FROBENIUS AND HODGE DEGENERATION

Aim of the seminar

We want to understand the results of

P. Deligne, L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. 89, 247-270 (1987).

Fortunately, there exists an expanded English version by Illusie,

L. Illusie, *Frobenius and Hodge degeneration*, in Introduction to Hodge theory, SMF/AMS Texts and Monographs 8, 99-149 (2002)

which not only gives the details of the previous article, but also introduces the techniques needed to understand it. This latter source will be the basis for our seminar - it is well-suited for graduate students and requires algebraic geometry at the level of Hartshorne's book.

For example, you can learn a lot about differental forms, smoothness, spectral sequences, derived categories, the Frobenius morphism and the Cartier operator...

What is this seminar about?

Let X is a smooth and projective variety over a field k. Then there exists a spectral sequence

$$E_2^{p,q} \;=\; H^q(X, \Omega_X^p) \;\Rightarrow\; H^*_{\mathrm{dR}}(X/k)$$

the so-called *Frölicher spectral sequence*, also called *Hodge-deRham spectral sequence*.

Spectral sequences will be introduced in this seminar, but let us mention for the moment that this is a tool that tells us how to compute the cohomology on the right side, deRham-cohomology $H^*_{\mathrm{dR}}(X/k)$ in this case, from the cohomology groups on the left side, i.e., $H^q(X, \Omega_X^p)$ in this case. Usually, a spectral sequence is a nightmare in book-keeping, *except* if something beautiful happens: namely that if it "denerates at E_2-level", in the jargon of spectral sequences. In our case, this means that for every $n \geq 1$ there exists a sequence of subvector spaces

$$0 = F^{n+1} \;\subseteq\; F^n \;\subseteq\; \ldots \;\subseteq\; F^0 = H^n_{\mathrm{dR}}(X/k)\,,$$

i.e., a filtration, and canonical isomorphisms

$$H^{n-p}(X,\Omega^p_X) \;\cong\; F^p/F^{p+1}$$

In particular, this implies for every $n \geq 1$

$$\dim_k H^n_{\mathrm{dR}}(X/k) \;=\; \sum_{p+q=n} \dim_k H^q(X,\Omega^p_X)$$

Example. Let X be an algebraic curve over the complex numbers. Then X is a complex manifold of complex dimension one, and thus an orientable differentiable manifold of real dimension two. Topologically, it is a two-dimensional sphere with g ”handles“ attached, where $g \geq 0$ is called the *genus* of X. Thus,

$$H^i_{\mathrm{sing}}(X,\mathbb{C}) \;\cong\; H^i_{\mathrm{dR}}(X/\mathbb{C}) \;\cong\; \begin{cases} \mathbb{C} & \text{for } i=0,2 \\ \mathbb{C}^{2g} & \text{for } i=1 \end{cases}$$

It is known (see below) that the Frölicher spectral sequence degenerates at E_2-level, which implies the short exact sequence

$$0 \;\to\; H^0(X,\Omega^1_X) \;\to\; H^1_{\mathrm{dR}}(X/\mathbb{C}) \;\to\; H^1(X,\mathcal{O}_X) \;\to\; 0\,.$$

Now, the cohomology group in the middle depends only on the topology of X, whereas the cohomology groups on the left and right can only be defined algebraically.

This is the starting point for extremely deep connections between topological and purely algebraic invariants in algebraic geometry. For example, in our case complex conjugation induces an isomorphism between $H^0(X,\Omega^1_X)$ and $H^1(X,\mathcal{O}_X)$ and we obtain

$$\dim H^0(X,\Omega^1_X) \;=\; \dim H^1(X,\mathcal{O}_X) \;=\; \frac{1}{2}\dim H^1_{\mathrm{dR}}(X/\mathbb{C}) \;=\; g.$$

Coming back to our seminar, there is the following theorem in complex geometry

Theorem. *Let X be a smooth projective variety over the complex numbers. Then its Frölicher spectral sequence degenerates at E_2-level.*

The classical proof uses quite involved techniques from differential geometry, partial differential equations and functional analysis.

However, Deligne and Illusie came up with a completely algebraic proof of this result, which is what we want to understand in this seminar.

Roughly their idea is as follows – **Spoiler alert!**

Suppose we have a smooth projective variety over $\mathbb{C}$. Then we may assume that it is in fact defined of a field that is finitely generated over $\mathbb{Q}$, where it makes sense to reduce X modulo p for various primes p. For almost all primes p, the reduction modulo p will be a smooth projective variety X_p over a field of characteristic p.

Over fields of positive characteristic p, one has the ubiquitous *Frobenius morphism* $F : x \mapsto x^p$, and differentiating has the remarkable property

$$\frac{\partial}{\partial x}F(x) \;=\; \frac{\partial}{\partial x}x^p \;=\; px^{p-1} \;=\; 0\,.$$

On the level of cohomology, this leads to the *Cartier operator*.

Now, the Frobenius morphism and the Cartier operator decompose the deRham-complex of X_p in characteristic p in a way that has no analogue in characteristic zero.

Using this and if the Frobenius morphism on the reduction X_p would be the reduction modulo p of some morphism on X, one can prove a (sort of) Hodge decompostion of the deRham-complex, which would imply the degeneration of the Frölicher spectral sequence.

Unfortunately, it is quite rare that Frobenius comes from a morphism in characteristic zero. However, one can always find a Zariski open cover of X where this is true. On this cover one has this (sort of) Hodge decomposition of the deRham-complex, and one can glue these local decompositions in the derived category, which ultimately implies the desired degeneration of the Frölicher spectral sequence.