THE AFFINE STRATIFICATION NUMBER AND THE MODULI SPACE OF CURVES

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ABSTRACT. We define the *affine stratification number* $\operatorname{asn} X$ of a scheme X. For X equidimensional, it is the minimal number $\operatorname{asn} X$ such that there is a stratification of X by locally closed affine subschemes of codimension at most $\operatorname{asn} X$. We show that the affine stratification number is well-behaved, and bounds many aspects of the topological complexity of the scheme, such as vanishing of cohomology groups of quasicoherent, constructible, and ℓ-adic sheaves. We explain how to bound $\operatorname{asn} X$ in practice. We give a series of conjectures (the first by E. Looijenga) bounding the affine stratification number of moduli spaces of pointed curves, in which the filtration by number of rational components (which first arose in [GV]) plays a role. This investigation is based on work and questions of Looijenga.

CONTENTS

1. INTRODUCTION

The *affine stratification* number of a scheme X bounds the "topological complexity" of a scheme. For example, it bounds the *cohomological dimension* $\operatorname{cd} X$ of X, which is the largest integer n such that $H^n(X, \mathcal{F}) \neq 0$ for some quasicoherent sheaf $\mathcal{F}$ (Prop. 4.12). Similarly, the cohomology of any constructible or ℓ-adic sheaf vanishes in degree greater than $\operatorname{asn} X + \dim X$ (Prop. 4.19). We expect that if the base field is $\mathbb{C}$, then X has the homotopy type of a finite complex of dimension at most $\operatorname{asn} X + \dim X$ (Conj. 4.21), but have not completed a proof. (Unless otherwise stated, all schemes and stacks are assumed to be separated and of finite type over an arbitrary base field.)

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A related, previously studied invariant is the *affine covering number* $\operatorname{acn} X$, which is one less than the minimal number of affine open sets required to cover X. The affine stratification number is bounded by $\operatorname{acn} X$, is better behaved (*e.g.* is bounded by dimension, *cf.* Example 4.8), and has the same topological consequences. We know of no interesting consequences of bounded acn that are not already consequences of the same bound on asn.

For equidimensional X, the definition is particularly simple.

1.1. *Definition.* The (equidimensional) affine stratification number of an equidimensional scheme X is the minimal number $\operatorname{easn} X$ such that there is a (finite) stratification of X by locally closed affine subschemes of codimension at most $\operatorname{easn} X$.

This is the form most likely to be of interest. The appropriate generalization to arbitrary schemes is only slightly more complicated.

1.2. *Definition.* An *affine stratification* of a scheme X is a finite decomposition $X = \bigsqcup_{k \in \mathbb{Z}^{\geq 0}, i} Y_{k,i}$ into disjoint locally closed affine subschemes $Y_{k,i}$, where for each $Y_{k,i}$,

$$\overline{Y}_{k,i} \setminus Y_{k,i} \subseteq \bigcup_{k' > k,\, j} Y_{k',j}. \tag{1}$$

The *length* of an affine stratification is the largest k such that $\cup_j Y_{k,j}$ is nonempty. The *affine stratification number* $\operatorname{asn} X$ of a scheme X is the minimum of the length over all possible affine stratifications of X.

The inclusion in (1) refers to the underlying set. We do not require that each $Y_{k,i}$ be irreducible. We also do not require any relation between k and the dimension or codimension of $Y_{k,i}$ in X. We will see however (Thm. 3.1) that it is always possible to assume that the stratification has a very nice form.

Strictly speaking, the term "stratification" is inappropriate, as $\overline{Y}_{k,i} \setminus Y_{k,i}$ need not be a union of $Y_{k',j}$: let X be the co-ordinate axes in $\mathbb{A}^2$, $Y_{0,1}$ the x-axis, and $Y_{1,1}$ the y-axis minus the origin. However, Theorem 3.1(a) shows that we may take (1) to be an actual stratification.

The affine stratification number has many good properties, including the following (Lem. 2.1, Props. 4.2, 4.6, 2.10).

- $\operatorname{asn} X = 0$ if and only if X is affine.
- $\operatorname{asn} X \leq \dim X$.
- $\operatorname{asn} X \leq \operatorname{acn} X$. (Equality does not always hold.)
- $\operatorname{asn}(X \times Y) \leq \operatorname{asn} X + \operatorname{asn} Y$.
- If D is an effective Cartier divisor on X, then $\operatorname{asn}(X - D) \leq \operatorname{asn} X$.
- If $Y \to X$ is an affine morphism, then $\operatorname{asn} Y \leq \operatorname{asn} X$.

Even if one is only interested in equidimensional schemes, the more general Definition 1.2 has advantages over Definition 1.1. For example, the last property is immediate using Definition 1.2, but not obvious using Definition 1.1.

In Section 2, we establish basic properties of affine stratifications. In Section 3, we show that affine stratifications can be reorganized into a particularly good form. In particular, if X is equidimensional, then $\operatorname{easn} X = \operatorname{asn} X$ (Prop. 3.7), so the notation easn may be discarded. In Section 4, we give topological consequences of bounded asn.

Our motivation is to bound the affine stratification number of moduli spaces (in particular, of pointed curves) to obtain topological and cohomological consequences. We describe our work in progress in the form of several conjectures in Section 5. For example, the conjectures bound the homotopy type of the moduli spaces of curves (a) of compact type, (b) with "rational tails", and (c) with at most k rational components (a locus introduced in [GV]), see Proposition 5.9.

1.3. *Acknowledgments.* This note arose from our ongoing efforts to prove a conjecture of E. Looijenga, and much of what is here derives from questions, ideas, and work of his. In particular, we suspect that he is aware of most of the results given here, and that we are following in his footsteps. We thank him for inspiration. We also thank J. Starr for Example 4.8, and T. Graber for helpful conversations. Finally, we are grateful to the organizers of the 2003 conference *Algebraic structures and moduli spaces*, at the Centre de Recherches Mathématiques (CRM), which led to this work.

2. Basic properties of affine stratifications

The most basic property is that an affine stratification always exists, and hence $\operatorname{asn} X$ is defined for any scheme X: if $\cup_{i=0}^n U_i$ is a covering of X by open affine sets, then

$$U_0 \sqcup (U_1 \setminus U_0) \sqcup (U_2 \setminus (U_0 \cup U_1)) \sqcup \cdots \tag{2}$$

gives an affine stratification of X.

The following lemma is trivial.

2.1. *Lemma.* —

(a) *The affine stratification number depends only on the reduced structure of X,* i.e. $\operatorname{asn} X = \operatorname{asn} X^{red}$.
(b) *If $X \to Y$ is an affine morphism, then* $\operatorname{asn} X \leq \operatorname{asn} Y$.
(c) $\operatorname{asn}(X \times Y) \leq \operatorname{asn} X + \operatorname{asn} Y$.
(d) *If D is an effective* Cartier *divisor on X, then* $\operatorname{asn}(X - D) \leq \operatorname{asn} X$.

Part (d) requires the following well-known fact.

2.2. *Lemma.* — *Any irreducible affine scheme X, minus an effective* Cartier *divisor D, is affine.*

(Reason: the inclusion $X \setminus D \hookrightarrow X$ is an affine morphism, since this can be verified locally. But X is affine.)

Here is a partial converse to Lemma 2.2. A more precise converse is given in Proposition 2.6.

2.3. *Lemma. — Suppose that V is an irreducible affine scheme, and that $U \subset V$ is an open affine subset. Then the complement $Z := V \setminus U$ is a Weil divisor in V.*

Proof. We first assume that V (and hence U) is normal. Let $Z = \cup_i Z_i$ be the decomposition of Z into irreducible components, and let $Z' = \cup_j Z_j$ be the union of those components of codimension one in V. We set $U' = V \setminus Z'$, and let $i : U \hookrightarrow U'$ be the natural open immersion. Since U' is normal, and the complement of U in U' is of codimension at least 2 in U', we have $i_*\mathcal{O}_U = \mathcal{O}_{U'}$. We will use this and the fact that both U and V are affine to see that $U = U'$.

Let $A = \Gamma(V, \mathcal{O}_V)$ and $B = \Gamma(U, \mathcal{O}_U) = \Gamma(U', \mathcal{O}_{U'})$. We have an inclusion of rings $A \hookrightarrow B$ corresponding to the opposite inclusion of open sets. Suppose that $U \neq U'$, and let x be any point of $U' \setminus U$. Since V is affine, x corresponds to a prime ideal $\mathcal{P}_x$ of A. Since $x \in U'$, no element of $\mathcal{P}_x$ can be a unit in $\Gamma(U', \mathcal{O}_{U'})$, and hence $\mathcal{P}_x$ remains a prime ideal in B, which is a localization of A. Therefore, since U is affine, $x \in U$, contrary to assumption.

Passing to the general case, we drop the assumption that V and U are normal, and let $\widetilde{V}$ and $\widetilde{U}$ be their normalizations. We have the commutative diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r, hook] \arrow[d] & \widetilde{V} \arrow[d] \\ U \arrow[r, hook] & V \end{tikzcd}$$

where the vertical arrows are the normalization maps, and the horizontal arrows are open immersions. By the first part of the lemma, the complement $\widetilde{Z}$ of $\widetilde{U}$ in $\widetilde{V}$ is of codimension one in $\widetilde{V}$. Since $\widetilde{Z}$ maps finitely and surjectively onto Z, $\dim(Z) = \dim(\widetilde{Z})$, and hence Z is of codimension one in V. $\square$

The next corollary follows immediately. (Note that X need not be equidimensional here.)

2.4. *Corollary. — The complement of a dense affine open subset in any scheme is of pure codimension one.*

2.5. *Examples.* *(a)* Let X be the affine cone over an elliptic curve, embedded in $\mathbb{CP}^2$ as a cubic. Let Z be the cone over any point of the curve of infinite order in the group law. Then $X \setminus Z$ is affine, but Z is not $\mathbb{Q}$-Cartier. This shows that the complement of an affine open set in an affine scheme need not be the support of a Cartier divisor: we cannot hope to improve the conclusion of Lemma 2.3 to match the hypothesis of Lemma 2.2.

(b) Let S be $\mathbb{P}^2$ blown up at a point, and let X be the affine cone over some projective embedding of S. Let $Z \subset X$ be the affine cone over the exceptional divisor of the blowup. Then Z is of codimension one in X, but $\operatorname{cd}(X \setminus Z) = 1$, so in particular it is not affine. This shows that, conversely, the complement of a Weil divisor in an affine scheme need not be affine: we can not hope to improve the hypothesis of Lemma 2.2 to match the conclusion of Lemma 2.3.

However, there is a more precise statement giving a necessary and sufficient condition on a closed subset Z of an affine scheme V for the complement $V \setminus Z$ to be affine.

2.6. *Proposition. — Let V be an affine scheme (possibly reducible) and Z a closed subset of V. Then $U := V \setminus Z$ is affine if and only if $H^i_Z(\mathcal{F}) = 0$ for all quasicoherent sheaves $\mathcal{F}$ on V and all $i \geq 2$.*

Here $H^i_Z(\mathcal{F})$ is the local cohomology group. This also implies the same fact for the local cohomology sheaves $\mathcal{H}^i_Z(\mathcal{F})$, see Corollary 2.7(a) below.

Proof. Let $\mathcal{F}$ be any quasicoherent sheaf on V. We have the long exact excision sequence of cohomology groups

$$\begin{array}{cccccccc} 0 & \longrightarrow & H^0_Z(\mathcal{F}) & \longrightarrow & H^0(V,\mathcal{F}) & \longrightarrow & H^0(U,\mathcal{F}|_U) & \\ & \longrightarrow & H^1_Z(\mathcal{F}) & \longrightarrow & H^1(V,\mathcal{F}) & \longrightarrow & H^1(U,\mathcal{F}|_U) & \\ & \longrightarrow & H^2_Z(\mathcal{F}) & \longrightarrow & H^2(V,\mathcal{F}) & \longrightarrow & H^2(U,\mathcal{F}|_U) & \longrightarrow \cdots . \end{array} \tag{3}$$

Since V is affine, we have $H^i(V, \mathcal{F}) = 0$ for all $i \geq 1$, so that $H^i(U, \mathcal{F}|_U) = H^{i+1}_Z(\mathcal{F})$ for all $i \geq 1$. Hence (using Serre's criterion for affineness) U is affine if and only if $H^i_Z(\mathcal{F}) = 0$ for all $i \geq 2$ and all quasicoherent sheaves $\mathcal{F}$. □

2.7. *Corollary. — Let X be a scheme (possibly reducible) and U a dense affine open subset. Let $Z := X \setminus U$. For any quasicoherent sheaf $\mathcal{F}$ on X,*

(a) $\mathcal{H}^i_Z(\mathcal{F}) = 0$ *for all* $i \geq 2$, *and*
(b) $H^i_Z(\mathcal{F}) = 0$ *for all* $i > \operatorname{cd} Z + 1$.

The notation cd denotes cohomological dimension, see Section 1.

Proof. (a) Since the local cohomology sheaf $\mathcal{H}^i_Z(\mathcal{F})$ is the sheafification of the functor $V \mapsto H^i_{Z \cap V}(\mathcal{F}|_V)$ [GH, Prop. 1.2], it is sufficient to check that the local cohomology group vanishes for sufficiently small V around any point of Z. But if V is any open affine set, then $V \cap U$ is nonempty (since U is dense) and also affine (by separatedness). Hence $H^i_{Z \cap V}(\mathcal{F}|_V) = 0$ for $i \geq 2$ by Proposition 2.6 and so $\mathcal{H}^i_Z(\mathcal{F}) = 0$ as well.

(b) The local cohomology sheaves $\mathcal{H}^i_Z(\mathcal{F})$ are quasicoherent and are supported on Z. The local cohomology groups can be computed by a spectral sequence with E_2^{pq} term $H^p(X, \mathcal{H}^q_Z(\mathcal{F})) = H^p(Z, \mathcal{H}^q_Z(\mathcal{F}))$. Since $H^p(Z, \cdot) = 0$ for $p > \operatorname{cd} Z$, and $\mathcal{H}^q_Z(\mathcal{F}) = 0$ for $q > 1$ by part (a), we have $H^i_Z(\mathcal{F}) = 0$ for $i > \operatorname{cd} Z + 1$. □

2.8. *Corollary. — Let X be a scheme, U a dense affine open subset, and set $Z := X \setminus U$. Then* $\operatorname{cd} X \le \operatorname{cd} Z + 1$.

Proof. For any any quasicoherent sheaf $\mathcal{F}$ on X, the excision sequence (3) and the fact that U is affine gives $H^i(X, \mathcal{F}) = H^i_Z(\mathcal{F})$ for all $i \ge 2$, and that $H^1(X, \mathcal{F})$ is a quotient of $H^1_Z(\mathcal{F})$. Hence, for any $i \ge 1$, $H^i_Z(\mathcal{F}) = 0$ implies that $H^i(X, \mathcal{F}) = 0$. Since $H^i_Z(\mathcal{F}) = 0$ for all $i > \operatorname{cd} Z + 1$ by Corollary 2.7(b), we have $\operatorname{cd} X \le \operatorname{cd} Z + 1$. □

2.9. Bounding asn **by finite flat covers.** The following result is useful to bound $\operatorname{asn} X$ by studying covers of X.

2.10. *Proposition. — Suppose $\pi : Y \to X$ is a surjective finite flat morphism of degree not divisible by the characteristic of the base field, and Y is affine. Then X is affine.*

Proof. The hypothesis implies that $\pi_* \mathcal{O}_Y$ is a coherent locally free sheaf on X. The trace map gives a splitting $\pi_* \mathcal{O}_Y \cong \mathcal{O}_X \oplus E$ for some vector bundle E on X. If $\mathcal{F}$ is any coherent sheaf on X, then the flatness of π gives $\pi_* \pi^* \mathcal{F} = \mathcal{F} \oplus (E \otimes \mathcal{F})$, and it then follows from the Leray spectral sequence and the finiteness of π that $H^i(X, \mathcal{F})$ is a direct summand of $H^i(Y, \pi^* \mathcal{F})$ for all $i \ge 0$. Since Y is affine, these vanish if $i \ge 1$, hence the cohomology groups on X do as well, and therefore X is affine by Serre's criterion for affineness. □

3. REORGANIZING AFFINE STRATIFICATIONS

We describe various ways that we can reorganize the stratification which are more convenient for analyzing X. The main results of this section are summarized in the following theorem.

3.1. *Theorem. — If X is any scheme and $\operatorname{asn} X = m$, then there exists an affine stratification $\{Z_0, \dots, Z_m\}$ of X such that for any $k \le m$:*

(a) $\overline{Z}_k = \cup_{k' \ge k} Z_{k'}$,
(b) *each Z_k is a dense open affine subset of $\overline{Z}_k$, and*
(c) *$\overline{Z}_k$ is of pure codimension one in $\overline{Z}_{k-1}$.*

If in addition X is equidimensional, then we also have

(d) *each $\overline{Z}_{k'}$ is of pure codimension $k' - k$ in $\overline{Z}_k$ for any $k' \ge k$.*

Even if X is not equidimensional, if we have an affine stratification $\{Y'_{k,l}\}$ of length M such that each $Y'_{k,i}$ is of pure codimension k in X, then setting $Z'_k := \cup_i Y'_{k,i}$ for $k = 0, \dots, M$ we have

(e) *the affine stratification $\{Z'_0, \dots, Z'_M\}$ satisfies (a)–(d) above.*

(We are not guaranteed that $M = m$, so this stratification may not be optimal.)

The proof is summarized in Section 3.9. In analogy with CW-complexes, we define an *affine cell decomposition* of a scheme X to be an affine stratification

$$X = \bigsqcup_k Z_k$$

where the Z_k's satisfy (a)–(c) of Theorem 3.1. The theorem guarantees that such a decomposition exists for any scheme X, with length $\operatorname{asn} X$.

3.2. *Lemma. — Let $\{Y_{k,i}\}$ be an affine stratification of a scheme X and let $Z_k := \cup_i Y_{k,i}$ be the union of all the affine pieces of index k. Then each Z_k is an open dense affine subset of $\overline{Z}_k$,* i.e. *Z_k is locally closed and affine.*

Proof. By definition, Z_k is a dense subset of $\overline{Z}_k$. We will see that it is an open subset, and most importantly, affine.

Since the affine stratification is finite, we have $\overline{\cup_i Y_{k,i}} = \cup_i \overline{Y}_{k,i}$. For any distinct $Y_{k,i}$ and $Y_{k,j}$ and any point $y \in \overline{Y}_{k,i} \cap \overline{Y}_{k,j}$, the fact that the Y's are disjoint, along with the stratification condition (1), implies that y must be in some $Z_{k'}$ with $k' > k$. In particular, y is in neither $Y_{k,i}$ nor $Y_{k,j}$.

If we let $C_i := \cup_{j \neq i} \overline{Y}_{k,j}$ be the closed subset consisting of the closures of other $Y_{k,j}$'s, and $V_i := X \setminus C_i$ the open complement, then the previous remark shows that $Y_{k,i} \subseteq V_i$, and therefore that $Z_k \cap V_i = Y_{k,i}$.

Since every locally closed subset is an open subset of its closure, $Y_{k,i}$ is an open subset of $\overline{Y}_{k,i} \cap V_i = \overline{Z}_k \cap V_i$. Since $\overline{Z}_k \cap V_i$ is an open subset of $\overline{Z}_k$, we see that $Y_{k,i}$ is an open subset of $\overline{Z}_k$, and therefore that $Z_k = \cup_i Y_{k,i}$ is an open subset of $\overline{Z}_k$.

Let $\widetilde{Z}_k$ be the disjoint union

$$\widetilde{Z}_k = \bigsqcup_i Y_{k,i},$$

and $f : \widetilde{Z}_k \longrightarrow X$ the natural morphism with image Z_k. The map f is one-to-one on points, and the fact that $Z_k \cap V_i = Y_{i,k}$ for each i implies that f is a homeomorphism, and in fact an immersion. Therefore, $\widetilde{Z}_k \cong Z_k$ as schemes, and so Z_k is affine since $\widetilde{Z}_k$ is. $\square$

3.3. *Proposition. — Let $\{Y_{k,i}\}$ be an affine stratification of X of length m. Then there exists an affine stratification $\{Y'_{k,j}\}$ of length at most m such that the generic points of all components of X are contained in the zero stratum $\cup_j Y'_{0,j}$ of $\{Y'_{k,j}\}$.*

Proof. We first set $Y'_{0,i} := Y_{0,i}$ for all valid indices i. Now let $Y_{k,i}$ be any piece of the stratification with $k \geq 1$. If $Y_{k,i}$ does not contain the generic point of any component of X then set $Y'_{k,i} := Y_{k,i}$. On the other hand, suppose that $Y_{k,i}$ contains $\eta_1, \ldots, \eta_r$ where each η_j is a generic point of X. In this case, for each $j \in \{1, \ldots, r\}$ choose an open affine subset

U_j of $Y_{k,i}$ containing η_j so that U_j intersects no components of X other than $\overline{\{\eta_j\}}$. Now set $Y'_{k,j} := Y_{k,j} \setminus (\cup_{j=1}^{r} U_j)$, and add the U_j in as elements of the zero stratum, $Y'_{0,i_j} := U_j$, where the i_j are chosen not to conflict with previously existing indices. It is straightforward to verify that this decomposition satisfies the affine stratification condition (1). □

3.4. *Lemma.* — *Let $\{Y_{k,i}\}$ be an affine stratification of X of length m. Then there exists an affine stratification $\{Y'_{k,i}\}$ of length at most m such that if $Z'_k := \cup_i Y'_{k,i}$ is the union of all affine pieces of index k, then for any k,*

$$\overline{Z}'_k = \cup_{k' \geq k} Z'_{k'}.$$

Proof. By Proposition 3.3 we may assume that all the generic points of components of X occur in the zero stratum of $\{Y_{k,i}\}$, and therefore that $\cup_i \overline{Y}_{0,i} = X$. We now proceed by induction on the length m of the stratification, the case $m = 0$ being trivial.

Let $U := \cup_i Y_{0,i}$ be the union of the pieces in the zero stratum, and $Z := \cup_{k \geq 1, i} Y_{k,i}$ the complement. Note that U is open and hence Z is closed by Lemma 3.2.

The $\{Y_{k,i}\}$ with $k \geq 1$ form an affine stratification of Z of length $m-1$ (after reindexing the k's to start with zero). Therefore by induction Z has an affine stratification of length at most $m-1$ satisfying the hypothesis of the lemma. Reindexing the k's again, and adding the $Y_{0,i}$'s as the zero stratum, we end up with an affine stratification $\{Y'_{k,i}\}$ of length at most m which also satisfies the hypothesis of the lemma, completing the inductive step. □

3.5. *Corollary.* — *For any scheme X, if $\operatorname{asn} X = m$ then there is an affine stratification $\{Z_0, \ldots, Z_m\}$ of X with $\overline{Z}_k = \cup_{k' \geq k} Z_{k'}$ for each k, and such that each Z_k is an open dense affine subset of $\overline{Z}_k$.*

Proof. Combine Lemmas 3.2 and 3.4. □

3.6. *Corollary.* — *For any scheme X, $\operatorname{asn} X \leq \dim X$.*

Proof. Let $m = \operatorname{asn} X$ and $\{Z_0, \ldots, Z_m\}$ be a stratification as in Corollary 3.5. By Corollary 2.4, each $\overline{Z}_{k+1}$ is of pure codimension one in $\overline{Z}_k$. If $\overline{Z}'_m$ is any irreducible component of $\overline{Z}_m$, then that means we can inductively find a chain of closed irreducible subsets $\overline{Z}'_m \subset \overline{Z}'_{m-1} \subset \overline{Z}'_{m-2} \subset \cdots \subset \overline{Z}'_1 \subset \overline{Z}'_0$, with each $\overline{Z}'_k$ an irreducible component of $\overline{Z}_k$. Then $\dim X \geq \dim \overline{Z}'_m + m \geq m$. □

If we assume an additional hypothesis about X or the stratification $\{Y_{k,i}\}$, we have slightly stronger results about the stratification $\{Z_0, \ldots, Z_m\}$ of Corollary 3.5.

3.7. *Proposition. — If X is an equidimensional scheme, and $\{Z_0, \dots, Z_m\}$ the stratification of Corollary 3.5, then we have in addition that $\overline{Z}_{k'}$ is of pure codimension $k' - k$ in $\overline{Z}_k$ for all $k' \geq k$. In particular,* $\operatorname{easn} X = \operatorname{asn} X$.

Proof. By Corollary 2.4 each $\overline{Z}_{k+1}$ is of pure codimension one in $\overline{Z}_k$. If $\overline{Z}_0 = X$ is equidimensional, then it follows that each $\overline{Z}_k$ is equidimensional as well, and from this that $\overline{Z}_{k'}$ is of pure codimension $k' - k$ in $\overline{Z}_k$ for any $k' \geq k$. □

Even if X is not equidimensional, if the affine stratification $\{Y_{k,i}\}$ satisfies a suitable condition we get a similar good result about the stratification by the Z_k's.

3.8. *Proposition. — Let $\{Y_{k,i}\}$ be an affine stratification of a scheme X with each $Y_{k,i}$ of pure codimension k in X. Let $Z_k := \cup_i Y_{k,i}$ be the union of all the affine pieces of codimension k. Then*

(i) *For $k' \geq k$, $\overline{Z}_{k'}$ is of pure codimension $k' - k$ in $\overline{Z}_k$; in particular, $\overline{Z}_{k'} \subseteq \overline{Z}_k$.*
(ii) $\overline{Z}_k = \cup_{k' \geq k} Z_{k'}$.

Proof. Since the decomposition is finite, the irreducible components of $\overline{Z}_m$ are all of the form $\overline{W}_m$ with W_m an irreducible component of some $Y_{m,j}$.

We prove (i) by induction on k. For $k = 0$ the result is obvious, since $\overline{Z}_0 = X$, and $\overline{Z}_{k'}$ is of pure codimension k' in X. So assume that $k > 0$ and that (i) is true for $k - 1$.

Let $\overline{W}_{k'}$ by any irreducible component of $\overline{Z}_{k'}$ with $k' \geq k$. By the induction hypothesis, $\overline{W}_{k'} \subset \overline{Z}_{k-1}$, and is of codimension $k' - k + 1$ in $\overline{Z}_{k-1}$. Let T_{k-1} be any irreducible component of Z_{k-1} whose closure contains $\overline{W}_{k'}$ and such that $\operatorname{codim}(\overline{W}_{k'}, \overline{T}_{k-1}) = k' - k + 1$. Lemma 3.2 gives us that Z_{k-1} is affine, and therefore T_{k-1} is affine also.

By Lemma 2.3, the closed set $\overline{T}_{k-1} \setminus T_{k-1}$ has codimension one in $\overline{T}_{k-1}$. Let η_k be the generic point of any component of $\overline{T}_{k-1} \setminus T_{k-1}$ containing $W_{k'}$; one exists by our choice of T_{k-1}. Since $\operatorname{codim}(\overline{W}_{k'}, \overline{T}_{k-1}) = k' - k + 1$ and $\operatorname{codim}(\overline{\{\eta_k\}}, \overline{T}_{k-1}) = 1$, we have $\operatorname{codim}(W_{k'}, \overline{\{\eta_k\}}) = k' - k$.

The Z_m's partition X, and so η_k must be in exactly one Z_m. We cannot have $m \leq k - 1$, since that would contradict the stratification condition. We cannot have $m \geq k + 1$, since this would contradict $\operatorname{codim}(\overline{Z}_m, \overline{Z}_{k-1}) = m - k + 1$, which holds by the induction hypothesis. Therefore η_k is in Z_k, and so $\overline{W}_{k-1} \subset \overline{Z}_k$.

We have already seen that $\operatorname{codim}(W_{k'}, \overline{\{\eta_k\}}) = k' - k$. Since

$$\operatorname{codim}\left(\overline{W}_{k'}, \overline{Z}_k\right) = \sup_i \left(\operatorname{codim}\left(\overline{W}_{k'}, \overline{Y}_{k,i}\right)\right),$$

with $\overline{Y}_{k,i}$ running over the components of $\overline{Z}_k$, we get $\operatorname{codim}(\overline{W}_{k'}, \overline{Z}_k) \geq k' - k$. But for any three closed schemes $\overline{W}$, $\overline{Z}$, and X with $\overline{W} \subseteq \overline{Z} \subseteq X$, we always have

$$\operatorname{codim}\left(\overline{W}, \overline{Z}\right) + \operatorname{codim}\left(\overline{Z}, X\right) \leq \operatorname{codim}\left(\overline{W}, X\right).$$

Since the codimensions of $\overline{W}_{k'}$ in X, and $\overline{Z}_k$ in X are k' and k by hypothesis, this gives $\operatorname{codim}(\overline{W}_{k'}, \overline{Z}_k) \leq k' - k$, and hence $\operatorname{codim}(\overline{W}_{k'}, \overline{Z}_k) = k' - k$. Therefore $\overline{Z}_{k'}$ is contained in $\overline{Z}_k$, and is of pure codimension $k' - k$, completing the inductive step for (i).

To prove (ii), the stratification condition gives $\overline{Z}_k \subseteq \cup_{k' \geq k} Z_{k'}$, while part (i) above gives the opposite inclusion. □

3.9. *Proof of Theorem 3.1.* Part (a) is Lemma 3.4, part (b) Lemma 3.2, part (c) Corollary 2.4, part (d) Proposition 3.7, and part (e) Proposition 3.8 and Lemma 3.2 again. □

4. Topological consequences of bounded affine stratification number

We now describe the topological consequences of bounded asn, in particular: relation to dimension (Sect. 4.1), affine covering number (Sect. 4.5), cohomological dimension (for quasicoherent sheaves, Sect. 4.11, as well as constructible and ℓ-adic sheaves, Sect. 4.15), dimension of largest proper subscheme (Sect. 4.14), and homotopy type (Sect. 4.20).

4.1. Relation to dimension.

4.2. *Proposition.* — $\operatorname{asn} X \leq \dim X$. *If one top dimensional component of X is proper, then equality holds.*

The first statement is Corollary 3.6. The second statement follows from Proposition 4.12 ($\operatorname{cd} \leq \operatorname{asn}$) and the following theorem, first conjectured by Lichtenbaum.

4.3. *Theorem (Grothendieck* [GH, 6.9], *Kleiman* [K, Main Thm.]*).* — *If $d = \dim X$, then $\operatorname{cd} X = d$ if and only if at least one d-dimensional component of X is proper.*

4.4. *Example: All values between 0 and $\dim X$ are possible.* Let $X_k = \mathbb{P}^n \setminus \{(n-k-1)\text{-plane})\}$, for k between 0 and $n-1$. Then clearly $\operatorname{cd} X_k = k$ and $\operatorname{asn} X_k \leq k$. We will see that $\operatorname{cd} \leq \operatorname{asn}$ (Prop. 4.12), from which the result follows.

4.5. Relation to affine covering number. Recall that the affine covering number $\operatorname{acn} X$ of a scheme X is the minimal number of affine open subsets required to cover X, minus 1. The invariant acn does not obviously behave as well as asn with respect to products (*cf.* Lemma 2.1(c)); it also is not bounded by dimension (Example 4.8 below).

The argument of (2) gives the following.

4.6. *Proposition.* — $\operatorname{asn} X \leq \operatorname{acn} X$.

4.7. *Example.* In general, $\operatorname{acn} X \neq \operatorname{asn} X$. As an example, let X be a complex K3 surface with Picard rank 1, minus a very general point. Then $\operatorname{acn} X = 2$: if for every point p of X, $\operatorname{acn}(X - p) = 1$, then (given the hypothesis that the Picard rank is 1) any two points of X are equivalent in $A_0(Y)$ (with $\mathbb{Q}$-coefficients), contradicting Mumford's theorem that $A_0(Y)$ is not countably generated [M]. Example 4.8 below gives another example (in light of Proposition 4.2).

4.8. *Example:* $\operatorname{acn} X$ *may be larger than* $\dim X$. When X is quasiprojective, $\operatorname{acn} X \leq \dim X$. (*Reason:* Let $\overline{X}$ be a projective compactification such that the complement $\overline{X} \backslash X$ is a Cartier divisor D. Consider an embedding $\overline{X} \hookrightarrow \mathbb{P}^n$ and let $H_0, \dots, H_{\dim X}$ be hypersurfaces so that $\overline{X} \cap H_0 \cap \cdots \cap H_{\dim X} = \emptyset$. Then the $\overline{U_i} := \overline{X} \setminus (\overline{X} \cap H_i)$ form an affine cover of $\overline{X}$. We conclude using Lemma 2.2.)

However, the following example, due to J. Starr, shows that $\operatorname{acn} X$ may be greater than $\dim X$. Given any n, we describe a reducible, reduced threefold that requires at least n affine open sets to cover it. Recall Hironaka's example (*e.g.* [H, Ex. B.3.4.1]) of a nonsingular proper nonprojective threefold X. Nonprojectivity is shown by exhibiting two curves ℓ and m whose sum is numerically trivial. Hence no affine open set can meet both ℓ and m; otherwise its complement would be a divisor (Lemma 2.3), hence Cartier (as X is nonsingular), which meets both ℓ and m positively. Now choose points p and ℓ and q on m. Consider $\binom{n}{2}$ copies of (X, p, q), corresponding to ordered pairs (i, j) $(1 \leq i < j \leq n)$; call these copies (X_{ij}, p_{ij}, q_{ij}). Let $r_1, \dots, r_n$ be copies of a reduced point. Glue r_i to p_{ji} and q_{ik}. Then no affine open can contain both r_i and r_j for $i < j$ (by considering X_{ij}).

4.9. *Example: a family of integral threefolds with arbitrary high affine covering number.* This leads to an example of an integral (but singular) threefold that requires at least n affine open sets to cover it. (Question: Is there a family of nonsingular irreducible varieties of fixed dimension with unbounded affine covering number?) Our example will be a blow-up of $\mathbb{P}^3$. Choose n curves $C_1, \dots, C_n$ in $\mathbb{P}^2 \subset \mathbb{P}^3$ that meet in n simple n-fold points $p_1, \dots,$ p_n (and possibly elsewhere). Away from $p_1, \dots, p_n$ blow up $C_1, \dots, C_n$ in some arbitrary order. In a neighborhood of p_i (not containing any other intersection of the C_j) blow up C_i first (giving a smooth threefold) and then blow up the local complete intersection $\cup_{j \neq i} C_j$ (or more precisely, the proper transform thereof), giving a threefold with a single singularity (call it q_i). The preimage of p_i is the union of two $\mathbb{P}^1$'s, one arising from the exceptional divisor of C_i (call it ℓ_i), and one from the exceptional divisor of $\cup_{j \neq i} C_j$; they meet at q_i. By Hironaka's argument, $\ell_i + \ell_j$ is numerically trivial for all $i \neq j$. Then no affine open U can contain both q_i and q_j: the complement of U would be a divisor, Cartier near $\ell_i - q_i$ and $\ell_j - q_j$ (as these are smooth points of our threefold), and the same contradiction applies.

This is also an example of a scheme which cannot be embedded in any smooth scheme. If $X \hookrightarrow W$ with W smooth, then for any divisor D (automatically Cartier) on W, $D \cdot \ell_i + D \cdot \ell_j = 0$ for all i, j. If $n \geq 3$, this implies that $D \cdot \ell_i = 0$ for all i. But for any affine open set U of W with $U \cap \ell_i \neq \emptyset$, the complement $D = W \setminus U$ would intersect ℓ_i properly, giving us the contradiction $D \cdot \ell_i > 0$. Hence no such embedding is possible.

By combining Proposition 4.2 with Proposition 4.6, we obtain the following.

4.10. *Proposition. — If X is proper, then* $\operatorname{acn} X \geq \dim X$.

We note that this also follows from Theorem 4.3. Example 4.7 shows that it is *not* true that $\operatorname{acn} X = \dim X$ if and only if X is proper, even for quasiprojective X.

4.11. Relation to cohomological dimension. Just as the dimensions of the cells in a CW-complex bounds the topological (co)homology, the length of the stratification into affine cells bounds the quasicoherent sheaf cohomology.

4.12. *Proposition. —* $\operatorname{cd} X \leq \operatorname{asn} X$.

Proof. We prove the result by induction on $\operatorname{asn} X$. It is clear for $\operatorname{asn} X = 0$, so assume that $m := \operatorname{asn} X > 0$ and that the result is proven for all schemes Z with $\operatorname{asn} Z < \operatorname{asn} X$. Let $\{Z_0, \dots, Z_m\}$ be an affine cell decomposition given by Theorem 3.1. Set $Z := X \setminus Z_0 = \overline{Z}_1 = \cup_{k \geq 1} Z_k$.

By Theorem 3.1(b) Z_0 is an open dense affine subset of X, so by Corollary 2.8, $\operatorname{cd} X \leq \operatorname{cd} Z + 1$. Next, $Z = \sqcup_{k=1}^{m-1} Z_k$ is (after reindexing) an affine stratification of Z of length $m-1$, so $\operatorname{asn} Z \leq \operatorname{asn} X - 1$. Finally, by the inductive hypothesis, $\operatorname{cd} Z \leq \operatorname{asn} Z$. Combining these three inequalities gives $\operatorname{cd} X \leq \operatorname{asn} X$, completing the inductive step. □

We remark in passing that by combining Proposition 4.12 with Corollary 3.6 we obtain another proof of Grothendieck's dimensional vanishing theorem ([G, Thm. 3.6.5], [H, Thm. III.2.7]).

We conclude with an obvious result.

4.13. *Proposition. —* $\operatorname{cd} X = 0$ *if and only if* $\operatorname{asn} X = 0$ *if and only if* $\operatorname{acn} X = 0$.

Proof. Each of the three is true if and only if X is affine (the first by Serre's criterion for affineness). □

4.14. Relation to dimension of largest complete subscheme. Motivated by Diaz' theorem [D], let $\operatorname{psv} X$ be the largest dimension of a proper closed subscheme of X. If Z is a proper closed subscheme of X (with inclusion $j : Z \hookrightarrow X$), and if $\mathcal{F}$ is a quasicoherent sheaf on Z, then $j_* \mathcal{F}$ is a quasicoherent sheaf on X, and $H^i(X, j_* \mathcal{F}) = H^i(Z, \mathcal{F})$ for all i. By Theorem 4.3 we can find a quasicoherent sheaf $\mathcal{F}$ on Z with $H^{\dim Z}(Z, \mathcal{F}) \neq 0$, and so this gives $\operatorname{psv} X \leq \operatorname{cd} X$. Hence by Proposition 4.12,

$$\operatorname{psv} X \leq \operatorname{asn} X.$$

4.15. Relation to cohomological vanishing for constructible and ℓ-adic sheaves. In this section all notions related to sheaves (including stalks, pushforwards, and cohomology groups) are with respect to the étale topology. For instance, "sheaf on X" means "sheaf on X in the étale topology".

To show how asn implies cohomological vanishing for constructible and ℓ-adic sheaves (Cor. 4.19), we first recall a theorem and some notation of Artin. For any (étale) sheaf F of abelian groups on X, let

$$\mathrm{d}(\mathsf{F}) := \sup \left\{ \dim \left(\overline{\{x\}} \right) \mid x \in X,\ \mathsf{F}_{\overline{x}} \neq 0 \right\}$$

be the dimension of the support of F.

4.16. *Artin's Theorem* [A2, Thm. 3.1]. — *Let $f : X \longrightarrow Y$ be an affine morphism of schemes of finite type over a field k, and F a torsion sheaf* (i.e. *sheaf of torsion groups) on X. Then $\mathrm{d}(R^q f_* \mathsf{F}) \leq \mathrm{d}(\mathsf{F}) - q$ for all $q \geq 0$.*

We will apply Artin's Theorem in the following form:

4.17. *Proposition.* — *Suppose that X is a scheme, U an affine open subset of X, and $Z := X \setminus U$ the complement. Then for any torsion sheaf F on X,*

$$\mathrm{d}(\mathcal{H}^q_{\text{ét},Z}(\mathsf{F})) \leq \mathrm{d}(\mathsf{F}) - q + 1.$$

Here the $\mathcal{H}^q_{\text{ét},Z}(\mathsf{F})$ are the local cohomology sheaves in the étale topology. The usual excision and spectral sequences for local cohomology remain true in the étale setting, see [V, Sec. 6].

Proof. If $i : U \hookrightarrow X$ is the inclusion, then for any sheaf F of abelian groups on X we have the exact sequence [V, Prop. 6.5]

$$0 \longrightarrow \mathcal{H}^0_{\text{ét},Z}(\mathsf{F}) \longrightarrow \mathsf{F} \longrightarrow i_*(\mathsf{F}|_U) \longrightarrow \mathcal{H}^1_{\text{ét},Z}(\mathsf{F}) \longrightarrow 0,$$

as well as isomorphisms

$$\mathcal{H}^q_{\text{ét},Z}(\mathsf{F}) \cong R^{q-1} i_*(\mathsf{F}|_U) \text{ for all } q \geq 2.$$

If $q \geq 2$ the proposition then follows from the above isomorphism and Artin's Theorem 4.16 applied to the inclusion morphism i, which is affine since U is.

If $q = 1$ it is enough to bound $\mathrm{d}(i_*(\mathsf{F}|_U))$, since $\mathcal{H}^1_{\text{ét},Z}(\mathsf{F})$ is a quotient of $i_*(\mathsf{F}|_U)$. The points $x \in X$ where $(i_*(\mathsf{F}|_U))_{\overline{x}} \neq 0$ are the points $x \in U$ with $\mathsf{F}_{\overline{x}} \neq 0$ and points $x \in Z$ such that there exists a point $x' \in U$, $x \in \overline{\{x'\}}$ with $\mathsf{F}_{\overline{x'}} \neq 0$. In particular, the support of $i_*(\mathsf{F}|_U)$ is contained in the support of F, so $\mathrm{d}(\mathcal{H}^1_{\text{ét},Z}(\mathsf{F})) \leq \mathrm{d}(\mathsf{F})$, which is exactly the statement of the proposition when $q = 1$.

If $q = 0$ note that $\mathrm{d}(\mathcal{H}^0_{\text{ét},Z}(\mathsf{F})) \leq \mathrm{d}(\mathsf{F})$ since $\mathcal{H}^0_{\text{ét},Z}(\mathsf{F})$ is a subsheaf of F, while the proposition only claims the weaker bound $\mathrm{d}(\mathcal{H}^0_{\text{ét},Z}(\mathsf{F})) \leq \mathrm{d}(\mathsf{F}) + 1$. □

4.18. *Lemma.* — *If F is a torsion sheaf on X, then $H^n_{\acute{e}t}(X, \mathsf{F}) = 0$ for all $n > \mathrm{d}(\mathsf{F}) + \operatorname{asn} X$.*

Proof. We show the result by induction on $\operatorname{asn} X$, the case $\operatorname{asn} X = 0$ being Artin's Theorem 4.16 again. Let $\{Z_0, \dots, Z_{\operatorname{asn} X}\}$ be an affine cell decomposition of X (as given by Theorem 3.1). Set $Z := X \setminus Z_0 = \cup_{k \geq 1} Z_k$.

We have $H^n_{\text{ét}}(Z_0, \mathsf{F}|_{Z_0}) = 0$ for all $n > \mathrm{d}(\mathsf{F}|_{Z_0})$ by Artin's Theorem, and since $\mathrm{d}(\mathsf{F}|_{Z_0}) \leq \mathrm{d}(\mathsf{F})$ the excision sequence (eqn. (3) holds in this context, [V, (6.5.3)]) shows that $H^n_{\text{ét}}(X, \mathsf{F})$ is a quotient of $H^n_{\text{ét},Z}(\mathsf{F})$ for all $n > \mathrm{d}(\mathsf{F})$. It is therefore enough to show that $H^n_{\text{ét},Z}(\mathsf{F}) = 0$ for $n > \mathrm{d}(\mathsf{F}) + \operatorname{asn} X$.

We can compute $H^n_{\text{ét},Z}(\mathsf{F})$ by a spectral sequence with E_2^{pq} term $H^p_{\text{ét}}(X, \mathcal{H}^q_{\text{ét},Z}(\mathsf{F}))$ ([V, Prop. 6.4]). We have $H^p_{\text{ét}}(X, \mathcal{H}^q_{\text{ét},Z}(\mathsf{F})) = H^p_{\text{ét}}(Z, \mathcal{H}^q_{\text{ét},Z}(\mathsf{F}))$ since $\mathcal{H}^q_{\text{ét},Z}(\mathsf{F})$ is supported on Z. By Proposition 4.17 we have $\mathrm{d}(\mathcal{H}^q_{\text{ét},Z}(\mathsf{F})) \leq \mathrm{d}(\mathsf{F}) - q + 1$. Since $\operatorname{asn} Z < \operatorname{asn} X$ we can apply the inductive hypothesis to conclude that $H^p_{\text{ét}}(Z, \mathcal{H}^q_{\text{ét},Z}(\mathsf{F})) = 0$ for $q > \mathrm{d}(\mathsf{F}) - q + 1 + \operatorname{asn} Z$, or $p + q > \mathrm{d}(\mathsf{F}) + \operatorname{asn} Z + 1$. Again using $\operatorname{asn} Z < \operatorname{asn} X$, this gives $H^n_{\text{ét},Z}(\mathsf{F}) = 0$ for $n > \mathrm{d}(\mathsf{F}) + \operatorname{asn} X$. □

4.19. *Corollary.* —

(a) *If* F *is a torsion sheaf, then* $H^n_{\text{ét}}(X, \mathsf{F}) = 0$ *for all* $n > \dim X + \operatorname{asn} X$.
(b) *If* F *is a constructible sheaf, then* $H^n_{\text{ét}}(X, \mathsf{F}) = 0$ *for all* $n > \dim X + \operatorname{asn} X$.
(c) *If* $\mathcal{F}_\ell$ *is an* ℓ*-adic sheaf on* X*, then* $H^n_{\text{ét}}(X, \mathcal{F}_\ell) = 0$ *for all* $n > \dim X + \operatorname{asn} X$.

Proof. *(a)* Clearly we have $\mathrm{d}(\mathsf{F}) \leq \dim X$. *(b)* A constructible sheaf is a special case of a torsion sheaf (compare [A1, Prop. 1.2(ii)] with [A1, Def. 2.3]). *(c)* follows from (a). □

4.20. Relation to homotopy type. We expect that the affine stratification number bounds the homotopy type as follows.

4.21. *Conjecture.* — *If the base field is* $\mathbb{C}$*, then* X *has the homotopy type of a finite complex of dimension at most* $\operatorname{asn} X + \dim X$.

5. Applications to moduli spaces of curves

One motivation for the definition of affine stratification number is the study of the moduli space of curves, and certain geometrically important open subsets. We will use Definition 1.1 (which we may, by Prop. 3.7).

5.1. *Preliminary aside: the affine stratification number of Deligne-Mumford stacks.* As we have only defined the affine stratification number of schemes, throughout this section, we will work with coarse moduli space of curves. One should presumably work instead with a more general definition for Deligne-Mumford stacks. One possible definition is to replace the notion of "affine" in the definition of affine stratification number with that of a Deligne-Mumford stack that has a surjective finite flat cover by an affine scheme (see Prop. 2.10).

5.2. Recall the following question of Looijenga's.

5.3. *Conjecture (Looijenga). — (a)* $\operatorname{acn} M_g \leq g-2$ *for* $g \geq 2$. *(b) More generally,* $\operatorname{acn} M_{g,n} \leq g-1-\delta_{n,0}$ *whenever* $g>0$, $(g,n) \neq (1,0)$.

The case $n=1$ of (b) implies the cases $n>1$, as the morphism $M_{g,n+1} \to M_{g,n}$ is affine for $n \geq 1$.

This suggests the following, weaker conjecture, which is straightforward to verify for small (g,n) (using Prop. 2.10 judiciously). We are currently pursuing a program to prove this (work in progress).

5.4. *Conjecture (Looijenga* [HL, p. 112, Prob. 6.5]*). —* $\operatorname{asn} M_g \leq g-2$ *for* $g \geq 2$.

From this statement (and properties of asn), we obtain a number of consequences.

5.5. *Proposition (Looijenga* [HL, p. 112]*). — Conjecture 5.4 implies that* $\operatorname{asn} M_{g,n} \leq g-1-\delta_{n,0}$ *whenever* $g>0$, $(g,n) \neq (1,0)$.

Proof. As $M_{g,n+1} \to M_{g,n}$ is affine for $n \geq 1$, it suffices to prove the result for $M_{0,3}$ and $M_{g,1}$ with $g>0$. The cases $g=0$ and $g=1$ are immediate. For $g>1$, let D be a multisection of $M_{g,1} \to M_g$ (*e.g.* a suitable Weierstrass divisor). Then the morphisms $D \to M_g$ and $(M_{g,1} \setminus D) \to M_g$ are affine and surjective, so pulling back the affine stratification of M_g to $M_{g,1}$ and intersecting with $(M_{g,1} \setminus D) \sqcup D$ yields the desired affine stratification of $M_{g,1}$. □

Examination of small genus cases suggests the following refinement of Conjecture 5.4.

5.6. *Conjecture. — There is an affine stratification of* $M_{g',n'}$ *preserved by the symmetric group acting on the* n' *points. The induced decomposition of* $\overline{M}_{g,n}$ *is a stratification.*

This leads to a bound on the affine stratification number of the open subset $\overline{M}_{g,n}^{\leq k}$, corresponding to stable n-pointed genus g curves with at most k genus 0 components, defined in [GV, Sect. 4].

5.7. *Proposition. — Conjecture 5.6 implies that* $\operatorname{asn} \overline{M}_{g,n}^{\leq k} \leq g-1+k$ *for all* $g>0$, $n>0$.

This is more evidence of the relevance of this strange filtration of the moduli space of curves. In particular, compare this to Theorem $\star$ of [GV], that the tautological ring of $\overline{M}_{g,n}^{\leq k}$ vanishes in codimension greater than $g-1+k$. (In [GV, p. 3], Looijenga asks precisely this question, with asn replaced by acn.)

Proof. We show that dimension of any stratum of $\overline{M}_{g,n}$ appearing in $\overline{M}_{g,n}^{\leq k}$ is at least

$$3g-3+n-(g-1+k)=2g-2+n-k.$$

This is true for strata in $M_{g,n}$ by Proposition 5.5. Consider any other boundary stratum, say with j rational components ($j \leq k$) with $m_1, \dots, m_j$ special points respectively; and s other components, with genus g_1, …, g_s and n_1, …, n_s special points respectively. By Proposition 2.10, it suffices to pass to the finite étale cover that is isomorphic to

$$\prod_{i=1}^{j} M_{0,m_i} \times \prod_{i=1}^{s} M_{g_i,n_i}.$$

By Proposition 5.5 (and using M_{0,m_i} affine), we can decompose this space into affine sets of dimension at least

$$\sum_{i=1}^{j}(m_i-3)+\sum_{i=1}^{s}(3g_i-3+n_i-(g_i-1)).$$

Now $\sum_{i=1}^{j}(m_i-2)+\sum_{i=1}^{s}(2g_i-2+n_i)=2g-2+n$, so each affine set has dimension at least

$$2g-2+n-j$$

and thus codimension in $\overline{M}_{g,n}^{\geq k}$ at most $g-1+j \leq g-1+k$ as desired. □

This leads to bounds on other spaces of interest. Let $M_{g,n}^{ct}$ be the open subset of $\overline{M}_{g,n}$ corresponding to curves of compact type (*i.e.* with compact Jacobian, or equivalently with dual graph containing no loops). Let $M_{g,n}^{rt}$ be the open subset corresponding to curves with rational tails (*i.e.* with a component a smooth genus g curve, or equivalently with dual graph with a genus g vertex).

5.8. *Corollary. — Conjecture 5.6 implies that* $\operatorname{asn} M_{g,n}^{ct} \leq 2g-3+n$ *and* $\operatorname{asn} M_{g,n}^{rt} \leq g+n-2$ *for* $g>0$, $n>0$.

Proof. $M_{g,n}^{ct}$ is obtained by removing boundary strata from $\overline{M}_{g,n}^{\leq g+n-2}$. $M_{g,n}^{rt}$ is obtained by removing boundary strata from $\overline{M}_{g,n}^{\leq n-1}$. □

5.9. *Corollary. — Conjectures 4.21 and 5.6 imply that* $\overline{M}_{g,n}^{\leq k}$ *(resp.* $M_{g,n}^{ct}$, $M_{g,n}^{rt}$*) has the homotopy type of a finite complex of dimension at most* $4g-4+n+k$ *(resp.* $5g-6+2n$, $4g-5+2n$*).*

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