ABSOLUTE GALOIS ACTS FAITHFULLY ON THE COMPONENTS OF THE MODULI SPACE OF SURFACES: A BELYI-TYPE THEOREM IN HIGHER DIMENSION

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ABSTRACT. Given an object over $\overline{\mathbb{Q}}$, there is often no reason for invariants of the corresponding holomorphic object to be preserved by the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and in general this is not true, although it is sometimes surprising to observe in practice. The case of covers of the projective line branched only over the points 0, 1, and ∞, through Belyi's theorem, leads to Grothendieck's *dessins d'enfants* program for understanding the absolute Galois group through its faithful action on such covers. This note is motivated by Catanese's question about a higher-dimensional analogue: does the absolute Galois group act faithfully on the deformation equivalence classes of smooth surfaces? (These equivalence classes are of course by definition the strongest deformation invariants.) We give a short proof of a weaker result: the absolute Galois group acts faithfully on the *irreducible* components of the moduli space of smooth surfaces (of general type, canonically polarized). Bauer, Catanese, and Grunewald have recently answered Catanese's original question using a different construction (using surfaces isogenous to a product) [BCG3].

1. INTRODUCTION

Given an object defined over $\overline{\mathbb{Q}}$, certain topological invariants of the corresponding holomorphic object are known to be preserved by the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is because these invariants are algebraic in nature. For example, if X is a nonsingular projective variety, the Betti numbers are algebraic (shown by Serre in his GAGA paper, [S1]). The profinite completion of the fundamental group of X is the étale fundamental group. More generally, Artin and Mazur showed that the profinite completion of the homotopy type of X is algebraic [AM].

It is thus natural to ask what topological invariants of the corresponding holomorphic object are preserved by conjugation. Indeed, given an object defined over $\overline{\mathbb{Q}}$, there is often no reason for topological invariants of the corresponding holomorphic object to be preserved by the absolute Galois group. In the case of covers of the projective line branched only over the points 0, 1, and ∞, this leads to Grothendieck's *dessins d'enfants* program for understanding the absolute Galois group [Gr], through its faithful action on such covers. In other words, given any nontrivial element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there is a cover $C \to \mathbb{P}^1$ (over $\overline{\mathbb{Q}}$) such that $\sigma(C) \to \mathbb{P}^1$ is a topologically different cover (where both covers are now considered over $\mathbb{C}$, as maps of Riemann surfaces).

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Similarly, returning to the case of smooth varieties, Serre gave an elegant example [S2] of a smooth variety X over $\overline{\mathbb{Q}}$ and an element $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the fundamental groups of the complex manifolds X and $\sigma(X)$ are different. (As the profinite completions $\pi_1^{\text{ét}}(X)$ and $\pi_1^{\text{ét}}(\sigma(X))$ are isomorphic, the fundamental groups are necessarily infinite.) Abelson [A] gave examples of conjugate (nonsingular projective) varieties with the *same* fundamental group yet of different homotopy types. He also gave examples of conjugate (nonsingular quasiprojective) varieties that are homotopy equivalent but not homeomorphic. More examples of nonhomeomorphic conjugate varieties have been given quite recently by Artal Bartolo, Carmona Ruber, and Cogolludo Agustín [ABCRCA], and Shimada [Shi]. Also, Charles has recently given an example of two conjugate smooth projective varieties with non-isomorphic cohomology algebras with real coefficients [Ch].

Surprising examples of a different flavor were given by Catanese earlier (see Theorem 21 and the discussion just before Question 4 in [BCP]; cf. [C3, Thm. 3.3] and [C2, Thm. 4.14]).

A potentially rich third family of examples arises from the theory of Shimura varieties, as described by Milne [Mi2, p. 7]. By a theorem of Baily and Borel [BB], the quotient of a bounded symmetric domain by an arithmetic subgroup of its analytic automorphism group has a canonical structure of a quasiprojective complex variety V. A conjecture of Langlands implies that if $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\sigma(V)$ is again such a quotient, and describes explicitly what the bounded symmetric domain and arithmetic subgroup are; this conjecture was proved by Borovoi and Milne [Bo, Mi1] using a theorem of Kazhdan and Nori-Raghunathan [K1, K2, NR]. One should be able to show that these arithmetic groups (the fundamental groups of the Shimura varieties in cases of good quotients) are not isomorphic (as abstract groups); to our knowledge, the details have not yet been worked out in the literature.

The strongest deformation-invariant discrete invariant is of course the deformation equivalence class. This note is motivated by a question of Catanese: does the absolute Galois group act faithfully on the deformation equivalence classes of surfaces (defined over $\overline{\mathbb{Q}}$)? In other words, given any nontrivial $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, can one produce a surface X such that $\sigma(X)$ is not deformation-equivalent to X? Catanese has shown that it is indeed true when σ is complex conjugation (see [C3, Thm. 3.5], as well as later numerous rigid examples by Bauer, Catanese, and Grunewald in [BCG2]). (One might speculate that every element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ other than the identity and complex conjugation can change the homeomorphism type of a $\overline{\mathbb{Q}}$-variety, and combined with Catanese's example this would answer Catanese's question. However, it is not clear from the examples produced to date that *every* $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ besides identity and complex conjugation has this property.)

In effect, Catanese's question translates to:

1.1. *Catanese's question.* *Does* $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *act faithfully on the connected components of the moduli space of surfaces of general type?*

We are able to show the following weaker result:

1.2. *Main Theorem. — The absolute Galois group acts faithfully on the* **irreducible** *components of the moduli space of surfaces of general type. More precisely, for* **any** *nontrivial $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we exhibit a surface X over $\overline{\mathbb{Q}}$, where X has ample canonical bundle (indeed very ample), and such that X and $\sigma(X)$ do not lie on the same component of the moduli space.*

Important remark. After we wrote this note, Bauer, Catanese, and Grunewald informed us that they have given a complete answer to Catanese's question using a different construction using 'surfaces isogenous to a product' (the quotient of a product of curves by the free action of a finite group). More precisely, given any element σ of the absolute Galois group different from the identity and complex conjugation, there is a surface S such that S and $\sigma(S)$ have different fundamental groups. Catanese announced their work at the Alghero conference of September 2006, and the paper is now publicly available [BCG3].

Strategy. We first choose $z \in \overline{\mathbb{Q}}$ not fixed by σ. Our surface $X = X_z$ will be constructed so that the number z will be "encoded" in it (and its infinitesimal deformations), and such that its conjugate $\sigma(X) = X_{\sigma(z)}$ (and its infinitesimal deformations) will encode the number $\sigma(z)$ in the same way. Thus there are *Zariski* neighborhoods of the points $[X_z]$ and $[X_{\sigma(z)}]$ of the moduli space that are disjoint.

We perform this encoding by first describing a configuration of points and lines on the plane (over $\overline{\mathbb{Q}}$) such that the combinatorics of incidences of points and lines encodes the number z, in such a way that the σ-conjugate encodes the number $\sigma(z)$. We do this as follows. There will be four distinguished ordered points on a line in the plane; they will be the four points with the most lines through them. The cross-ratio of these four points on the line will be z. Hence the Galois conjugate will have cross-ratio $\sigma(z)$.

Then (as in [V]) we let X be a branched cover of the blow-up of the plane at the points, where the branch locus consists of the proper transforms of the lines, as well as several high-degree curves. This positivity will force the vanishing of certain cohomology groups, which will allow us to ensure that the deformations of X correspond exactly with the deformations of the point-line configuration on the plane. More precisely, from X (or any infinitesimal deformation), we can recover the branched cover, and hence the data of the point-line configuration. These constructions "commute with σ", yielding the result.

1.3. Miscellaneous remarks. *(a)* We do not know if the two surfaces are homeomorphic (if σ is not complex conjugation), and we have no reason to expect that they are. If they are not homeomorphic, then this would answer Catanese's question 1.1 in the affirmative. Moreover, they are constructed so that it is possible in theory to compute their fundamental groups. If one could do so, and show that they are different, this would answer Catanese's question completely.

(b) González-Diez [Go] and Paranjape [P] have given other higher-dimensional analogues of Belyi's theorem.

(c) This is vaguely reminiscent of dessins d'enfants. In the case of covers of $\mathbb{P}^1$, the covers were encoded by graphs — not just the incidences of vertices and edges, but also the embedding in the surface. In this case, the surfaces are encoded by lines in the plane —

not just the combinatorial data of incidences of points and lines, but also the embedding in the (complex) plane.

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2. THE ARGUMENT

2.1. Branched covers background. We first review some results about branched covers, due to Catanese, Pardini, Fantechi, and Manetti.

Suppose $G = (\mathbb{Z}/p\mathbb{Z})^n$ with p prime. Let $G^\vee = \operatorname{Hom}(G, \mathbb{C}^*)$ be the group of complex characters of G, and for each $\chi \in G^\vee$, define $(\chi, g) \in \{0, \dots, p-1\}$ by $\chi(g) = e^{\frac{2\pi i}{p}(\chi, g)}$. Let S be any nonsingular surface, and suppose $\{D_g\}_{g \in G}$, $\{M_\chi\}_{\chi \in G^\vee}$ are divisors in S satisfying $D_0 = \emptyset$ and

$$pM_\chi \equiv \sum_{g \in G} (\chi, g) D_g$$

in $\operatorname{Pic}(S)$ for all $\chi \in G^\vee$. Moreover, suppose the D_g are all nonsingular curves, no three intersect in a point, and $D_g \cap D_{g'} \neq \emptyset$ only if g and g' are independent in G (i.e. g' is not a multiple of g). Then:

2.2. *Theorem. — There exists a nonsingular G-cover $\pi \colon X \to S$ with branch divisor $D = \bigcup_{g \in G} D_g$. Moreover, if $n \geq 3$ and M_χ is sufficiently ample for all nonzero $\chi \in G^\vee$, then:*

(a) *K_X is very ample;*
(b) *deformations of $(S, \{D_g\})$ are equivalent to deformations of X, i.e. the natural map*

$$\operatorname{Def}(S, \{D_g\}) \to \operatorname{Def}(X)$$

is an isomorphism; and
(c) *$\operatorname{Aut}(X) \cong G$.*

Part (a) is given in the e-print version of [V] (Theorem 4.4), and the idea is due to Catanese. (The argument for bidouble covers is given in [C1, p. 502].) Part (b) is [V, Thm. 4.4(c)], and the argument is due to Manetti [Ma, Cor. 3.23]; indeed, the case $p = 2$ that we will actually use is Manetti's original result. Part (c) is due to Fantechi and Pardini [FP, Thm. 4.6].

2.3. The construction. Take any nontrivial $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and fix some $z \in \bar{\mathbb{Q}}$ with $\sigma(z) \neq z$. Choose a line ℓ on the plane, and three distinct points on it, which we name

0, 1, ∞. Then the points of $\ell - \{\infty\}$ are naturally identified with elements of $\bar{\mathbb{Q}}$. As such, identify $z \in \bar{\mathbb{Q}}$ with the corresponding point on ℓ. It is well-known there exists a configuration of lines in the plane, containing ℓ, which "represents" z. (Given three points a, b and c on $\ell - \{\infty\}$, considered as numbers, it is straightforward to construct a point-line configuration through these points that forces precisely the equation $a + b = c$; and a different configuration that forces $ab = c$; and a third that forces $a = -b$. We combine these operations suitably so as to force $p(z) = 0$. See for example [Sha, p. 13]. The apotheosis of this idea is Mnëv's Universality Theorem [Mn1, Mn2].) Let $\mathcal{L}'_z$ denote this configuration representing z.

We now modify $\mathcal{L}'_z$ so as to produce a configuration $\mathcal{L}_z \supset \mathcal{L}'_z$ over which a branched G-cover is readily constructed, with $G = (\mathbb{Z}/2)^3$. First, add a general line through each point in the plane through which an odd number of lines (greater than one) in $\mathcal{L}'_z$ pass. Next, add general lines through the points 0, 1, ∞, and z so that an even number pass through each and so that these four points, in order, have the greatest number of lines in the configuration passing through them. Let $\mathcal{L}_z$ denote this final configuration. The marked points in our point-line configuration consist of all points of intersection of pairs of lines in $\mathcal{L}_z$.

Note that this configuration has the following properties. First, forgetting the marked points, we can recover z by finding the four points with the greatest number of lines through them, observing that they lie on a common line, and taking their cross-ratio on this line. Secondly, acting on $\mathcal{L}_z$ with σ yields a configuration encoding $\sigma(z)$ in the same way. Thus the first point-line configuration may not be deformed to the second (while preserving the point-line incidences). Lastly, the marked points in the configuration each have an even number of lines in $\mathcal{L}_z$ passing through them, a fact which will be used shortly.

We now let S_z be the blow-up of $\mathbb{P}^2$ at our marked points, and C_z be the strict transform of the union of the lines in $\mathcal{L}_z$. As mentioned above, we will construct a branched G-cover with $G = (\mathbb{Z}/2\mathbb{Z})^3$. First, we define maps $D : G \to \mathrm{Div}(S_z)$ and $M : G^\vee \to \mathrm{Pic}(S_z)$ satisfying the conditions necessary to construct a branched G-cover. Let $D_0 = \emptyset$. Fix any nonzero $\alpha \in G$ and let $D_\alpha = C_z$. Then fix any map $m : G \to \mathbb{Z}^+$ such that $m_0 = 0, m_\alpha = L$, and $\sum_{g \in G} m_g g = 0$ in G, where L is the number of lines in $\mathcal{L}_z$. For $g \in G - \{0, \alpha\}$, define D_g to be the pull-back of a general (nonsingular) curve in $\mathbb{P}^2$ of degree m_g. By our choice of m_g, we then have

$$\sum_{g \in G} (\chi, g) D_g \equiv -(\chi, \alpha) \sum_q e_q(\mathcal{L}_z) E_q + \sum_{g \in G} (\chi, g) m_g H$$

in $\mathrm{Pic}(S_z)$, where H is the hyperplane class in $\mathbb{P}^2$ and $e_q(\mathcal{L}_z)$ is the number of lines in our configuration passing through the marked point q. By construction of the configuration $\mathcal{L}_z$, the numbers $e_q(\mathcal{L}_z)$ are all even. By our choice of the map m, the number $\sum_{g \in G} (\chi, g) m_g$ is even for every $\chi \in G^\vee$:

$$1 = \chi(0) = \chi\left(\sum_g m_g g\right) = \prod_g \chi(g)^{m_g} = e^{\frac{2\pi i}{2} \sum_g (\chi, g) m_g} = (-1)^{\sum_g (\chi, g) m_g}.$$

Hence we may define $M : G^\vee \to \mathrm{Pic}(S_z)$ by $M_\chi = \frac{1}{2}\sum_g (\chi, g) D_g$. Note that the M_χ can be made arbitrarily ample by an appropriate choice of the map m. For such a choice, by Theorem 2.2 we obtain a nonsingular general type G-cover $\pi \colon X_z \to S_z$ with branch divisor $D = \bigcup D_g \supset C_z$. The same construction mutatis mutandis produces a conjugate nonsingular general type G-cover $\pi_{\sigma(z)} \colon X_{\sigma(z)} \to S_{\sigma(z)}$ with branch divisor $D_{\sigma(z)} = \sigma(D_z) \supset \sigma(C_z) = C_{\sigma(z)}$. It then follows from Theorem 2.2(b) that the deformations of X_z (resp. $X_{\sigma(z)}$) are equivalent to the deformations of (S_z, D_z) (resp. $(S_{\sigma(z)}, D_{\sigma(z)})$).

We now describe how to recover the number z from X_z and any infinitesimal deformation (and similarly for $X_{\sigma(z)}$). To make the idea clearer, we first describe how to extract the number z from the surface X_z. By Theorem 2.2(c), $G \to \mathrm{Aut}(X_z)$ is an isomorphism, from which we may recover $X_z \to X_z/G = S_z$. The components of the branch divisor of $X_z \to X_z/G$ are the divisors $\{D_g\}_{g \neq 0}$. All but one of them (all except D_α) are $\mathbb{Q}$-multiples of each other; they are all equivalent to multiples of H, the pullback of the hyperplane divisor in $\mathbb{P}^2$. We may therefore use any divisor from this distinguished collection to recover the blow-down to $\mathbb{P}^2$. Under this blow-down, the components of the remaining branch divisor D_α recover the configuration $\mathcal{L}_z \subset \mathbb{P}^2$. As noted previously, from this configuration we may recover z by taking the cross-ratio of the four distinguished ("highest-multiple") points on the distinguished line.

With a little care, the same argument extends to the (formal) deformation space Δ around $[X_z]$ in the moduli space, as follows. Let $\mathcal{X}_z \to \Delta$ be the total (flat) family of the deformation. Then $\mathrm{Aut}(\mathcal{X}_z/\Delta)$ is the trivial group scheme G over Δ by Theorem 2.2 (b) and (c). (Part (c) ensures that the central fiber is G, and part (b) ensures that all $|G|$ automorphisms extend over Δ.) Let $\mathcal{S}_z = \mathcal{X}_z/G$. Then as $\mathcal{X}_z/\mathcal{S}_z$ is faithfully flat (it is a finite group quotient), and $\mathcal{X}_z/\Delta$ is flat, we have that $\mathcal{S}_z/\Delta$ is flat. As the central fiber S_z is smooth over 0, $\mathcal{S}_z/\Delta$ is smooth. From the components of the branch divisor of $X_z \to S_z$ (in the central fiber), we obtain the divisors D_g on S_z. All but one are $\mathbb{Q}$-multiples of each other, so we choose any such D_g, which gives an invertible sheaf on the central fiber S_z. This invertible sheaf extends uniquely to an invertible sheaf $\mathcal{M}$ on the family $\mathcal{S}_z$, as $\mathrm{Pic}\, S_z$ is discrete. This invertible sheaf is relatively base-point-free, as it is base-point-free on the central fiber. The image of $\mathcal{S}_z$ under the corresponding linear series $|\mathcal{M}|$ over Δ is a deformation of $\mathbb{P}^2$ over the central fiber. As $\mathbb{P}^2$ is rigid, the image of the map can be (noncanonically) identified with $\mathbb{P}^2 \times \Delta$. The components of the remaining branch divisor $\mathcal{D}_\alpha$ (on the total family $\mathbb{P}^2 \times \Delta$) recover the family of point-line configurations $\mathcal{L}_z \subset \mathbb{P}^2 \times \Delta$. The cross-ratio map (the map to $\overline{\mathcal{M}}_{0,4}$) of the four distinguished points on the distinguished line gives a map $\Delta \to \mathbb{P}^1$. This map is necessarily the constant map, as the point-line configuation was chosen so that the z is forced to satisfy a given algebraic equation.

2.4. Closing remarks. This result suggests an approach to answering Catanese's question 1.1 in general, by producing a rigid surface as such a branched cover. One might attempt to do so by rigidifying the point-line configuration $\mathcal{L}'_z$ of the start of §2 by adding judiciously chosen additional lines, using a theorem of Paranjape [P, Thm. 2]. One would then have to modify the argument to ensure that *(a)* the four points "marking z" remain distinguished, and *(b)* Theorem 2.2 continues to hold, without the assistance of the positivity of M_χ.

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