THE GEOMETRY OF SPACES OF EIGHT POINTS IN PROJECTIVE SPACE (DUALITIES, REPRESENTATION THEORY, AND LIE THEORY)

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ABSTRACT. **To be written.**

CONTENTS

1. INTRODUCTION

To do next: Go through the substance of the paper, then come back to the introduction and abstract. It is not clear N_8' gives something of dimension 9,, i.e. that N_8 is finite over it, but we don't need this? But note that this is known, see Ben's email in July 2010. Mention Sam G's comments on the cubic as sum of 35 things, see Berkeley talk. [Define I_* at the start –R] This note discusses the geometry of the spaces

$$(\mathbb{P}^1)^8//SL(2), \quad (\mathbb{P}^3)^8//SL(4) \quad \text{and} \quad (\mathbb{P}^5)^8//SL(6),$$

each the GIT quotient with respect to the "usual" linearization $\mathcal{O}(1,\dots,1)$. For each of these quotients Q, let $R_*(Q)$ be the corresponding ring of invariants. (Coble–)Gale duality gives a canonical isomorphism between the first and third, via canonical isomorphism of the graded rings of invariants. Call this modular fivefold M_8 — we will use only its first incarnation. Let N_8 be the modular ninefold $(\mathbb{P}^3)^8//PGL(4)$. Gale duality gives an involution on N_8, through an involution of its underling graded ring $R_*(N_8)$. Our goal is to study and relate M_8 and N_8 (and its Gale-quotient N_8') and their extrinsic geometry. The key constructions are dual hypersurfaces C and Q in $\mathbb{P}^{13}$ of degree three and five respectively; for example, $M_8 = \mathrm{Sing}(C)$ and $N_8' = \mathrm{Sing}(Q)$.**[internal ref –R]** The partial derivatives of C, which cut out M_8, will be referred to as "the 14 quadrics"; they span an irreducible S_8-representation, so up to symmetry there is only one quadric (given in appropriate coordinates by (1)).

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In [HMSV4], we give *all* relations among generators of the graded rings for $(\mathbb{P}^1)^n//SL(2)$, with *any* linearization. In each case the graded rings are generated in one degree, so each quotient naturally comes with a natural projective embedding. The general case reduces to the linearizations 1^n, with n even. In this 1^n case, with the single exception of $n = 6$, there is (up to S_n-symmetry) a single quadratic equation, which is binomial in the Kempe generators (Specht polynomials). The quadratic for the case $n \geq 8$ is pulled back from the (unique up to symmetry) $n = 8$ quadratic discussed here, which forms the base case of an induction. We will indicate in §1.4 how the only case of smaller n with interesting geometry ($n = 6$, related to Gale duality, and projective duality of the Segre cubic and the Igusa quartic) also is visible in the boundary of the structure we describe here. Thus various beautiful structures of GIT quotients of n points on $\mathbb{P}^1$ are all consequences of the beautiful geometry in the 8-point space M_8.

The main results are outlined in §1.1, and come in logically independent parts.

(A) In §3, we describe the intricate relationship between M_8 and N_8, which will be summarized in Figure 1. We note that this section does not use the fact that the ideal cutting out M_8 is generated by the 14 quadrics (established in (B)); only that it lies the intersection of the 14 quadrics, which is immediate from their description **[internal ref –R]**. We also do not use that N_8' is the Gale-quotient of N_8 (established in (C)); only that N_8' corresponds to the subring of the ring of invariants of $(\mathbb{P}^3)^8//SL(4)$ generated in degree 1.

(B) In §4–6, we give a Lie-theoretic proof of the fact that there are no linear syzygies among the 14 quadrics (Theorem 4.1). As observed at the start of §4, this (in combination with results of [HMSV4]) implies that the 14 quadrics generate the ideal of M_8, the base case of the main induction of [HMSV4]. This was known earlier by brute force computer calculation by a number of authors (Maclagan, private communication; Koike [Koi]; and Freitag and Salvati-Manni [FS2]), but we wished to show the structural reasons for this result in order to make the main theorem of [HMSV4] (giving all relations for all GIT quotients of $(\mathbb{P}^1)^n//SL(2)$) computer-independent. (Strictly speaking, in [HMSV4], computers were used to deal with the character theory of small-dimensional S_6-representations, but this could certainly be done by hand with some effort.) In §7, we use the absence of linear syzygies to determine the graded free resolution of (the ring of invariants of) M_8.

(C) In the short section §8, we verify with the aid of a computer that the subring $R_*(N_8')$ of $R_*(N_8)$ generated in degree 1 is indeed the ring of Gale invariants.

To be clear on the use of computer calculation (as opposed to pure thought): in §3, we use a computer only to intersect two curves in $\mathbb{P}^2$; in §4–6, computers are not used; and computer calculation is essential to §8.

We describe other manifestations of the ring of invariants of M_8 in §1.2. Miscellaneous algebraic results about M_8 that may be useful to others are given in §1.3. We sketch how the beautiful classical geometry of the six point case is visible at the boundary in §1.4. The justifications of the statements made in §1.1 are given in the rest of the paper.

In general, we work over a field k of characteristic 0.**[Be careful throughout with language. Should we worry about using k in different contexts? One way out is to have the field k in blackboard bold. –R]** Many statements work away from a known finite list of primes, so we occasionally give characteristic-specific statements. If λ is a partition of 8, we denote the irreducible S_8-representation corresponding to λ by V_λ. The representations of main interest to us are the trivial (V_n) and sign ($\mathrm{sgn} = V_{1^n}$) representations, and the two 8-dimensional representations $V_{4,4}$ and $V_{2,2,2,2}$. The latter two are skew-dual: $V_{4,4} \otimes \mathrm{sgn} \cong V_{2,2,2,2}$.

1.1. **Main constructions (see Figure 1).** Justifications will be given in later sections of the paper.**[Give subsection refs for everything in this intro section; I've begun doing that –R]**

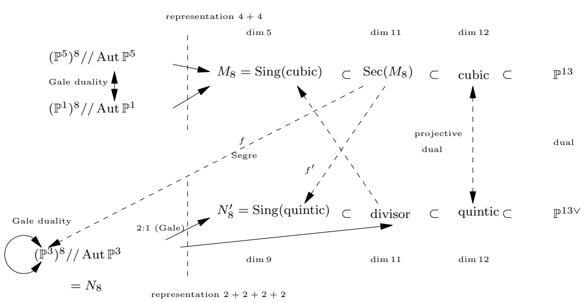

FIGURE 1. Diagram of interrelationships

The ring $R_*(M_8) = \operatorname{Proj} \oplus_k \Gamma(\mathcal{O}_{\mathbb{P}^1}(k)^{\otimes 8})^{SL_2}$ is generated in degree 1 (Kempe's 1894 theorem, see for example [HMSV1, Thm. 2.3]), and $\dim R_1(M_8) = 14$. We thus have a natural embedding $M_8 \hookrightarrow \mathbb{P}^{13}$. As an S_8-representation, $R_1(M_8) \cong V_{4,4}$ by Schur-Weyl duality (§2.2), and the tableau description of $R_1(M_8)$ is precisely the tableau description of $V_{4,4}$ **[internal ref –R]**.

The ideal of relations of M_8, $I_*(M_8) \subset \operatorname{Sym}^* R_1(M_8)$, is generated by 14 quadrics (**[internal ref –R]**, known earlier by computer calculation as described in (B) above). There is (up to multiplication by non-zero scalar) a unique skew-invariant cubic (an element of $\operatorname{Sym}^3 R_1(M_8)$), **[internal ref –R]**. We call this cubic the *skew cubic* C, and by abuse of notation call the hypersurface C as well. Then the fivefold

M_8 is the singular locus of C in a strong sense: $I_*(M_8)$ is the Jacobian ideal of C **[internal ref –R]**. The 14 partial derivatives of the skew cubic C generate $I_*(M_8)$ **[internal ref –R]** and are of course the 14 quadrics described in the previous paragraph. (In fact, this result holds away from characteristic 3. In characteristic 3, the Euler formula yields a linear syzygy among the 14 quadrics, and the skew cubic C can be taken as the remaining generator of the ideal.)

The ring $R_*(N_8) = \operatorname{Proj} \oplus_k \Gamma(\mathcal{O}_{\mathbb{P}^3}(k)^{\boxtimes 8})$ is generated in degree 1 and 2,**[internal ref to §8 –R]** and $\dim R_1(N_8) = 14$; $R_1(N_8) \cong V_{2,2,2,2}$ (as with $R_1(M_8)$, by Schur-Weyl duality, §2.2, or by direct comparison of the tableau description). The Gale-invariant subalgebra is the subalgebra $R_*(N_8') \subset R_*(N_8)$ generated by $R_1(N_8)$.**[internal ref to §8 –R]** More precisely: we *define* the graded ring $R_*(N_8')$ as the subalgebra of $R_*(N_8)$ generated in degree 1 (by $R_1(N_8)$), and *define* $N_8' = \operatorname{Proj} R_*(N_8')$, and *show* **[internal ref to §8 –R]** that $R_*(N_8')$ is the Gale-invariant subalgebra.

Bezout's theorem implies that $\mathrm{Sec}(M_8) \subset C$: restricting the cubic form C to any line joining two distinct points of M_8 yields a cubic vanishing to order 2 at those two points (as $M_8 = \operatorname{Sing} C$), so this cubic must be 0. The secant variety $\mathrm{Sec}(M_8)$ has the "expected dimension" 11, and is thus a divisor C.

Let $I_*(N_8') \subset \operatorname{Sym}^* R_1(N_8')$ be the ideal of relations of N_8'. By comparing the readily computable S_8-representations of $\operatorname{Sym}^n(R_1(N_8'))$ and $R_n(N_8')$, we find that there is a unique skew quintic *relation* Q in $I_5(N_8')$. Then the ninefold N_8' is the singular locus of Q.

Moreover, Q and C are projective dual hypersurfaces. Every secant line $\overline{pq}$ to M_8 (where $p, q \in M_8$) is contracted by the dual map: the dual map is given by the 14 partials of C, which are quadratic; their restriction to $\overline{pq}$ are 14 quadrics vanishing at the same two points p, q, so they are the same up to scalar. Thus $\mathrm{Sec}(M_8)$ is contained in the exceptional divisor of the dual map $C \dashrightarrow Q$, and in fact is the entire exceptional divisor. Thus the dual map contracts $\mathrm{Sec}(M_8)$ to $\mathrm{Sing}(Q) = N_8'$. Furthermore, this map $\mathrm{Sec}(M_8) \dashrightarrow N_8'$ lifts to $\mathrm{Sec}(M_8) \dashrightarrow N_8$, and this map can be interpreted geometrically as follows (see Figure 2). Suppose we are given a point of $\mathrm{Sec}(M_8)$ on a line connecting two general points of M_8. This corresponds to two ordered octuples of points on $\mathbb{P}^1$, or equivalently an ordered octuple of points on $\mathbb{P}^1 \times \mathbb{P}^1$. Embedding $\mathbb{P}^1 \times \mathbb{P}^1$ by the Segre map yields 8 points in $\mathbb{P}^3$, and hence a point of N_8. The rational map $\mathrm{Sec}(M_8) \dashrightarrow N_8$ must contract 2 dimensions ($\dim \mathrm{Sec}(M_8) = 11$ while $\dim N_8' = 9$); one is the contraction of the secant line, and the other corresponds to the fact that there is a pencil of quadrics passing through 8 points in $\mathbb{P}^3$.

The interrelationships of Figure 1 can be conjecturally completed as follows.

Conjecture 1.1. *The skew quintic is the trisecant variety (the union of trisecant* lines*) of N_8'. The divisor contracted to M_8 by the dual map is the quadrisecant variety (union of 4-secant lines) of N_8'.*

Note that the trisecant variety to N_8' lies in the skew quintic, by Bezout's theorem, and a naive dimension count suggests that the trisecants should "easily cover" all of $\mathbb{P}^{13}$". Similarly, Bezout's theorem implies that the quadrisecant variety to N_8' lies in the contracted divisor (analogous to the above argument showing that secant lines to M_8 are contracted by the dual map), and a naive dimension count suggests that the quadrisecants should "easily cover" all of $\mathbb{P}^{13}$".

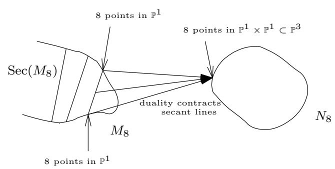

FIGURE 2. The contraction of the secant variety of $M_8 = (\mathbb{P}^1)^8//SL(2)$ to $N_8 = (\mathbb{P}^3)^9//SL(4)$

1.2. **Other manifestations of this space, and this graded ring.** The extrinsic and intrinsic geometry of M_n for small n has special meaning often related to the representation theory of S_n. For example, M_4 relates to the cross ratio, M_5 is the quintic del Pezzo surface, and the geometry of the Segre cubic M_6 is well known (see [HMSV2] for further discussion). The space M_8 might be the last of the M_n with such individual personality. For example, over $\mathbb{C}$, the space may be interpreted as a ball quotient in two ways:

(1) Deligne and Mostow [DM] showed that M_8 is isomorphic to the Satake-Baily-Borel compactification of an arithmetic quotient of the 5-dimensional complex ball, using the theory of periods of a family of curves that are fourfold cyclic covers of $\mathbb{P}^1$ branched at the 8 points.
(2) Kondo [Kon] showed that M_8 may also be interpreted in terms of moduli of certain K3 surfaces, and thus M_8 is isomorphic to the Satake-Baily-Borel compactification of a quotient of the complex 5-ball by $\Gamma(1-i)$, an arithmetic subgroup of a unitary group of a hermitian form of signature $(1,5)$ defined over the Gaussian integers. See also [FS2, p. 12] for further clarification and discussion.

Both interpretations are S_8-equivariant (see [Kon, p. 8] for the second).

Similarly, the graded ring R_8 we study has a number of manifestations:

(1) It is the ring of genus 3 hyperelliptic modular forms of level 2.
(2) Freitag and Salvati Manni showed that R_8 is isomorphic to the full ring of modular forms of $\Gamma(1-i)$ [FS2, p. 2], via the Borcherds additive lifting.
(3) The space of sections of multiples of a certain line bundle on $\overline{\mathcal{M}}_{0,8}$ (as there is a morphism $\overline{\mathcal{M}}_{0,8} \to M_8$, [Ka], see also [AL]).
(4) Igusa [I] showed that there is a natural map $A(\Gamma_3[2])/\mathcal{I}_3[2]^0 \to R_8$, where $A(\Gamma_3[2])$ is the ring of Siegel modular forms of weight 2 and genus 3. (See [FS2, §3] for more discussion.)
(5) It is a quotient of the third in a sequence of algebras related to the orthogonal group $\mathrm{O}(2m, \mathbb{F}_2)$ defined by Freitag and Salvati Manni (see [FS1], [FS2, §2]). (The cases $m=5$ and $m=6$ are related to Enriques surfaces.)

One reason for M_8 to be special is the coincidence $S_8 \cong \mathrm{O}(6, \mathbb{F}_2)$. A geometric description of this isomorphism in this context is given in [FS2, §4]. Another reason is Deligne and Mostow's table [DM, p. 86].

1.3. **Miscellaneous facts about M_8 and N_8.** The Hilbert function $f(k) = \dim R_k(M_8)$ was computed classically (see for example [Ho, p. 155, §5.4.2.3]):

$$f(k) = \frac{1}{3}\left(k^5 + 5k^4 + 11k^3 + 13k^2 + 9k + 3\right),$$

from which the Hilbert series $\sum_{k=0}^{\infty} f(k)t^k$ is

$$\frac{1 + 8t + 22t^2 + 8t^3 + t^4}{(1-t)^6}.$$

(Both formulas are given in [FS2, p. 7].) The degree of M_8 is 40 [HMSV1, p. approx. 12].**[fix –R]** Of course M_8 is projectively normal, by the first fundamental theorem of invariant theory. It is arithmetically Gorenstein, as the numerator of the Hilbert series is symmetric.**[Later ask Ben if this is right. –R]** It doesn't satisfy the N_2 condition of Green and Lazarsfeld **[[add Jul 29 08 e, John gives short proof] –R]**, and it is not Koszul (as the dual Hilbert series $1/H(-t)$ has negative coefficients). [Give ref for this?]

The graded free resolution is [put it here; gradedness appears to be new; F-SM don't do this — Ben said Jul 13 that F-SM undoubtedly had this, but didn't publish it]. Thus the graded Hilbert function is [here], and the a-invariant is -2 [explain].

The Hilbert series for N_8' is [get it from Ben, see Nov. 10 email from some year.] $\deg N_8' = 56$.

July 10 asked Ben and Andrew to add what is relevant. [worth saying: no linear syzygies?]

Ben says in his July 12, 2010 email that the Hilbert series for N_8 is

$$(1 + 4t + 31t^2 + 40t^3 + 31t^4 + 4t^5 + t^6)/(1-t)^{10},$$

so it is Gorenstein, and the a-invariant is -4. He has a guess for the generator as a tableau or GT-pattern. He also says (Jul 13) that it another way to see that it is Gorenstein is to apply the result of F. Knop (1989?) that if you havea group acting linearly on affines space, and it preserves volume (subgroup of SL), and if the unstable locus has codimension ≥ 2, then the subring of invariants is Gorenstein. I asked him for a precise ref on August 3, 2010.

1.4. **Relation to the six-point case.** (We will not need this picture, so we omit all details.) The classical geometry of six points in projective space, Figure 3, shows strong similarities to Figure 1. This can be made somewhat precise in a number of ways. Here is one way to see Figure 3 "at the boundary" of Figure 1. In the space of 8 points in $\mathbb{P}^3$ (the bottom left of Figure 1), consider the locus where the two given points (of the eight) coincide. Projecting from that point of $\mathbb{P}^3$, the remaining six points (generally) give six points in $\mathbb{P}^2$ (the bottom left of Figure 3). This can be extended to all parts of the two Figures, in a way respecting the Gale and projective dualities.

1.5. **Acknowledgments.** Foremost we thank Igor Dolgachev, who predicted the existence of the cubic of C to us. Diane Maclagan and Greg Smith gave essential advice on computational issues at key points in this project. We also thank Sam Grushevsky, Shrawan Kumar and Riccardo Salvati Manni for helpful comments.

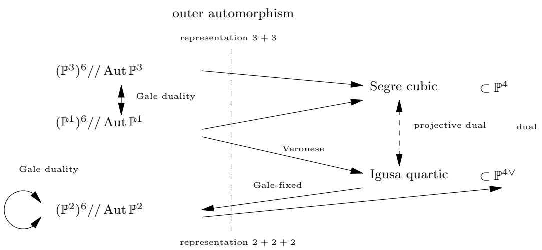

FIGURE 3. The classical geometry of six points in projective space (cf. Figure 1)**Line up right side later — move projective spaces to the right**

2. PRELIMINARIES ON INVARIANT THEORY AND REPRESENTATION THEORY

2.1. **Invariants of n points in $\mathbb{P}^{m-1}$ (with linearization $1,\dots,1$).** (See [D] for a thorough introduction to all invariant theory facts we need.) The degree d invariants of n points in $\mathbb{P}^{m-1}$ are generated (as a vector space over the ground field k, or more generally as a module over the ground ring) by invariants corresponding to certain tableaux: $m \times (dn/m)$ matrices, with entries consisting of the numbers 1 through n, each appearing d times. To such a tableau, we associate a product of $m \times m$ determinants, one for each column. To each column, we associate the $m \times m$ determinant, whose ith row are the projective coordinates of the point indexed by the entry in that row. For example, if $m = d = 2$ and $n = 4$, and the four points in $\mathbb{P}^1$ have coordinates $[x_i; y_i]$ $(1 \le i \le 4)$, then corresponding to

1	2	1	4
3	3	4	2

we associate the $SL(2)$-invariant

$$(x_1y_3 - x_3y_1)(x_2y_3 - x_3y_2)(x_1y_4 - x_4y_1)(x_4y_2 - x_2y_4).$$

The linear relations among these invariants are spanned by three basic types: (i) columns can clearly be rearranged without changing the invariant; (ii) swapping two entries in the same column clearly change the sign of the invariant; and (iii) Plücker relations, which we don't describe here (but see [HMSV4] **[external ref –R]** for a graphical description, which we use in the proof of Proposition 3.1). The straightening algorithm implies that for fixed n, m, d, the semistable tableaux (where the entries are increasing vertically and weakly increasing horizontally) form a basis.

If $m = 2$ and n is even, it is not hard to see (and a theorem of Kempe) that the ring of invariants is generated in degree 1. Thus the GIT quotient $(\mathbb{P}^1)^n//SL(2)$ naturally comes with a projective embedding, whose coordinates correspond to

$2 \times (n/2)$ tableaux; we call the corresponding coordinates *tableaux coordinates*. In this ($m = 2$, n even) case, it is helpful to interpret the invariants as directed graphs n vertices, where for each column $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ we have the edge $\vec{ij}$ (see [HMSV4] **[external ref –R]**). In this language, the straightening algorithm gives as a basis the upwards-oriented non-crossing graphs (those graphs with only edges $\vec{ij}$ with $j > i$, where when represented with the vertices cyclically arranged around a circle, no two edges cross). For example, Figure 4 gives a basis for $R_1(M_8)$. The following information is omitted to highlight the symmetries: the vertices are labeled cyclically 1 through 8 (it doesn't matter to us where one starts), and edges are upwards-oriented (if $i < j$, edge ij is oriented $\vec{ij}$).

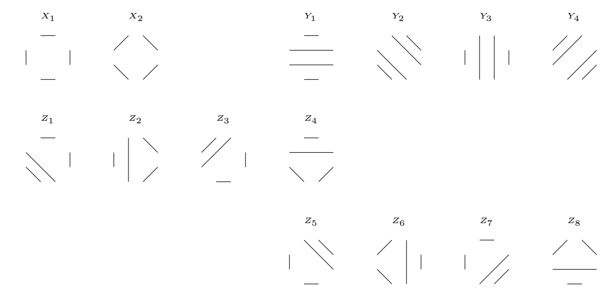

FIGURE 4. The fourteen non-crossing matchings on eight points.**[add pts –R]**

In m is arbitrary, $d = 1$, and n is divisible by m, the description of the degree 1 invariants, with its S_n action, is precisely usual description of the irreducible S_n-representation $V_{(n/m)^m}$. If $n = 8$ and $m = 2$ or $m = 4$, the corresponding representation has dimension 14, so $\dim R_1(M_8) = \dim R_1(N_8) = 14$.

If $n = 8$ and $m = 2$, we have the quadratic relation

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\ \hline 2&4&6&8\\ \hline\end{array} \times \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\ \hline 3&4&7&8\\ \hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2&4&7&8\\ \hline\end{array} \times \begin{array}{|c|c|c|c|}\hline 1&2&5&7\\ \hline 3&4&6&8\\ \hline\end{array}.$$

(This is clearly a relation: each column appears the same number of times on each side.) All four tableaux are semistandard, so this equation is nonzero. This is an example of a *simple binomial relation*, central to [HMSV4]. With appropirate labelling of vertices, in terms of the variables of Figure 4, the relation is

$$X_2Y_1 = Z_4Z_8. \tag{1}$$

We will meet this relation again in the proof of Proposition 3.2, which will show that the simple binomials span an irreducible 14-dimensional representation of type $V_{2,2,2,2}$.

2.2. **Representation-theoretic preliminaries: S_8-decomposition of ideals.** Recall that we are working over a field k of characteristic 0 (although all statements hold over $\mathbb{Z}[1/30]$). We will repeatedly use Schur-Weyl duality: if W is a vector space, then $W^{\otimes n}$ is the direct sum over partitions λ of n with at most $\dim W$ parts of $S_\lambda(W) \otimes V_\lambda$, where S_λ is the Schur functor associated to W and V_λ as described above is an irreducible S_n-representation. Each such summand appears with multiplicity one. For example, let $n = 8$ (which will be the case throughout this paper) and let $W = \mathbb{C}^2$. Then $S_\lambda W$ are distinct irreducible representations of $GL(2)$. The underlying $SL(2)$-representation on S_{λ_1,λ_2} corresponds to $\lambda_1 - \lambda_2$. (Translation: in the partition, remove all vertical columns of "maximum length", i.e. two.) Thus $R_1(M_8) = (W^{\otimes 8})^{SL_2}$ corresponds to those partitions of 8 into 2 parts λ_1, λ_2, where $\lambda_1 = \lambda_2$. In other words, as an S_8-representation, $R_1(M_8) = V_{4,4}$, as described earlier.

The decomposition of $I_d(M_8)$ into irreducible S_8-representations can be readily determined as follows:

$$I_d(M_8) = \ker(\mathrm{Sym}^d(R_1(M_8)) \twoheadrightarrow R_d(M_8)),$$

and $\mathrm{Sym}^d(R_1(M_8))$ may be determined from character theory (using the fact that $R_1(M_8)$ carries the representation $V_{4,4}$ (§2.1), and the representation on

$$R_d(M_8) = \Gamma((\mathbb{P}^1)^8, \mathbb{P}(d, \dots, d))^{SL(2)}$$

can be determined by Schur-Weyl duality.

Similarly, information about the decomposition $I_d(N_8)$ into irreducible S_8-representations can be readily determined by the map

$$I_d(N_8) = \ker(\mathrm{Sym}^d(R_1(N_8)) \to R_d(N_8)).$$

Caution: the map $\mathrm{Sym}^d(R_1(N_8)) \to R_d(N_8))$ is *not* in general a surjection — the analogue of Kempe's theorem doesn't hold.**[Refer forward to subsection of §8 –R]**

The particular facts we need are the following. The first was proved with 8 replaced by arbitrary even n in [HMSV4, Prop. 6.5], but can be verified for $n = 8$ as described above, or using the methods of Proposition 2.2(a) below.

Proposition 2.1. *We work over a characteristic 0 field k. In the following table, each representation is multiplicity free. The set of irreducibles it contains corresponds to the given set of partitions.*

S_8-representation	Set of partitions of n
$\mathrm{Sym}^2(R_1(M_8))$	at most four parts, all even
$\bigwedge^2 R_1(M_8)$	exactly four parts, all odd
$R_1(M_8)^{\otimes 2}$	union of previous two sets
$I_2(M_8)$	exactly four parts, all even

Proposition 2.2. *We work over a characteristic 0 field k. All statements refer to S_8-representations.*

(a) "The skew cubic." *Up to scalar, there is a single skew-invariant in* $\mathrm{Sym}^3 R_3(M_8)$, *and it is a relation, i.e., lies in*

$$\ker\left(\mathrm{Sym}^3 R_1(M_8) \twoheadrightarrow R_3(M_8)\right).$$

(b) "The fourteen quadrics." *There is a single representation of type $V_{2,2,2,2}$ in the degree 2 part of the ideal of M_8, i.e. in*

$$\ker\left(\mathrm{Sym}^2 R_1(M_8) \twoheadrightarrow R_2(M_8)\right).$$

(c) "The skew quintic." *There is a non-zero skew-invariant relation in* $\mathrm{Sym}^5 R_1(N_8)$ *vanishing on N_8, i.e.*

$$\ker\left(\mathrm{Sym}^5 R_1(N_8) \to R_5(N_8)\right)$$

has a skew-quintic in the kernel.

(d) "The fourteen quartics." *There is a representation of type $V_{4,4}$ in the degree 4 part of the ideal of N_8, i.e. in*

$$\ker\left(\mathrm{Sym}^4 R_1(N_8) \to \mathrm{Sym}(N_8)\right).$$

We will verify "uniqueness" in (c) in (d) in the proof of Proposition 3.7: (c) there is (up to scalar) there is precisely one skew-invariant relation in $I_5(N_8)$, and (d) precisely one representation of type $V_{4,4}$ in $I_4(N_8)$.

Proof. (a) First verify that $\mathrm{Sym}^3 R_1(M_8)$ has a single *sgn* component. Then note that $R_3(M_8)$ has no *sgn* component: $(\mathrm{Sym}^3(k^2))^{\otimes 8})$ has no sign component because by Schur-Weyl duality it contains no S_8-representation with more than $4 = \dim \mathrm{Sym}^3(k^2)$ rows. (Alternatively, as in [HMSV3, §2], use the fact that the Vandermonde has too high degree.)

(b) follows from Proposition 2.1. Alternatively, use the method of part (a).

(c) and (d) follow from comparing the appropriate representations in $\mathrm{Sym}^d(R_1(N_8))$ and $R_d(N_8)$ for $d = 4, 5$, using Schur-Weyl duality for the latter. □

3. The web of relationships between M_8 and N_8, via the skew cubic C and the skew quintic Q

Consider the (nonzero) cubic C, written in terms of the variables of Figure 4:

$$\begin{aligned} C \;=\;\; & X_1X_2(X_1 + X_2) + X_1X_2(Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8) \\ & - (X_1Y_2Y_4 + X_2Y_3Y_1) + (X_1Z_2Z_6 + X_2Z_3Z_7 + X_1Z_4Z_8 + X_2Z_5Z_1) \\ & + (Y_1Z_2Z_6 + Y_2Z_3Z_7 + Y_3Z_4Z_8 + Y_4Z_5Z_1) - (Z_1Z_2Z_3 + Z_2Z_3Z_4 \\ & + Z_3Z_4Z_5 + Z_4Z_5Z_6 + Z_5Z_6Z_7 + Z_6Z_7Z_8 + Z_7Z_8Z_1 + Z_8Z_1Z_2). \end{aligned}$$

Proposition 3.1. *The cubic C is skew-invariant.*

By Proposition 2.2(a), C is a relation; this will also follow from Proposition 3.2 and the Euler formula. We have thus found the skew cubic of Proposition 2.2(a).
Proof. Cyclically rotating the labels on the eight vertices of the graphs of Figure 4 clearly changes the sign of C. One readily checks by hand that swapping two chosen adjacent labels changes the sign of C (using the Plücker relations once). □

Proposition 3.2. *The fourteen partial derivatives of C span the same vector space as the simple binomial quadrics (the S_8-orbit of* (1)*). This representation is of type $V_{2,2,2,2}$.*

Proof. As $R_1(M_8)$ carries an S_8-representation of type $V_{4,4}$ (§2.1), the partial derivatives of the skew cubic C form an irreducible representation $V_{4,4} \otimes \mathrm{sgn} = V_{2,2,2,2}$. Observe that $\partial C/\partial Y_3 = -X_2Y_1 + Z_4Z_8$ is the simple binomial relation (1). Hence by applying the S_8-action, each simple binomial can be described as a partial derivative; and conversely each partial derivative can be interpreted as a combination of simple binomials. The last sentence of the statement then follows. $\square$

Thus the fivefold M_8 is contained in the singular locus of C. We will later show **[internal ref –R]** that M_8 *is* the singular locus of C in a strong sense: the 14 partial derivatives of the skew cubic C generate $I_*(M_8)$. Translation: $I_*(M_8)$ is the Jacobian ideal. But we won't need this fact in this section.

Proposition 3.3. *Suppose the ground field k is $\mathbb{Q}$. Let H be the Hessian of C (the determinant of the 14×14 Hessian matrix, hence degree 14). Then H does not contain C (so $\deg H \cap C = 42$), and the irreducible components of $H \cap C$ have degree 21 or 42.*

(This proof uses the single computer calculation we need in §3.)

Note that the proof makes essential use of the fact that k is not algebraically closed. We will later **[internal ref –R]** deduce that $H \cap C$ has multiplicity 2, and its reduction has degree 21, and that this holds over *any* field k of characteristic 0

Proof. We suitably choose a plane $\mathbb{P}^2 \subset \mathbb{P}^{13}$, and observe by computer that the intersection of $H \cap \mathbb{P}^2$ with $C \cap \mathbb{P}^2$ has is an irreducible degree 21 (dimension 0) subscheme, appearing with multiplicity 2. (Short Macaulay2 code is given in `http://math.stanford.edu/~vakil/files/cubic.m2`.) The desired result follows. $\square$

3.1. **The (projective) dual map from C contracts $\mathrm{Sec}(M_8)$.** The projective dual map $D : C \dashrightarrow \mathbb{P}^{13\vee}$ (sending a smooth point of C to its tangent space) blows up its singular locus, which includes M_8. The dual map is given by the fourteen partials of C, which form a $V_{2,2,2,2}$-representation (Prop. 3.2), so the dual map naturally maps to $\mathbb{P}V_{2,2,2,2}$.

The dual to the cubic C is a hypersurface: $H \cap C \neq C$ by Proposition 3.3 (this can also be done directly, without a computer), so C is not contracted by the dual map.

As argued in §1.1, Bezout's theorem implies that $\mathrm{Sec}(M_8) \subset C$, and every secant line ℓ to M_8 is contracted by the dual map, so $\mathrm{Sec}(M_8)$ is contained in the exceptional divisor of the dual map $C \dashrightarrow \mathbb{P}V_{2,2,2,2}$.

Theorem 3.4. *The rational map $\mathrm{Sec}(M_8) \dashrightarrow \mathbb{P}V_{2,2,2,2}$ maps dominantly onto N_8'.*

Proof. We do this by describing a lift of $\mathrm{Sec}(M_8) \dashrightarrow \mathbb{P}V_{2,2,2,2}$ to $\mathrm{Sec}(M_8) \dashrightarrow N_8$, in terms of the Segre map $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ (as promised in §1.1). We know the secant lines to M_8 are contracted, so we need only describe the rational map $M_8 \times M_8 \dashrightarrow N_8$.

Consider two general points of M_8, interpreted as octuples $(x_1, \dots, x_8)$ and $(y_1, \dots y_8)$ of points in $\mathbb{P}^1$. For simplicity of algebra we consider these as inhomogeneous coordinates (so for example by x_i we mean $[1; x_i]$).

We will show the dual map sends the secant line joining these two points of M_8 to the image in $\mathbb{P}V_{2,2,2,2}$ corresponding to the 8 points in $\mathbb{P}^3$ given by $p_i := [1; x_i; y_i; x_i y_i]$ — the image of $([1; x_i], [1; y_i])$ under the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Because the secant line is contracted, it suffices to check the image of the point of $\mathrm{Sec}(M_8)$ whose tableaux coordinates (§2.1) are given by $(T(\vec{x}_i) + T(\vec{y}_i))_T$, where T runs through the 2×4 tableaux. In order to show the claimed isomorphism between two irreducible representations of type $V_{2,2,2,2}$ it suffices to check a single

vector in each corresponding to a fixed partition. We will check that **[how to get rid of first 3 numbers? –R]**

$$\left(\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array}(1;x_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\\hline 2&4&6&8\\\hline\end{array}(1;y_i)_{1\le i\le 8}\right) \tag{2}$$

$$\left(\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array}(1;x_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&8\\\hline\end{array}(1;y_i)_{1\le i\le 8}\right) \tag{3}$$

$$-\left(\begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline\end{array}(1;x_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline\end{array}(1;y_i)_{1\le i\le 8}\right) \tag{4}$$

$$\left(\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&7&8\\\hline\end{array}(1;x_i)_{1\le i\le 8} + t\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&7&8\\\hline\end{array}(1;y_i)_{1\le i\le 8}\right) \tag{5}$$

(the evaluation of the partial derivative (1) of the skew cubic C, at a particular point of the secant line corresponding to t) equals

$$t\begin{array}{|c|c|}\hline 1&5\\\hline 2&6\\\hline 3&7\\\hline 4&8\\\hline\end{array}(1; x_i; y_i; x_i y_i)_{1 \le i \le 8}. \tag{6}$$

(Because the secant line is contracted, we are not surprised to see that the t's will largely fall out of (5).) For simplicity, let $x_{ij} := x_i - x_j$, and $y_{ij} = y_i - y_j$. Then (5) is

$$\begin{aligned} & (x_{21}x_{43}x_{65}x_{87} + t\, y_{21}y_{43}y_{65}y_{87})(x_{31}x_{42}x_{75}x_{86} + t\, y_{31}y_{42}y_{75}y_{86}) \\ - & (x_{21}x_{43}x_{75}x_{86} + t\, y_{21}y_{43}y_{75}y_{86})(x_{31}x_{42}x_{65}x_{87} + t\, y_{31}y_{42}y_{65}y_{87}). \end{aligned}$$

When we expand this out, all terms involving only x-variables cancel (expected, as (1) is a relation satisfied by points of M_8), and similarly for the y-variables. What remains is t times

$$\begin{array}{rcl} x_{21}x_{43}x_{65}x_{87}y_{31}y_{42}y_{75}y_{86} & + & y_{21}y_{43}y_{65}y_{87}x_{31}x_{42}x_{75}x_{86} \\ -x_{21}x_{43}x_{75}x_{86}y_{31}y_{42}y_{65}y_{87} & - & y_{21}y_{43}y_{75}y_{86}x_{31}x_{42}x_{65}x_{87} \end{array}$$

$$= (x_{21}x_{43}y_{31}y_{42} - x_{31}x_{42}y_{21}y_{43})(x_{65}x_{87}y_{75}y_{86} - y_{65}y_{87}x_{75}x_{86})$$

By comparing this to (6), it suffices to show that

$$x_{21}x_{43}y_{31}y_{42} - x_{31}x_{42}y_{21}y_{43} = \det(1\ x_i\ y_i\ x_i y_i)_{i=1,2,3,4}$$

which can be verified by hand. (More elegantly: both sides are quartic. On each side, there are only monomials of the form $x_i x_k y_j y_k$, i.e., of the 4 possible subscripts, 3 appear, one both on the x side and on the y side, and they form one S_4 orbit, with a sign representation.) $\square$

Corollary 3.5. *We have (a)* $\dim \mathrm{Sec}(M_8) = 11$ *and (b)* $\deg \mathrm{Sec}(M_8) = 21$ *or* 42.

Proof. (a) Of course $\dim \mathrm{Sec}(M_8) \le 2 \dim(M_8) + 1 = 11$. For the opposite inequality, note that $\dim N_8 = 9$, and we see that the preimage (under the dominant rational map $\mathrm{Sec}(M_8) \dashrightarrow N_8$) of a general point of N_8 has dimension at least 2: one corresponding to the one-parameter family of quadrics through 8 general points in $\mathbb{P}^3$, and one corresponding to the secant line joining those two point of M_8 (corresponding to the two octuples of points in $\mathbb{P}^1$). Thus $\dim \mathrm{Sec}(M_8) \ge 11$.

Part (b) then follows from Proposition 3.3. Note that Proposition 3.3 assumes the base field k is $\mathbb{Q}$, but it suffices to show Corollary 3.5 in this case, as the statement of Corollary 3.5 behaves well with respect to extension of base field. □

We pause to take stock of where we are. We now know that $H \cap C$, which has degree 42, contains $\operatorname{Sec}(M_8)$ (which has degree 21 or 42) as a component. We will soon see that $H \cap C$ contains $\operatorname{Sec}(M_8)$ with multiplicity 2 (and $\deg \operatorname{Sec}(M_8) = 21$).**[internal ref –R]**

3.2. **There is a unique skew quintic relation Q in $I_*(N'_*)$, and it defines the dual hypersurface to C.** Let $Q' \subset \mathbb{P}V_{2,2,2,2}$ be the dual hypersurface to C. Let $y_1, \ldots, y_{14}$ be projective coordinates on $V_{2,2,2,2}$. Let $D' : Q' \to C$ be the inverse projective dual map to $D : C \dashrightarrow \mathbb{P}V_{2,2,2,2}$.

Proposition 3.6. *The degree d of the dual hypersurface Q' is at least 5.*

Proof. Suppose $d \leq 4$. The map D contracts $\operatorname{Sec}(M_8)$, and hence sends $\operatorname{Sec}(M_8)$ into the singular locus of the dual hypersurface. The partial derivatives Q'_i of the dual hypersurface have degree $d-1$, and vanish on $\operatorname{Sing} Q'$ but not on all of Q'. Thus $D^*Q'_i$ vanishes on $\operatorname{Sec}(M_8)$ but not C. Now $D^*Q'_i$ has degree $2(d-1)$. But $\deg(\operatorname{Sec}(M_8)) \geq 21$ (Proposition 3.3), so by Bezout's theorem, $\deg D^*Q'_i \cap C = 2(d-1) \times 3 \geq 21$, from which $d \geq 5$. □

Proposition 3.7. *There is a unique skew quintic relation Q, and it defines the dual hypersurface to C.*

[WAVE FRONT. Also, we don't know that $\dim N'_8 = 9$, so work on N_8. –R]

Proof. The dual rational map $D' : Q' \dashrightarrow C$ is given by $[\partial Q'/\partial y_1; \ldots; \partial Q'/\partial y_{14}]$. Choose a $V_{4,4}$-subrepresentation in the space of quartic relations of N_8, as described in Proposition 2.2(d). (By the end of the proof, we will see that there is only one such subrepresentation.) Let $Q_1, \ldots, Q_{14}$ be a basis of this vector space, corresponding (via isomorphism of $V_{4,4}$-representations, which is unique up to scalar by Schur's lemma) to the basis $\partial Q'/\partial y_1, \ldots, \partial Q'/\partial y_{14}$ of partials **[WAIT. Why do we have the right representations? Why is Q' skew? –R]**. Now Q_i vanishes on N_8, and hence D^*Q_i vanishes on $D^*N_8 = \operatorname{Sec}(M_8)$. By Proposition 3.6, D^*Q_i doesn't vanish on all of C (as otherwise the dual of C would lie in the quartic $Q_i = 0$). Now D^*Q_i has degree 8, so by Bezout's theorem, $D^*Q_i \cap C$ has degree 24. Now $D^*Q_i \cap C$ contains $\operatorname{Sec}(M_8)$ (D maps $\operatorname{Sec}(M_8)$ into N_8, and Q_i vanishes on N_8), so by Proposition 3.5(b), $\deg \operatorname{Sec}(M_8) = 21$, and the residual divisor to $\operatorname{Sec}(M_8)$ in D^*Q_i has degree 3.

Now $\operatorname{Pic} \mathbb{P}V_{4,4} \to \operatorname{Pic} C$ is an isomorphism [G-SGA2]**[get ref from Hartshorne Ex. III.11.6 –R]**. Also, C is factorial (by the Lefschetz hyperplane theorem for hypersurfaces, [G-SGA2, Exp. XII, Cor. 3.6] — this can also be shown by hand using Nagata's criterion for factoriality [E, Lem. 19.20] applied to the explicit description of the cubic (3)). Thus the residual divisor to $\operatorname{Sec}(M_8)$ in $D^*Q_i \cap C$ may be identified with a section of $\mathcal{O}_C(1)$, and Schur's lemma identifies (up to scalar) the following two $V_{4,4}$-representations:

- the 14-dimensional vector space of quartics Q_i
- the 14-dimensional vector space $\Gamma(\mathbb{P}V_{4,4}, \mathcal{O}(1))$.

Thus the dual map $D' : Q' \dashrightarrow C$ is given by $[Q_1; \ldots; Q_{14}]$. Thus the rational maps

$$[Q_1; \ldots; Q_{14}] \text{ and } [\partial Q'/\partial y_1; \ldots; \partial Q'/\partial y_{14}]$$

are the same. Now the quartics Q_i have no common divisorial component on the hypersurface Q' (or else their quotient by this divisorial relation would be a lower-degree polynomial, which when pulled back to C, would vanish on $\operatorname{Sec}(M_8)$ but not on all of C, which is impossible by Bezout's theorem). Thus it must be true that there is an invariant homogeneous polynomial P such that $\partial Q'/\partial y_i = PQ_i$ for all i. By Euler's formula, $0 \neq (\deg Q')Q' = \sum_{i=1}^{14} y_i \partial Q'/\partial y_i = P \sum_1^{14} y_i Q_i$. In particular, $\sum y_i Q_i \neq 0$ is a quintic skew relation. Call this quintic skew relation Q.

We finally claim that D maps C into Q: on C, we have

$$\begin{aligned} D^*Q &= D^* \sum y_i Q_i \\ &= \sum (D^* y_i)(D^* Q_i) \\ &= \sum \frac{\partial C}{\partial x_i}(x_i \cdot \operatorname{Sec} M_8) \end{aligned}$$

[This needs explaining; that last line is incomprehensible, and the x_i are not yet defined –R] (from our earlier identification of D^*Q_i as the hyperplane $x_i = 0$ union $\operatorname{Sec} M_8$). But on C $\sum x_i C_i = 3C = 0$. Thus as the dual of C is of degree at most 5 and contained in Q which has degree 5, the proof is complete. □

[Remark: uniqueness of Q and Q_i. –R]

Furthermore, the dual map $D' : Q \dashrightarrow C$ will blow up precisely $\operatorname{Sing} Q$, cut out by the 14 quartics. The exceptional divisor on C is $\cap_{i=1}^{14} D^*Q_i$, which we have established is $\operatorname{Sec} M_8$. But $D(M_8) = N'_8$. Also, the Hessian of C vanishes precisely along the exceptional divisor. Thus we have:

Proposition 3.8. *(a)* $\operatorname{Sing}(Q) = N'_8$.
(b) The Hessian vanishes to order 2 along $\operatorname{Sec}(M_8)$.

This leads to a natural question: the Hessian is a perfect square modulo C? Is there some Pfaffian interpretation of the Hessian? [See septic discussion, not yet added.]

I haven't yet discussed how H is double along $\operatorname{Sec} M_8$.

[Further questions and speculation: What is the septic? –R]

4. The partial derivatives of C have no linear syzygies I

The goal of §4–6 is to establish the following:

Theorem 4.1. *The partial derivatives of C have no linear syzygies.*

This proposition means that if $\sum_{i=1}^{14} x_i \partial_i s = 0$ with x_i in $\operatorname{Sym}^1(V)$ then $x_i = 0$ for all i. We will not prove Theorem 4.1 in this section but we will reduce the proof to a problem that we will solve in §6.

Remark 4.2. Theorem 4.1 implies that the ideal $I_*(M_8)$ is cut out by quadrics. Reason: by [HMSV4] **[external ref –R]**, $I_*(M_8)$ is cut out by quaics and cubics. One can readily check by hand (counting noncrossing graphs) that $\dim(I_3(M_8)) = 14^2$. By Theorem 4.1, the map of 196-dimensional vector spaces $R_1(M_8) \otimes I_2(M_8) \to I_3(M_8)$ has no kernel and is thus surjective.

Remark 4.3. In fact, Theorem 4.1 holds away from characteristic 3. In characteristic 3, the Euler formula yields a linear syzygy among the 14 quadrics, and the skew cubic C is the remaining generator of the ideal. **[[Refer to the e-print version, Thm 1.2 and section 9.] [This is actually stated above.] [Note that this requires use of a computer, ulike the proof of Theorem 4.1.] –R]**

We prove Theorem 4.1 by the following strategy.

(a) Let $\Psi : \mathrm{End}(R_8^{(1)}) \to I_8^{(3)}$ be the map given by $A \mapsto AC$, where AC is defined via the natural action of the Lie algebra $\mathrm{End}(R_8^{(1)}) \cong \mathfrak{gl}(14)$ on $\mathrm{Sym}^3(R_8^{(1)})$. We first observe that the space of linear syzygies between the partial derivaties of C is exactly $\mathfrak{g} = \ker \Psi$. We note that $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{End}(R_8^{(1)})$ and is stable under the action of S_8.

(b) Next, using general theory developed in §5 concerning G-stable Lie subalgebras of $\mathrm{End}(V)$, where V is a representation of G, and the classification of simple Lie algebras, we show that the only S_8-stable Lie subalgebras of $\mathrm{End}(R_8^{(1)})$ are 0, $\mathfrak{so}(14)$ and $\mathfrak{sl}(14)$ (ignoring the center). Thus $\mathfrak{g}$ must be one of these three Lie algebras.

(c) Finally, we show that $\mathfrak{so}(14)$ does not annihilate any non-zero cubic. As $\mathfrak{g}$ is the annihilator of C we conclude $\mathfrak{g} = 0$.

We now implement this strategy.

Consider the composition

$$\widetilde{\Psi} : \mathrm{End}(V) \otimes \mathrm{Sym}^3(V) = V \otimes V^* \otimes \mathrm{Sym}^3(V) \to V \otimes \mathrm{Sym}^2(V) \to \mathrm{Sym}^3(V)$$

where the first map is the partial derivative map and the second map is the multiplication map. One easily verifies that $\widetilde{\Psi}$ is just the map which expresses the action of the Lie algebra $\mathfrak{gl}(V) = \mathrm{End}(V)$ on the third symmetric power of its standard representation V. We are trying to show that $\widetilde{\Psi}$ induces an injection

$$\Psi : \mathrm{End}(V) \otimes kC \to I^{(3)}.$$

(We know that Ψ maps $\mathrm{End}(V) \otimes kC$ into $I^{(3)}$ since we know that the partial derivatives of C belong to $I^{(2)}$.) Indeed, the kernel of Ψ is the space of linear syzygies between the partial derivatives of C. Now, the kernel of Ψ is equal to $\mathfrak{g} \otimes kC$, where $\mathfrak{g}$ is the annihilator in $\mathfrak{gl}(V)$ of C. Thus Theorem 4.1 is equivalent to the following:

Proposition 4.4. *We have* $\mathfrak{g} = 0$.

We know two important things about $\mathfrak{g}$: first, $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, as it is the annihilator of some element in a representation of $\mathfrak{gl}(V)$; and second, $\mathfrak{g}$ is stable under the group G, as the action map Ψ is G-equivariant and kC is stable under G. We will prove Proposition 4.4 by first classifying the G-stable Lie subalgebras of $\mathfrak{gl}(V)$ and then proving that $\mathfrak{g}$ cannot be any of them except zero.

Before continuing, we note a few results:

Proposition 4.5. *The skew-invariant cubic C belongs to $I^{(3)}$.*

Proof. We have already remarked that any element of the Lie algebra $\mathfrak{gl}(V)$ takes C into $I^{(3)}$. Now, the identity matrix in $\mathfrak{gl}(V)$ acts by multiplication by 3 on $\mathrm{Sym}^3(V)$, and thus $3s$, and thus C, belongs to $I^{(3)}$. □

Proposition 4.6. *The Lie algebra $\mathfrak{g}$ is contained in $\mathfrak{sl}(V)$.*

Proof. The trace map $\mathfrak{gl}(V) \to k$ is G-equivariant, where G acts trivially on the target. Thus if $\mathfrak{g}$ contained an element of non-zero trace it would have to contain a copy of the trivial representation. Thanks to Proposition 2.1, we know that $\mathfrak{gl}(V) \cong V^{\otimes 2}$ is multiplicity free as a representation of G. Thus the one-dimensional space spanned by the identity matrix is the only copy of the trivial representation in $\mathfrak{gl}(V)$. Therefore, if $\mathfrak{g}$ were not contained in $\mathfrak{sl}(V)$ then it would contain the center of $\mathfrak{gl}(V)$. However, we know that the identity matrix does not annihilate C. Thus $\mathfrak{g}$ must be contained in $\mathfrak{sl}(V)$. □

Proposition 4.7. *Theorem 4.1 implies that $I^{(2)}$ generates $I^{(3)}$.*

Proof. The image of Ψ is exactly the subspace of $I^{(3)}$ generated by $I^{(2)}$. Thus $I^{(2)}$ generates $I^{(3)}$ if and only if Ψ is surjective. Now, V being 14 dimensional, the dimension of $\mathrm{End}(V)$ is 196. It happens that this is exactly the dimension of $I^{(3)}$ as well. Thus the domain and target of Ψ have the same dimension, and so surjectivity is equivalent to injectivity. □

5. Interlude: G-stable Lie subalgebras of $\mathfrak{sl}(V)$

In this section G will denote an arbitrary finite group and V an irreducible representation of G over an algebraically closed field k of characteristic zero. We investigate the following general problem:

Problem 5.1. *Determine the G-stable Lie subalgebras of $\mathfrak{sl}(V)$.*

We do not obtain a complete answer to this question, but we do prove strong enough results to determine the answer in our specific situation. We will use the term *G-subalgebra* to mean a G-stable Lie subalgebra.

5.1. Some structure theory.

Our first result is the following:

Proposition 5.2. *Let V be an irreducible representation of G. Then every solvable G-subalgebra of $\mathfrak{sl}(V)$ is abelian and consists solely of semi-simple elements.*

Proof. Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{sl}(V)$. By Lie's theorem, $\mathfrak{g}$ preserves a complete flag $0 = V_0 \subset \cdots \subset V_n = V$. The action of $\mathfrak{g}$ on each one-dimensional space V_i/V_{i-1} must factor through $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$; thus $[\mathfrak{g},\mathfrak{g}]$ acts by zero on V_i/V_{i-1} and so carries V_i into V_{i-1}. The space $[\mathfrak{g},\mathfrak{g}]V$ is therefore not all of V. On the other hand, $[\mathfrak{g},\mathfrak{g}]$ is G-stable and therefore so is $[\mathfrak{g},\mathfrak{g}]V$. From the irreducibility of V we conclude $[\mathfrak{g},\mathfrak{g}]V = 0$, from which it follows that $[\mathfrak{g},\mathfrak{g}] = 0$. Thus $\mathfrak{g}$ is abelian.

Now let R be the subalgebra of $\mathrm{End}(V)$ generated (under the usual multiplication) by $\mathfrak{g}$. Let R_s (resp. R_n) denote the set of semi-simple (resp. nilpotent) elements of R. Then R_s is a subring of R, R_n is an ideal of R and $R = R_s \oplus R_n$. As $R_n^m = 0$ for some m, the space R_nV is not all of V. As it is G-stable it must be zero, and so $R_n = 0$. We thus find that $R = R_s$ and so all elements of R, and thus all elements of $\mathfrak{g}$, are semi-simple. □

Let V be a representation of G. We say that V is *imprimitive* if there is a decomposition $V = \bigoplus_{i \in I} V_i$ of V into non-zero subspaces, at least two in number, such that each element of G carries each V_i into some V_j. We say that V is *primitive* if it is not imprimitive. Note that primitive implies irreducible. An irreducible representation is imprimitive if and only if it is induced from a proper subgroup.

Proposition 5.3. *Let V be an irreducible representation of G. Then V is primitive if and only if the only abelian G-subalgebra of $\mathfrak{sl}(V)$ is zero.*

Proof. Let V be an irreducible representation of G and let $\mathfrak{g}$ be a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$. We will show that V is imprimitive. By Proposition 5.2 all elements of $\mathfrak{g}$ are semi-simple. We thus get a decomposition $V = \bigoplus V_\lambda$ of V into eigenspaces of $\mathfrak{g}$ (each λ is a linear map $\mathfrak{g} \to k$). As $\mathfrak{g}$ is G-stable, each element of G must carry each V_λ into some $V_{\lambda'}$. Note that if $V = V_\lambda$ for some λ then $\mathfrak{g}$ would consist of scalar matrices, which is impossible as $\mathfrak{g}$ is contained in $\mathfrak{sl}(V)$. Thus there must be at least two non-zero V_λ and so V is imprimitive.

We now establish the other direction. Thus let V be an irreducible imprimitive representation of G. We construct a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$. Write $V = \bigoplus V_i$ where the elements of G permute the V_i. Let p_i be the endomorphism of V given by projecting onto V_i and then including back into V and let $\mathfrak{g}$ be the subspace of $\mathfrak{gl}(V)$ spanned by the p_i. Then $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{gl}(V)$ since $p_i p_j = 0$ for $i \neq j$. Furthermore, $\mathfrak{g}$ is G-stable since for each i we have $g p_i g^{-1} = p_j$ for some j. Intersecting $\mathfrak{g}$ with $\mathfrak{sl}(V)$ gives a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$ (the intersection is non-zero because $\mathfrak{g}$ has dimension at least two and $\mathfrak{sl}(V)$ has codimension one). □

We have the following important consequence of Proposition 5.3:

Corollary 5.4. *Let V be a primitive representation of G. Then every G-subalgebra of $\mathfrak{sl}(V)$ is semi-simple.*

Proof. Let $\mathfrak{g}$ be a G-subalgebra of $\mathfrak{sl}(V)$. The radical of $\mathfrak{g}$ is then a solvable G-subalgebra and therefore vanishes. Thus $\mathfrak{g}$ is semi-simple. □

Proposition 5.3 can also be used to give a criterion for primitivity.

Corollary 5.5. *Let V be an irreducible representation of G such that each non-zero G-submodule of $\mathfrak{sl}(V)$ has dimension at least that of V. Then V is primitive.*

Proof. Let $\mathfrak{g}$ be a abelian G-subalgebra of $\mathfrak{sl}(V)$. We will show that $\mathfrak{g}$ is zero. By Proposition 5.2 $\mathfrak{g}$ consists of semi-simple elements and is therefore contained in some Cartan subalgebra of $\mathfrak{sl}(V)$. This shows that $\dim \mathfrak{g} < \dim V$. Thus, by our hypothesis, $\mathfrak{g} = 0$. □

Let V be a primitive G-module and let $\mathfrak{g}$ be a G-subalgebra. As $\mathfrak{g}$ is semi-simple it decomposes as $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ where each $\mathfrak{g}_i$ is a simple Lie algebra. The $\mathfrak{g}_i$ are called the *simple factors* of $\mathfrak{g}$ and are unique. As the simple factors are unique, G must permute them. We call $\mathfrak{g}$ *prime* if the action of G on its simple factors is transitive. Note that in this case the $\mathfrak{g}_i$'s are isomorphic and so $\mathfrak{g}$ is "isotypic." Clearly, every G-subalgebra of $\mathfrak{sl}(V)$ breaks up into a sum of prime subalgebras and so it suffices to understand these.

5.2. **The action of a G-subalgebra on V.** We now consider how a G-stable subalgebra acts on V:

Proposition 5.6. *Let V be a primitive G-module, let $\mathfrak{g}$ be a G-subalgebra of $\mathfrak{sl}(V)$ and let $\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into simple factors.*

(1) *The representation of $\mathfrak{g}$ on V is isotypic, that is, it is of the form $V_0^{\oplus m}$ for some irreducible $\mathfrak{g}$-module V_0.*

(2) *We have a decomposition $V_0 = \bigotimes_{i\in I} W_i$ where each W_i is a faithful irreducible representation of $\mathfrak{g}_i$.*

(3) *We have $V_0 \cong V_0^g$ for each element g of G. (Here V_0^g denotes the $\mathfrak{g}$-module obtained by twisting V_0 by the automorphism g induces on $\mathfrak{g}$.)*

(4) *If $\mathfrak{g}$ is a prime subalgebra then for any i and j one can choose an isomorphism $f : \mathfrak{g}_i \to \mathfrak{g}_j$ so that W_i and $f^* W_j$ become isomorphic as $\mathfrak{g}_i$-modules.*

Proof. (1) Since $\mathfrak{g}$ is semi-simple we get a decomposition $V = \bigoplus V_i^{\oplus m_i}$ of V as a $\mathfrak{g}$-module, where the V_i are pairwise non-isomorphic simple $\mathfrak{g}$-modules. Each element g of G must take each isotypic piece $V_i^{\oplus m_i}$ to some other isotypic piece $V_j^{\oplus m_j}$ since the map $g : V \to V^g$ is $\mathfrak{g}$-equivariant. As V is primitive for G, we conclude that it must be isotypic for $\mathfrak{g}$, and so we may write $V = V_0^{\oplus m}$ for some irreducible $\mathfrak{g}$-module V_0.

(2) As V_0 is irreducible, it necessarily decomposes as a tensor product $V_0 = \bigotimes_{i\in I} W_i$ where each W_i is an irreducible $\mathfrak{g}_i$-module. Since the representation of $\mathfrak{g}$ on $V = V_0^{\oplus m}$ is faithful so too must be the representation of $\mathfrak{g}$ on V_0. From this, we conclude that each W_i must be a faithful representation of $\mathfrak{g}_i$.

(3) For any $g \in G$ the map $g : V \to V^g$ is an isomorphism of $\mathfrak{g}$-modules and so $V_0^{\oplus m}$ is isomorphic to $(V_0^{\oplus m})^g = (V_0^g)^{\oplus m}$, from which it follows that V_0 is isomorphic to V_0^g.

(4) Since G acts transitively on the simple factors, given i and j we can pick $g \in G$ such that $g\mathfrak{g}_i = \mathfrak{g}_j$. The isomorphism of V_0 with V_0^g then gives the isomorphism of W_i and W_j as $\mathfrak{g}_i$-modules. □

This proposition gives a strong numerical constraint on prime subalgebras:

Corollary 5.7. *Let V be a primitive representation of G and let $\mathfrak{g} = \mathfrak{g}_0^n$ be a prime subalgebra of $\mathfrak{sl}(V)$, where $\mathfrak{g}_0$ is a simple Lie algebra. Then $\dim V$ is divisible by d^n where d is the dimension of some faithful representation of $\mathfrak{g}_0$. In particular, $\dim V \geq d_0^n$ where d_0 is the minimal dimension of a faithful representation of $\mathfrak{g}_0$.*

5.3. **Self-dual representations.** Let V be an irreducible self-dual G-module. Thus we have a non-degenerate G-invariant form $\langle , \rangle : V \otimes V \to k$. Such a form is unique up to scaling, and either symmetric or anti-symmetric. We accordingly call V *orthogonal* or *symplectic*.

Let A be an endomorphism of V. We define the *transpose* of A, denoted A^t, by the formula

$$\langle A^t v, u\rangle = \langle v, Au\rangle.$$

It is easily verified that $(AB)^t = B^t A^t$ and $({}^g A)^t = {}^g(A^t)$. We call an endomorphism A *symmetric* if $A = A^t$ and *anti-symmetric* if $A = -A^t$. One easily verifies that the commutator of two anti-symmetric endomorphisms is again anti-symmetric. Thus the set of all anti-symmetric endomorphisms forms a G-subalgebra of $\mathfrak{sl}(V)$ which we denote by $\mathfrak{sl}(V)^-$. In the orthogonal case $\mathfrak{sl}(V)^-$ is isomorphic to $\mathfrak{so}(V)$ as a Lie algebra and $\bigwedge^2 V$ as a G-module, while in the symplectic case it is isomorphic to $\mathfrak{sp}(V)$ as a Lie algebra and $\mathrm{Sym}^2(V)$ as a G-module. We let $\mathfrak{sl}(V)^+$ denote the space of symmetric endomorphisms.

Proposition 5.8. *Let V be an irreducible self-dual G-module. Assume that:*

- *$\mathrm{Sym}^2(V)$ and $\bigwedge^2 V$ have no isomorphic G-submodules; and*
- *$\mathfrak{sl}(V)^-$ has no proper non-zero G-subalgebras.*

Then any proper non-zero G-subalgebra of $\mathfrak{sl}(V)$ other than $\mathfrak{sl}(V)^-$ is commutative. In particular, if V is primitive then the G-subalgebras of $\mathfrak{sl}(V)$ are exactly 0, $\mathfrak{sl}(V)^-$ and $\mathfrak{sl}(V)$.

Proof. Let $\mathfrak{g}$ be a non-zero G-subalgebra of $\mathfrak{sl}(V)$. The intersection of $\mathfrak{g}$ with $\mathfrak{sl}(V)^-$ is a G-subalgebra of $\mathfrak{sl}(V)^-$ and therefore either 0 or all of $\mathfrak{sl}(V)^-$. First assume that the intersection is zero. Since the spaces of symmetric and anti-symmetric elements of $\mathfrak{sl}(V)$ have no isomorphic G-submodules, it follows that $\mathfrak{g}$ is contained in the space of symmetric elements of $\mathfrak{sl}(V)$. However, two symmetric elements bracket to an anti-symmetric element. It thus follows that all brackets in $\mathfrak{g}$ vanish and so $\mathfrak{g}$ is commutative. Now assume that $\mathfrak{g}$ contains all of $\mathfrak{sl}(V)^-$. It is then a standard fact that $\mathfrak{sl}(V)^-$ is a maximal subalgebra of $\mathfrak{sl}(V)$ and so $\mathfrak{g}$ is either $\mathfrak{sl}(V)^-$ or $\mathfrak{sl}(V)$. (To see this, note that $\mathfrak{sl}(V) = \mathfrak{sl}(V)^- \oplus \mathfrak{sl}(V)^+$ and so to prove the maximality of $\mathfrak{sl}(V)^-$ it suffices to show that $\mathfrak{sl}(V)^+$ is an irreducible representation of $\mathfrak{sl}(V)^-$. In the orthogonal case this amounts to the fact that, as a representation of $\mathfrak{so}(V)$, the space $\mathrm{Sym}^2(V)/W$ is irreducible, where W is the line spanned by the orthogonal form on V. The symplectic case is similar.) □

6. The partial derivatives of C have no linear syzygies II

We now complete the proof of Theorem 4.1. We return to our previous notation. We begin with the following:

Proposition 6.1. *Assume k is algebraically closed. The G-subalgebras of $\mathfrak{sl}(V)$ are exactly 0, $\mathfrak{so}(V)$ and $\mathfrak{sl}(V)$.*

Proof. We begin by noting that any irreducible representation of the symmetric group is defined over the reals (in fact, the rationals) and is therefore orthogonal self-dual. Thus $\mathfrak{so}(V) = \mathfrak{sl}(V)^-$ makes sense as a G-subalgebra.

For our particular representation V, Proposition 2.1 shows that $\mathrm{Sym}^2(V)$ has five irreducible submodules of dimensions 1, 14, 14, 20 and 56, while $\bigwedge^2 V$ has two irreducible submodules of dimensions 35 and 56. Furthermore, none of these seven irreducibles are isomorphic. As all irreducible submodules of $\mathfrak{sl}(V)$ have dimension at least that of V (which in this case is 14), we see from Corollary 5.5 that V is primitive. (Note that the one-dimensional representation occurring in $\mathrm{Sym}^2(V)$ is the center of $\mathfrak{gl}(V)$ and does not occur in $\mathfrak{sl}(V)$.)

As V is primitive, multiplicity free and self-dual, we can apply Proposition 5.8. This shows that to prove the present proposition we need only show that $\mathfrak{so}(V)$ has no proper non-zero G-subalgebras. Thus assume that $\mathfrak{g}'$ is a proper non-zero G-subalgebra of $\mathfrak{so}(V)$. As $\mathfrak{so}(V) = \bigwedge^2 V$ has two irreducible submodules we see that $\mathfrak{g}'$ must be one of these two irreducibles. In particular, this shows that $\mathfrak{g}'$ must be prime and so therefore isotypic. Now, by examining the list of all simple Lie algebras, we see that there are exactly four isotypic Lie algebras of dimension either 35 or 56:

$$\mathfrak{g}_2^4, \qquad \mathfrak{so}(8)^2, \qquad \mathfrak{sl}(3)^7, \qquad \mathfrak{sl}(6).$$

The minimal dimensions of faithful representations of $\mathfrak{g}_2$, $\mathfrak{so}(8)$ and $\mathfrak{sl}(3)$ are 7, 8 and 3. As 7^4, 8^2 and 3^7 are all bigger than $\dim V$, Corollary 5.7 rules out the first three Lie algebras above. (One can also rule out $\mathfrak{g}_2^4$ and $\mathfrak{sl}(3)^7$ by noting that the alternating group A_8 does not act non-trivially on them.) We rule out $\mathfrak{sl}(6)$ by using Proposition 5.6 and noting that $\mathfrak{sl}(6)$ has no faithful 14 dimensional isotypic representation — this is proved in Lemma 6.2 below. (One can also rule out $\mathfrak{sl}(6)$ by noting that A_8 does not act on it.) This shows that $\mathfrak{g}'$ cannot exist, and proves the proposition. □

Lemma 6.2. *The Lie algebra $\mathfrak{sl}(6)$ has exactly two non-trivial irreducible representations of dimension ≤ 14: the standard representation and its dual. It has no 14-dimensional faithful isotypic representation.*

Proof. For a dominant weight λ let V_λ denote the irreducible representation with highest weight λ. If λ and λ' are two dominant weights then a general fact valid for any semi-simple Lie algebra states

$$\dim V_{\lambda+\lambda'} \geq \max(\dim V_\lambda, \dim V_{\lambda'}).$$

(To see this, recall the Weyl dimension formula:

$$\dim V_\lambda = \prod_{\alpha^\vee > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where ρ is half the sum of the positive roots and the product is taken over the positive co-roots $\alpha^\vee$. Then note that $\langle \lambda, \alpha^\vee \rangle$ is positive for any dominant weight λ and any positive co-root $\alpha^\vee$. Thus $\dim V_{\lambda+\lambda'} \geq \dim V_\lambda$.)

Now, let $\varpi_1, \ldots, \varpi_5$ be the fundamental weights for $\mathfrak{sl}(6)$. The representation V_{ϖ_i} is just $\bigwedge^i V$, where V is the standard representation. For $2 \leq i \leq 4$ the space V_{ϖ_i} has dimension ≥ 15. Furthermore, a simple calculation shows that

$$\dim V_{2\varpi_1} = 21, \qquad \dim V_{\varpi_1+\varpi_5} = 168, \qquad \dim V_{2\varpi_5} = 21.$$

(Note that $V_{2\varpi_1}$ is $\mathrm{Sym}^2(V)$, while $V_{2\varpi_5}$ is its dual. This shows why they are 21-dimensional. To compute the dimension of $V_{\varpi_1+\varpi_5}$ we use the formula for the dimension of the relevant Schur functor, [FuH, Ex. 6.4].) Thus only V_{ϖ_1} and V_{ϖ_5} have dimension at most 14, and they each have dimension 6. Since 6 does not divide 14 we find that there are no non-trivial 14-dimensional isotypic representations. □

Remark 6.3. We can prove Proposition 6.1 whenever L has cardinality at most 14. Perhaps it is true for all L.

We now have the following:

Proposition 6.4. *The only element of $\mathrm{Sym}^3(V)$ annihilated by $\mathfrak{so}(V)$ is zero.*

Proof. As mentioned, V has a canonical non-degenerate symmetric inner product. Pick an orthonormal basis $\{x_i\}$ of V and let $\{x_i^*\}$ be the dual basis of V^*. We can think of $\mathrm{Sym}(V)$ as the polynomial ring in the x_i. The space $\mathfrak{so}(V)$ is spanned by elements of the form $E_{ij} = x_i \otimes x_j^* - x_j \otimes x_i^*$. Recall that, for an element C of $\mathrm{Sym}(V)$, the element $x_i \otimes x_j^*$ of $\mathrm{End}(V)$ acts on C by $x_i \partial_j s$, where ∂_j denotes differentiation with respect to x_j. Thus we see that C is annihilated by E_{ij} if and only if it satisfies the equation

$$x_i \partial_j s = x_j \partial_i s. \tag{7}$$

Therefore C is annihilated by all of $\mathfrak{so}(V)$ if and only if the above equation holds for all i and j.

Let C be an element of $\mathrm{Sym}^3(V)$. We now consider (7) for a fixed i and j. Write

$$s = g_3(x_j) + g_2(x_j)x_i + g_1(x_j)x_i^2 + g_0(x_j)x_i^3$$

where each g_i is a polynomial in x_j whose coefficients are polynomials in the x_k with $k \neq i, j$. Note that g_0 must be a constant by degree considerations. We have

$$x_i \partial_j s = g_3'(x_j)x_i + g_2'(x_j)x_i^2 + g_1'(x_j)x_i^3$$
$$x_j \partial_i s = x_j g_2(x_j) + 2x_j g_1(x_j)x_i + 3x_j g_0(x_j)x_i^2.$$

We thus find

$$g_2 = 0, \quad 2x_j g_1 = g_3', \quad 3x_j g_0 = g_2', \quad g_1' = 0.$$

From this we deduce that $g_0 = g_2 = 0$ and that g_1 is determined from g_3. The constraint on g_3 is that it must satisfy

$$g_3'(x_j) = x_j g_3''(x_j). \tag{8}$$

Putting

$$g_3(x_j) = a + bx_j + cx_j^2 + dx_j^3$$

we see that (8) is equivalent to $b = d = 0$. We thus have

$$g_3(x_j) = a + cx_j^2, \qquad \text{and} \qquad g_1(x_j) = c$$

and so

$$s = a + c(x_i^2 + x_j^2)$$

is the general solution to (7).

We thus see that if C satisfies (7) for a particular i and j then x_i and x_j occur in C with only even powers. Thus if C satisfies (7) for all i and j then all variables appear to an even power. This is impossible, unless $C = 0$, since C has degree three. Thus we see that zero is the only solution to (7) which holds for all i and j. □

Remark 6.5. The above computational proof can be made more conceptual. By considering the equation (7) for a fixed i and j we are considering the invariants of $\mathrm{Sym}^3(V)$ under a certain copy of $\mathfrak{so}(2)$ sitting inside of $\mathfrak{so}(V)$. The representation V restricted to $\mathfrak{so}(2)$ decomposes as $V' \oplus T$ where V' is the standard representation of $\mathfrak{so}(2)$ and T is a 12-dimensional trivial representation of $\mathfrak{so}(2)$. We then have

$$\mathrm{Sym}^3(V)^{\mathfrak{so}(2)} = \bigoplus_{i=0}^{3} \mathrm{Sym}^i(V')^{\mathfrak{so}(2)} \otimes \mathrm{Sym}^{3-i}(T).$$

Finally, our general solution to (7) amounts to the fact that the ring of invariant $\mathrm{Sym}(V')^{\mathfrak{so}(2)}$ is generated by the norm form $x_i^2 + x_j^2$.

We can now prove Proposition 4.4, which will establish Theorem 4.1.

Proof of Proposition 4.4. To prove $\mathfrak{g} = 0$ we may pass to the algebraic closure of k; we thus assume k is algebraically closed. By Proposition 6.1, the Lie algebra $\mathfrak{g}$ must be 0, $\mathfrak{so}(V)$ or $\mathfrak{sl}(V)$. By Proposition 6.4 $\mathfrak{g}$ cannot be $\mathfrak{so}(V)$ or $\mathfrak{sl}(V)$ since it annihilates C and C is non-zero. Thus $\mathfrak{g} = 0$. □

7. The minimal graded free resolution of the graded ring of M_8

An intro is necessary here. Before doing so, we need to review some commutative algebra. In this section we work over an algebraically closed field k of characteristic zero.

7.1. **Betti numbers of modules over polynomial rings.** Let P be a graded polynomial ring over k in finitely many indeterminates, each of positive degree. Let M be a finite P-module. One can then find a surjection $F \to M$ with F a finite free module having the following property: if $F' \to M$ is another surjection from a finite free module then there is a surjection $F' \to F$ making the obvious diagram commute. We call $F \to M$ a *free envelope* of M. It is unique up to non-unique isomorphism. As an example, if M is generated by its degree d piece then we can take F to be $P[-d] \otimes M^{(d)}$ where the tensor product is over k and $P[-d]$ is the free P-module with one generator in degree d.

Let M be a finite free P-module. We can build a resolution of M by using free envelopes:

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

Here F_0 is the free envelope of M and F_{i+1} is the free envelope of $\ker(F_i \to F_{i-1})$. Define integers $b_{i,j}$ by

$$F_i = \bigoplus_{j \in \mathbb{Z}} P[-i-j]^{\oplus b_{i,j}}.$$

These integers are called the *Betti numbers* of M and the collection of them all the *Betti diagram* of M. They are independent of the choice of free envelopes, as $b_{i,j}$ is also the dimension of the jth graded piece of $\mathrm{Tor}_i^P(M, P/I)$, where I is ideal of positive degree elements. The Betti numbers have the following properties:

- (B1) We have $b_{i,j} = 0$ for all but finitely many i and j. This is because each F_i is finitely generated and $F_i = 0$ for i large by Hilbert's theorem on syzygies.
- (B2) We have $b_{i,j} = 0$ for $i < 0$. This follows from the definition.
- (B3) If $b_{i_0,j} = 0$ for $j \leq j_0$ then $b_{i,j} = 0$ for all $i \geq i_0$ and $j \leq j_0$. This follows from the fact that if d is the lowest degree occurring in a module M and $F \to M$ is a free envelope then $F^{(d)} \to M^{(d)}$ is an isomorphism, and thus the lowest degree occurring in $\ker(F \to M)$ is $d + 1$.
- (B4) In particular, if M is in non-negative degrees then $b_{i,j} = 0$ for $j < 0$.
- (B5) Let $f(k) = \dim M^{(k)}$ (resp. $g(k) = \dim P^{(k)}$) denote the Hilbert function of M (resp. P). Then

$$f(k) = \sum_{i,j \in \mathbb{Z}} (-1)^i \cdot b_{i,j} \cdot g(k - i - j).$$

This follows by taking the Euler characteristic of the kth graded piece of $F_\bullet \to M$.

In particular we see that if M is in non-negative degrees then its Betti diagram is contained in a bounded subset of the first quadrant.

7.2. **Betti numbers of graded algebras.** Let R be a finitely generated graded k-algebra, generated in degree one. We let $P = \mathrm{Sym}(R^{(1)})$ be the graded polynomial algbera on the first graded piece. We have a natural surjective map $P \to R$ and so R is a P-module. We can thus speak of the Betti numbers of R as a P-module. We call these the Betti numbers of R.

Assume now that the ring R is Gorenstein and a domain. The canonical module ω_R of R is then naturally a graded module. Furthermore, there exists an integer a, called the *a-invariant* of R, such that ω_R is isomorphic to $R[a]$. We now have the following important property of the Betti numbers of R:

(B6) We have $b_{i,j} = b_{r-i,d+a-j}$ where $d = \dim R$ is the Krull dimension of R, $r = \dim P - \dim R$ is the codimension of $\mathrm{Spec}(R)$ in $\mathrm{Spec}(P)$ and a is the a-invariant of R.

No doubt this formula appears in the literature, but we derive it here for completeness. We have $\mathrm{Ext}^i_P(R, \omega_P) \cong \omega_R$ if $i = r$ and 0 if $i \neq r$. If n is the dimension of P, then $\omega_P \cong P[-n]$. Since R is Gorenstein we have $\omega_R \cong R[a]$. Therefore we obtain a minimal free resolution $G_\bullet$ of $R[a]$ by $G_i = \mathrm{Hom}_P(F_{r-i}, P[-n])$. We have $G_\bullet[-a]$ is a minimal free resolution of R, and by uniqueness of the resolution we therefore have $G_i[-a] \cong F_i$ for each i. Now $G_i[-a] \cong \oplus_{j'} P[-n-r+i+j'-a]$, and so

$$\oplus_{j'} P[-n+r-i+j'-a]^{b_{r-i,j'}} \cong \oplus_j P[-i-j]^{b_{i,j}}.$$

Equating components of the same degree gives $-n+r-i+j'-a = -i-j$, or $j' = n-r+a-j$. Hence $b_{i,j} = b_{r-i,n-r+a-j} = b_{r-i,d+a-j}$.

7.3. **This needs a title.** We now return to our previous notation. Thus L is a fixed eight element set, $R = R_L$, k is a field of characteristic zero, *etc.* **FIX THIS** We begin with the following:

Proposition 7.1. *The ring R is Gorenstein with a-invariant -2.*

Proof. We first recall a theorem of Hochster-Roberts [BH, Theorem 6.5.1]: if V is a representation of the reductive group G (over a field k of characteristic zero) then the ring of invariants $(\mathrm{Sym}\, V)^G$ is Cohen-Macaulay. As our ring R can be realized in this manner, with V being the space of 2×8 matrices and $G = \mathrm{SL}(2) \times T$, where T is the maximal torus in $\mathrm{SL}(8)$, we see that R is Cohen-Macaulay. We now recall a theorem of Stanley [BH, Corollary 4.4.6]: if R is a Cohen-Macaulay ring generated in degree one with Hilbert series $f(t)/(1-t)^d$, where d is the Krull dimension of R, then R is Gorenstein if and only if the polynomial f is symmetric. Furthermore, if f is symmetric then the a-invariant of R is given by $\deg f - d$. Going back to our situation, the Hilbert series of our ring was given in [**Howe ref**]. The numerator is symmetric of degree four and the denominator has degree six. We thus see that R is Gorenstein with $a = -2$. $\square$

We can now deduce the Betti diagram of R:

Proposition 7.2. *The Betti diagram of R is given by:*

	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	14	0	0	0	0	0	0	0
2	0	0	175	512	700	512	175	0	0
3	0	0	0	0	0	0	0	14	0
4	0	0	0	0	0	0	0	0	1

The i-axis is horizontal and the j-axis vertical. All $b_{i,j}$ outside of the above range are zero.

Proof. We first note that (B6) gives $b_{8-i,4-j} = b_{i,j}$ as $r = 8$, $d = 6$ and $a = -2$ in our situation. We thus have the symmetry of the table. Now, by (B2) and (B4) we have $b_{i,j} = 0$ if either i or j is negative. We thus see that $b_{i,j} = 0$ if $i > 8$ or $j > 4$ by symmetry. Next, observe that $P \to R$ is the free envelope of R, where $P = \mathrm{Sym}(V)$. This gives the $i = 0$ column of the table. We now look at the $i = 1$ column. We know that the 14 generators have no linear relations and so $b_{1,0} = 0$. By (B3) we have $b_{i,0} = 0$ for $i \geq 1$. We also know that there are 14 quadric relations and so $b_{1,1} = 14$. We now look at the $i = 2$ column of the table. We have proved (Theorem 4.1) that the 14 quadric relations have no linear syzygies; this gives $b_{2,1} = 0$. Using (B3) again, we conclude $b_{i,1} = 0$ for $i \geq 2$. We have thus completed the first two rows of the table. The last two rows can then be completed by symmetry. The middle row can now be determined from (B5) by evaluating both sides at $k = 2, \ldots, 10$ and solving the resulting upper triangular system of equations for $b_{i,2}$. (In fact, the computation is simpler than that since $b_{i,2} = b_{8-i,2}$ and we know $b_{0,2} = b_{1,2} = 0$, the latter vanishing coming from Proposition 4.7.) $\square$

Proposition 7.2 — in particular, the $i = 1$ column of the table — shows that I_8 is generated by its degree two piece. **Thus we have proved the old main theorem.**

Remark 7.3. The resolution of R as a P-module, without any consideration of grading, is given by Freitag and Salvati Manni [FS2, Lemma 1.3, Theorem 1.5]. It was obtained by computer.

8. The invariants of N_8 generated in degree one are the Gale-invariants

Anything interesting here should be stated in the introduction.

Ben has a theorem that generalizes an earlier theorem that all generators are integral over degree 1, meaning that $N_8 \to N_8'$ is already finite.

I'd asked (Feb. 12 2010: if we knew a little more, e.g. no linear syzygies among the quartics, would that be enough to know the entire free resolution of N_8'? No one had any ideas. Ben knows no degree 2 relations, see August 11, 2008 and Oct. 23, 2008. Also no degree 3 relations, 10/23/08. He knows the relations are in degree at most 12, 11/3/08. I'm not sure if that includes the degree 2 generators.

I believe Ben knows that $\deg(N_8') = 56$, see Jan 12, 2010.

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