\}$. Note that if xy is an edge of Δ, then $\phi(x) \neq \phi(y)$, as there exists a matching Ξ of $\{a, \dots, f\}$ such that $X'_{\Delta \cdot \Xi} \neq 0$.

The following result will confirm that (a) $\Rightarrow$ (b) $\Rightarrow$ (a) is the identity, concluding the proof of (II).

4.7. *Claim.* — We have the equality $X_\Gamma = X'_\Gamma$ for all Γ.

Proof. This proof will occupy us until the end of §4.14.

4.8. *One-overlap argument.* As in §4.3, the result holds for those Γ sharing an edge yz with Δ: by considering only those variables $X_{\Gamma'}$ containing the edge yz (including both X_Γ and X_Δ), we obtain a point of M_{n-2}. This point of M_{n-2} is the one given by ϕ (this was part of how ϕ was defined), so $[X_{\Gamma'}]_{yz\in\Gamma'} = [X'_{\Gamma'}]_{yz\in\Gamma'}$. By choosing a matching Ξ on $\{a,\dots,f\}$ so that $X_{\Delta\cdot\Xi} \neq 0$, we have that $X_\Gamma X'_{\Delta\cdot\Xi} = X_{\Delta\cdot\Xi}X'_\Gamma$. Using $X_{\Delta\cdot\Xi} = X'_{\Delta\cdot\Xi} \neq 0$, we have $X_\Gamma = X'_\Gamma$, as desired.

We now deal with the remaining case, where Γ and Δ share no edge.

4.9. *Reduction to Γ with $X'_\Gamma \neq 0$ (cf. §4.4).* It suffices to prove the result for those graphs such that $X'_\Gamma \neq 0$, or equivalently that for each edge xy of Γ, $\phi(x) \neq \phi(y)$. We show this by showing that any X_Γ is a linear combination of such graphs, by induction on the number of edges xy of Γ with $\phi(x) = \phi(y)$. For the purposes of this paragraph, call these *bad edges*. The base case $i = 0$ is tautological. For the inductive step, choose a bad edge $wx \in \Gamma$ (with $\phi(w) = \phi(x)$), and another edge yz such that $\phi(y), \phi(z) \neq \phi(w)$. (Such an edge exists, as by stability, less than $n/2$ elements of $\{1,\dots,n\}$ take the same value in $\mathbb{P}^1$.) Then the Plücker relation using Γ with respect to $wxyz$ is

$$\pm X_\Gamma \pm X_{\Gamma-wx-yz+wy+xz} \pm X_{\Gamma-wx-yz+wz+xy} = 0,$$

and both $\Gamma - wx - yz + wy + xz$ and $\Gamma - wx - yz + wz + xy$ have at most $i-1$ bad edges, and the result follows.

Recall that we are proceeding by induction. We first deal with the case $n \geq 14$, assuming the cases $n = 10$ and $n = 12$. We will then deal with these two stray cases. This logically backward, but the $n \geq 14$ case is cleaner, and the two other cases are similar but more ad hoc.

4.10. The case $n \geq 14$. *$pqrs$ argument, second version.* As $n \geq 14$, there is an edge qr of Γ not meeting $abcdef$. See Figure 14. By §4.9, we may assume $\phi(q) \neq \phi(r)$. Let pq and rs be the edges of Δ meeting q and r respectively (so $\phi(p) \neq \phi(q)$ and $\phi(r) \neq \phi(s)$. (i) (cf. the similar argument of §4.5) If $\phi(p) \neq \phi(s)$, then let $\Delta' = \Delta - pq - rs + qr + ps$; then Δ' defines the same family of n points as Δ by the two-overlap argument §4.6, and Γ and Δ' share an edge, so we are done by the one-overlap argument §4.8. (More precisely, this argument applies on the open subset of B where $\phi(p) \neq \phi(s)$.) (ii) If $\phi(p) = \phi(s)$, then $\phi(p) \neq \phi(r)$. (More precisely, this argument applies on the open set where $\phi(p) \neq \phi(r)$.) Let st be the edge of Γ containing s. (It is possible that $t = p$.) Let $\Gamma' = \Gamma - qr - st + rs + qt$ and $\Gamma'' = \Gamma - qr - st + qs + rt$ be the other two terms in the Plücker relation for Γ for $pqrs$. Then Γ' shares edge rs with Δ, so $X'_{\Gamma'} = X_{\Gamma'}$ by the one-overlap argument §4.8, and by applying (i) to Γ'' (swapping the names of r and s), $X'_{\Gamma''} = X_{\Gamma''}$, so by the Plücker relation, $X'_\Gamma = X_\Gamma$ as desired.

4.11. The cases $n = 10$ and $n = 12$. We are assuming that Γ and Δ share no edges. If there is an edge of Γ not meeting $\{a,\dots,f\}$, the $pqrs$-argument §4.10 applies, so assume otherwise. Divide $\{1,\dots,n\}$ into two subsets $abcdef$ and $ghij$ (resp. $ghijkl$) if $n = 10$ (resp. $n = 12$), where the edges of Δ are gh, ij, and (if $n = 12$) kl. By renaming $abcdef$, we

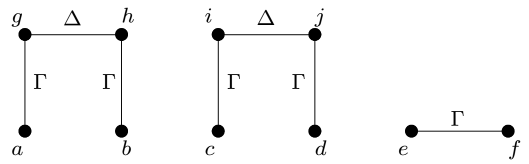

FIGURE 15. The problematic graphs for $n = 10$

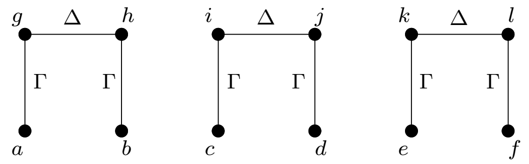

FIGURE 16. The problematic graphs for $n = 12$

may assume the edges of Γ are ag, bh, ci, dj, and either ef (if $n = 10$, see Figure 15) or ek and fl (if $n = 12$, see Figure 16).

4.12. Suppose that $\phi(a) \neq \phi(b)$. Note that we will only use that $ag, bh \in \Gamma$, $gh \in \Delta$, and $\phi(a) \neq \phi(b)$ — we will use this argument again below. There is a matching Ξ of $cdef$ so that if $xy \in \Xi$, then $\phi(x) \neq \phi(y)$. (This is a statement about stable configurations of 6 points on $\mathbb{P}^1$: if we have a stable set of 6 points on $\mathbb{P}^1$, then no three of them are the same point. Hence for any four of them $cdef$, we can find a matching of this sort.) Let $\Delta' = \Xi \cdot ab \cdot \Delta$. Then by the simple binomial relations (our first invocation!) $X_{\Delta'} X_\Gamma = X_{\Delta'-ab-gh+ag+bh} X_{\Gamma+ab+gh-ag-bh}$ and $X'_{\Delta'} X'_\Gamma = X'_{\Delta'-ab-gh+ag+bh} X'_{\Gamma+ab+gh-ag-bh}$. However, by the one-overlap argument §4.8, $X_{\Delta'} = X'_{\Delta'} \neq 0$ (Δ' and Δ share edge ij), $X_{\Delta'-ab-gh+ag+bh} = X'_{\Delta'-ab-gh+ag+bh}$ ($\Delta' - ab - gh + ag + bh$ and Δ share edge ij), and $X_{\Gamma+ab+gh-ag-bh} = X'_{\Gamma+ab+gh-ag-bh}$ ($\Gamma + ab + gh - ag - bh$ and Δ share edge gh), so we are done.

We are left with the case $\phi(a) = \phi(b)$.

4.13. Suppose now that $n = 10$. As $\phi(a) = \phi(b)$, $\phi(b)$ is distinct from $\phi(e)$ and $\phi(f)$ (as $\phi(a)$, ..., $\phi(f)$ are a stable set of six points on $\mathbb{P}^1$). By the Plücker relations for Γ (using $agef$),

$$\pm X_\Gamma \pm X_{\Gamma-ag-ef+ae+gf} \pm X_{\Gamma-ag-ef+af+eg} = 0,$$

and similarly for the X' variables. By applying the argument of §4.12 with e and a swapped, we have $X'_{\Gamma-ag-ef+af+eg} = X_{\Gamma-ag-ef+af+eg}$, and by applying the argument of §4.12 with f and a swapped, we have $X'_{\Gamma-ag-ef+ae+gf} = X_{\Gamma-ag-ef+ae+gf}$, from which $X'_\Gamma = X_\Gamma$, concluding the $n = 10$ case.

4.14. Suppose finally that $n = 12$. If $\phi(c) \neq \phi(d)$, we are done (by the same argument as §4.12, with ab replaced by cd), and similarly if $\phi(e) \neq \phi(f)$. Hence the only case left is if

$\phi(a) = \phi(b)$, $\phi(c) = \phi(d)$, and $\phi(e) = \phi(f)$, and (by stability of the 6 points $\phi(a), \ldots, \phi(f)$) these are three distinct points of $\mathbb{P}^1$. Consider the Plücker relation for Γ with respect to $bhci$. One of the other two terms is $\Gamma - bh - ci + bi + ch$, and $X'_{\Gamma-bh-ci+bi+ch} = X_{\Gamma-bh-ci+bi+ch}$ (by the same argument as in §4.12, as $\phi(a) \neq \phi(c)$). We thus have to prove that $X_{\Gamma'} = X'_{\Gamma'}$ for the third term in the Plücker relation, where

$$\Gamma' = ag \cdot bc \cdot hi \cdot dj \cdot ek \cdot fl.$$

For this, apply the argument of §4.13, with $abghef$ replaced by $felkbc$ respectively. □

References

[BP] V. Batyrev and O. Popov, *The Cox ring of a del Pezzo surface*, in *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, 85–103, Progr. Math., 226, Birkhäuser Boston, Boston, MA, 2004.

[D] I. Dolgachev, *Lectures on Invariant Theory*, LMS Lecture Note Series, 296, Cambridge U.P., Cambridge, 2003.

[DO] I. Dolgachev and D. Oortland, *Point sets in projective spaces and theta functions*, Astérisque No. 165, (1988), 210 pp. (1989).

[HMSV] B. Howard, J. Millson, A. Snowden and R. Vakil, *The projective invariants of ordered points on the line*, math.AG/0505096v6, submitted for publication.

[Ho] R. Howe, *The classical groups and invariants of binary forms*, in *The Mathematical Heritage of Hermann Weyl (Durham, NC, 1987)*, 133–166, Proc. Sympos. Pure Math. **48**, Amer. Math. Soc., Providence, RI, 1988.

[KT] S. Keel and J. Tevelev, *Equations for $\overline{M}_{0,n}$*, preprint 2005, math.AG/0507093.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Springer-Verlag, Berlin, 1994.

[MS] D. Mumford and K. Suominen, Introduction to the theory of moduli, in *Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer-School in Math.)*, pp. 171–222, Wolters-Noordhoff, Groningen, 1972.

[N] P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, Tata Inst. of Fund. Res. Lectures on Math. and Phys., **51**, Tata Inst. of Fund. Res. Bombay; by the Narosa Publ. House, New Delhi, 1978.

[Sh] T. Shioda, *On the graded ring of invariants of binary octavics*, Amer. J. Math. **89** 1967 1022–1046.

[Sl] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available electronically at `http://www.research.att.com/~njas/sequences/`, 2005.

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