ABSOLUTE GALOIS ACTS FAITHFULLY ON THE COMPONENTS OF THE MODULI SPACE OF SURFACES: A BELYI-TYPE THEOREM IN HIGHER DIMENSION

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ABSTRACT. Given an object over $\overline{\mathbb{Q}}$, there is often no reason for invariants of the corresponding holomorphic object to be preserved by the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and in general this is not true, although it is sometimes surprising to observe in practice. The case of covers of the projective line branched only over the points 0, 1, and ∞, through Belyi's theorem, leads to Grothendieck's *dessins d'enfants* program for understanding the absolute Galois group through its faithful action on such covers. This note is motivated by Catanese's question about a higher-dimensional analogue: does the absolute Galois group act faithfully on the deformation equivalence classes of smooth surfaces? (These equivalence classes are of course by definition the strongest deformation invariants.) We prove a weaker result: the absolute Galois group acts faithfully on the *irreducible* components of the moduli space of smooth surfaces (of general type, canonically polarized).

1. INTRODUCTION

Given a object defined over $\overline{\mathbb{Q}}$, certain topological invariants of the corresponding holomorphic object are known to be preserved by the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is because these invariants are algebraic in nature. For example, if X is a nonsingular projective variety, the Betti numbers are algebraic (shown by Serre in his GAGA paper, [S1]). The profinite completion of the fundamental group of X is the étale fundamental group. More generally, Artin and Mazur showed that the profinite completion of the homotopy type of X is algebraic [AM].

It is thus natural to ask what topological invariants of the corresponding holomorphic object are preserved by conjugation. Indeed, given an object defined over $\overline{\mathbb{Q}}$, there is often no reason for topological invariants of the corresponding holomorphic object to be preserved by the absolute Galois group. In the case of covers of the projective line branched only over the points 0, 1, and ∞, this leads to Grothendieck's *dessins d'enfants* program for understanding the absolute Galois group [Gr], through its faithful action on such covers. In other words, given any nontrivial element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there is a cover $C \to \mathbb{P}^1$ (over $\overline{\mathbb{Q}}$) such that $\sigma(C) \to \mathbb{P}^1$ is a topologically different cover (where both covers are now considered over $\mathbb{C}$, as maps of Riemann surfaces).

Similarly, returning to the case of smooth varieties, Serre gave an elegant example [S2] of a smooth variety X over $\overline{\mathbb{Q}}$ and an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the fundamental

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groups of the complex manifolds X and $\sigma(X)$ are different. (As the profinite completions $\pi_1^{\text{ét}}(X)$ and $\pi_1^{\text{ét}}(\sigma(X))$ are isomorphic, the fundamental groups are necessarily infinite.) Abelson [A] gave examples of conjugate (nonsingular projective) varieties with the *same* fundamental group yet of different homotopy types. He also gave examples of conjugate (nonsingular quasiprojective) varieties that are homotopy equivalent but not homeomorphic. More examples of nonhomeomorphic conjugate varieties have been given quite recently by Artal Bartolo, Carmona Ruber, and Cogolludo Agustín [ABCRCA], and Shimada [Shi].

Surprising examples of a different flavor, using Beauville surfaces, were given by Catanese earlier (see Theorem 21 and the discussion just before Question 4 in [BCP]; cf. [C3, Thm. 3.3] and [C2, Thm. 4.14]).

A potentially rich third family of examples arises from the theory of Shimura varieties, as described by Milne [Mi2, p. 7]. By a theorem of Bailey and Borel [BB], the quotient of a bounded symmetric domain by an arithmetic subgroup of its analytic automorphism group has a canonical structure of a quasiprojective complex variety V. A conjecture of Langlands implies that if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\sigma(V)$ is again such a quotient, and describes explicitly what the bounded symmetric domain and arithmetic subgroup are; this conjecture was proved by Borovoi and Milne, and Kazhdan and Nori-Raghunathan [Bo, Mi1, K1, K2, NR]. One should be able to show that these arithmetic groups (the fundamental groups of the Shimura varieties in cases of good quotients) are not isomorphic (as abstract groups); to our knowledge, the details have not yet been worked out in the literature.

The strongest deformation-invariant discrete invariant is of course the deformation equivalence class. This note is motivated by a question of Catanese: does the absolute Galois group act faithfully on the deformation equivalence classes of surfaces (defined over $\overline{\mathbb{Q}}$)? In other words, given any nontrivial $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, can one produce a surface X such that $\sigma(X)$ is not deformation-equivalent to X? Catanese has shown that it is indeed true when σ is complex conjugation, by exhibiting a surface that is rigid (has no deformations), such that $\sigma(X) \ncong X$ (see for example [C3, Thm. 3.5]). (One might speculate that every element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ other than the identity and complex conjugation can change the homeomorphism type of a $\overline{\mathbb{Q}}$-variety, and combined with Catanese's example this would answer Catanese's question. However, it is not clear from the examples produced to date that *every* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ besides identity and complex conjugation has this property.)

In effect, Catanese's question translates to:

1.1. *Catanese's question.* *Does* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *act faithfully on the connected components of the moduli space of surfaces of general type?*

We are able to show the following weaker result:

1.2. *Main Theorem. — The absolute Galois group acts faithfully on the* **irreducible** *components of the moduli space of surfaces of general type. More precisely, for* **any** *nontrivial* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *we exhibit a surface X over $\overline{\mathbb{Q}}$, where X has ample canonical bundle (indeed very ample), and such that X and $\sigma(X)$ do not lie on the same component of the moduli space.*

Strategy. We first choose $z \in \overline{\mathbb{Q}}$ not fixed by σ. Our surface $X = X_z$ will be constructed so that the number z will be "encoded" in it (and its infinitesimal deformations), and such that its conjugate $\sigma(X) = X_{\sigma(z)}$ (and its infinitesimal deformations) will encode the number $\sigma(z)$ in the same way. Thus there are *Zariski* neighborhoods of the points $[X_z]$ and $[X_{\sigma(z)}]$ of the moduli space that are disjoint.

We perform this encoding by first describing a configuration of points and lines on the plane (over $\overline{\mathbb{Q}}$) such that the combinatorics of incidences of points and lines encodes the number z, in such a way that the σ-conjugate encodes the number $\sigma(z)$. We do this as follows. There will be four distinguished ordered points on a line in the plane; they will be the four points with the most lines through them. The cross-ratio of these four points on the line will be z. Hence the Galois conjugate will have cross-ratio $\sigma(z)$.

Then (as in [V]) we let X be a branched cover of the blow-up of the plane at the points, where the branch locus consists of the proper transforms of the lines, as well as several high-degree curves. This positivity will force the vanishing of certain cohomology groups, which will allow us to ensure that the deformations of X correspond exactly with the deformations of the point-line configuration on the plane. More precisely, from X (or any infinitesimal deformation), we can recover the branched cover, and hence the data of the point-line configuration. These constructions "commute with σ", yielding the result.

1.3. Miscellaneous remarks. *(a)* We do not know if the two surfaces are homeomorphic (if σ is not complex conjugation), and we have no reason to expect that they are. If they are not homeomorphic, then this would answer Catanese's question 1.1 in the affirmative. Moreover, they are constructed so that it is possible in theory to compute their fundamental groups. If one could do so, and show that they are different, this would answer Catanese's question completely.

(b) González-Diez [Go] and Paranjape [P] have given other higher-dimensional analogues of Belyi's theorem.

(c) This is vaguely reminiscent of dessins d'enfants. In the case of covers of $\mathbb{P}^1$, the covers were encoded by graphs — not just the incidences of vertices and edges, but also the embedding in the surface. In this case, the surfaces are encoded by lines in the plane — not just the combinatorial data of incidences of points and lines, but also the embedding in the (complex) plane.

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2. THE ARGUMENT

Take any nontrivial $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and fix some $z \in \bar{\mathbb{Q}}$ with $\sigma(z) \neq z$.

2.1. Point-line configurations. We construct a point-line configuration encoding the algebraic number z as follows. Let $p(z)$ be the minimal polynomial of z (with relatively prime integer coefficients). Choose a line ℓ on the plane, and three distinct points on it, which we name $0, 1, \infty$. Then the points of $\ell - \{\infty\}$ are naturally identified with numbers; i.e., elements of $\bar{\mathbb{Q}}$. Given three points a, b and c on $\ell - \{\infty\}$ (considered as numbers), it is straightforward to construct a point-line configuration through these points that forces precisely the equation $a+b=c$; and a different configuration that forces $ab = c$; and a third that forces $a = -b$. We combine these operations suitably so as to force $p(z) = 0$. (This sort of recipe is well-known and straightforward, so we omit the details. See for example [Sha, p. 13]. The apotheosis of this idea is Mnëv's Universality Theorem [Mn1, Mn2].)

Let $\mathcal{L}'_z$ denote this configuration representing z. We modify $\mathcal{L}'_z$ so as to produce a configuration $\mathcal{L}_z \supset \mathcal{L}'_z$ over which a branched $(\mathbb{Z}/2)^3$-cover is readily constructed. First, we add a general line through each point in the plane through which an odd number of lines (greater than one) in $\mathcal{L}_z$ pass. Finally, we add general lines through the points 0, 1, ∞, and z (an even number through each), so that the points on the plane with the most lines through them are, in order, these four points. The points in our point-line configuration will consist of all points of intersection of pairs of such lines. Note that from a general such configuration, we can recover z by finding the four points with the most lines through them, observing that they lie on a common line, and taking their cross-ratio on this line. Also, acting on such a configuration with σ will yield a configuration encoding $\sigma(z)$ in the same way. Thus the first point-line configuration may not be deformed to the second (while preserving the point-line incidences). The "even valence" condition will be used later.

2.2. Branched covers background. This portion of our argument follows [V]. We will obtain X by blowing up the plane at our marked points, and taking a suitable branched cover of the resulting surface.

We first review some results about branched covers, due to Catanese, Pardini, Fantechi, and Manetti. Suppose $G = (\mathbb{Z}/p\mathbb{Z})^n$ with p prime. Let $G^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$ be the group of complex characters of G, and for each $\chi \in G^\vee$, define $(\chi, g) \in \{0, \dots, p-1\}$ by $\chi(g) = e^{\frac{2\pi i}{p}(\chi,g)}$. Let S be any nonsingular surface, and suppose $\{D_g\}_{g \in G}, \{M_\chi\}_{\chi \in G^\vee}$ are divisors in S satisfying $D_0 = \emptyset$ and

$$pM_\chi \equiv \sum_{g \in G} (\chi, g) D_g$$

in $\mathrm{Pic}(S)$ for all $\chi \in G^\vee$. Moreover, suppose the D_g are all nonsingular curves, no three intersect in a point, and $D_g \cap D_{g'} \neq \emptyset$ only if g and g' are independent in G (i.e. g' is not a multiple of g). Then:

2.3. *Theorem. — There exists a nonsingular G-cover $\pi\colon X \to S$ with branch divisor $D = \bigcup_{g\in G} D_g$. Moreover, if $n \geq 3$ and M_χ is sufficiently ample for all nonzero $\chi \in G^\vee$, then:*

(a) *K_X is very ample;*
(b) *deformations of $(S, \{D_g\})$ are equivalent to deformations of X, i.e. the natural map*
$$\mathrm{Def}(S, \{D_g\}) \to \mathrm{Def}(X)$$
is an isomorphism; and
(c) *$\mathrm{Aut}(X) \cong G$.*

Part (a) is given in the e-print version of [V] (Theorem 4.4), and the idea is due to Catanese. (The argument for bidouble covers is given in [C1, p. 502].) Part (b) is [V, Thm. 4.4(c)], and the argument is due to Manetti [Ma, Cor. 3.23]; indeed, the case $p = 2$ that we will actually use is Manetti's original result. Part (c) is due to Fantechi and Pardini [FP, Thm. 4.6].

2.4. Branched covers of the blow-up of the plane. We now let S_z be the blow-up of $\mathbb{P}^2$ at our marked points, and C_z be the strict transform of the union of our lines. We will construct a branched cover with $G = (\mathbb{Z}/2\mathbb{Z})^3$. First, we define maps $D : G \to \mathrm{Div}(S_z)$ and $M : G^\vee \to \mathrm{Pic}(S_z)$. Let $D_0 = \emptyset$. Fix any nonzero $\alpha \in G$ and let $D_\alpha = C_z$. Let L denote the number of lines in our configuration. Fix any map $m : G \to \mathbb{Z}^+$ such that $m_0 = 0, m_\alpha = L$, and $\sum_{g\in G} m_g g = 0$ in G. Then $\sum_{g\in G}(\chi, g)m_g$ is even for every $\chi \in G^\vee$:
$$1 = \chi(0) = \chi\left(\sum_g m_g g\right) = \prod_g \chi(g)^{m_g} = e^{\frac{2\pi i}{2}\sum_g(\chi,g)m_g} = (-1)^{\sum_g(\chi,g)m_g}.$$
We will use this fact momentarily.

For $g \in G - \{0, \alpha\}$, define D_g to be the pull-back of a general (nonsingular) curve in $\mathbb{P}^2$ of degree m_g. By our choice of m_g, we then have
$$\sum_{g\in G}(\chi, g)D_g \equiv -(\chi, \alpha)\sum_q e_q(\mathcal{L}_z)E_q + \sum_{g\in G}(\chi, g)m_g H,$$
in $\mathrm{Pic}(S_z)$, where H is the hyperplane class in $\mathbb{P}^2$ and $e_q(\mathcal{L}_z)$ is the number of lines in our configuration passing through the point q. By construction, the above divisor is even, and hence we may define $\chi : G^\vee \to \mathrm{Pic}(S_z)$ by $M_\chi = \frac{1}{2}\sum_g(\chi, g)D_g$. Note that the M_χ can be made arbitrarily ample by an appropriate choice of the map m. For such a choice, by Theorem 2.3 we obtain a nonsingular general type G-cover $\pi\colon X_z \to S_z$ with branch divisor $D = \bigcup D_g \supset C_z$. The same construction mutatis mutandis produces a conjugate nonsingular general type G-cover $\pi_{\sigma(z)}\colon X_{\sigma(z)} \to S_{\sigma(z)}$ with branch divisor $D_{\sigma(z)} = \sigma(D_z) \supset \sigma(C_z) = C_{\sigma(z)}$. It then follows from Theorem 2.3(b) that the deformations of X_z (resp. $X_{\sigma(z)}$) are equivalent to the deformations of (S_z, D_z) (resp. $(S_{\sigma(z)}, D_{\sigma(z)})$).

We now describe how to recover the number z from S_z and any infinitesimal deformation (and similarly for $S_{\sigma(z)}$). By Theorem 2.3(c), $G \to \mathrm{Aut}(X_z)$ is an isomorphism, from which we may recover $X_z \to X_z/G = S_z$. The components of the branch divisor of $X_z \to X_z/G$ are the divisors $\{D_g\}_{g\neq 0}$. All but one of them (all except D_α) are $\mathbb{Q}$-multiples of each other; they are all equivalent to multiples of H, the pullback of the hyperplane divisor in $\mathbb{P}^2$. We may therefore use any divisor from this distinguished collection to recover the blow-down to $\mathbb{P}^2$. The components of the remaining branch divisor D_α give the lines $\mathcal{L}_z \subset \mathbb{P}^2$. From this we may recover z. This discussion clearly extends to the (formal) deformation space around $[X_z]$ in the moduli space. (The argument begins: Let Δ be the formal deformation space, and $\mathcal{X}_z$ be the total family of the deformation. Then $\operatorname{Aut}(\mathcal{X}_z/\Delta)$ is the trivial group scheme G over Δ, from which we obtain $\mathcal{X}_z \to \mathcal{S}_z$, etc.)

2.5. Closing remarks. This result suggests an approach to answering Catanese's question 1.1 in general, by producing a rigid surface as such a branched cover. One might attempt to do so by rigidifying the point-line configuration $\mathcal{L}'_z$ of the start of §2 by adding judiciously chosen additional lines, using a theorem of Paranjape [P, Thm. 2]. One would then have to modify the argument to ensure that *(a)* the four points "marking z" remain distinguished, and *(b)* Theorem 2.3 continues to hold, without the assistance of the positivity of M_χ.

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