book-by-committee approach will prevent this becoming a widely used text, but the more discrete math texts we have, the more likely the American mathematical community will finally find a worthy place for discrete mathematics in the undergraduate mathematics curriculum.

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A Mathematical Mosaic: Patterns and Problem Solving. By Ravi Vakil. Brendan Kelly Publishing Inc., 1997, 254 pp., $16.95.

Reviewed by **Andre Toom**

There are different books on my shelf. Some are large like dinosaurs; these are textbooks. Others are much smaller, but their educational value may be greater. For example, Kordemsky's book [**3**] contributed a lot to Russian childrens' interest in mathematics, although its English edition fits in a hand. The book I am going to discuss fits in a coat pocket, but it speaks in an interesting and understandable way about number theory, combinatorics, game theory, geometry, and calculus, to say nothing about magic tricks, puzzles and other digressions. What is most important is that whenever Vakil starts to discuss something, he never leaves the reader without a piece of exact, rigorous knowledge. This is a book about mathematics, not about its fuzzy placebo.

Ravi Vakil received several olympiad prizes and now is an instructor at Princeton University. This is his first book, and in it he tries to share his expertise with his readers. He tries to encourage curiosity, a sense of beauty, and the love of knowledge. This is a book I would like to have read as a boy. Why? Because it addresses the normal curiosity of children. It contains many good problems, facts, and stories. It is a mixture of just those ingredients which are most useful for children. Vakil enjoys ideas that seem simple if you already know them, but may seem paradoxical if you don't. One of them is presented as a card trick (p. 44):

> I ask you to shuffle a deck of cards thoroughly. Then I ask for them back (face down). Carefully examining the backs of the cards, I separate them into two piles. I then claim that, through the power of magic, I've made sure that the number of black cards in the first pile is the number of red cards in the second pile!

The explanation starts as follows: "While pretending to examine the backs of the cards, I was simply... ." Can you complete this explanation? I remember that as a boy I was quite fond of various tricks and had a notebook where I wrote as many of them as I could find.

Although Vakil's book is intended to be recreative and facultative, it contains many facts that are indispensable for mathematical literacy, including

- Criteria for divisibility. Those for 2, 3 and 7 are proved; other proofs are left for the reader (pp. 23–29).
- Let $g(x)$ denote any polynomial in x. Then the remainder when $g(x)$ is divided by $x - a$ is $g(a)$. (p. 26).
- $\sqrt{2}$ is irrational (p. 121).
- There are infinitely many primes (p. 124).
- Heron's formula for the area of a triangle (p. 160).
- The harmonic series diverges (p. 180).

Each of these facts is not only proved, but accompanied with several variations that are presented as problems or comments. Sometimes they lead into quite substantial mathematics. There are many "local" proofs, which also help to develop the readers' "proof sense." I like the following most:

> The "hypervolume" of a "four-dimensional sphere" of radius r is $H = \pi^2 r^4/2$. Can you use a method similar to that of Part 1 and Part 4 to find the "surface volume?" (p. 81).

The book contains several "personal profiles" of gifted youngsters with whom Vakil became acquainted at olympiads. Vakil writes several lines about how they found their way into mathematics, for example: "J. P.'s curiosity is typical of the young mathematicians profiled in this book" (p. 41), "It was at this time that Katy discovered the tremendous enjoyment she gains from solving problems" (p. 54). Vakil cites one talented student's advice: "Do math for math's sake, not because your parents will be proud of you, or because people will think you are smart" (p. 143).

"Do math for math's sake"—this goes against the mentality of those educators who desperately look for external reasons to study math. Some educators naively expect that so-called "real-world problems" are a good way to arouse interest in mathematics. My experience tells the opposite: young people are mostly interested in problems that can be solved without much erudition. Interest in applications comes later because it takes a much broader experience. For example, I obtained my first result in mathematics as an undergraduate. It was an algorithm that produced the product of two n-digit numbers, whose complexity grew slower than $n^{1+\varepsilon}$ for any positive ε. Now this algorithm is classified as computer science [2, vol. 2, pp. 280–289], but for me it was just an intellectual challenge. I did not care about applications at that time.

Another example: my daughter. Her main interest is art. However, she takes honors algebra and precalculus in her high school. Her textbooks abound in "real-world problem situations". According to the prevailing educational theory, my daughter is supposed to be attracted by them more than by abstractions, but she is not. If a professional educator ever reads this, please, notice my testimony: my daughter likes the classical abstract mathematics MORE than those far-fetched cumbersome "real" situations that are so fashionable now!

The same is true of my students (I have more than thirty years of teaching experience). When students are interested in mathematics at all, they are interested in the intrinsic mathematical content rather than in external decorations. In this context the following question arises: has anybody ever cared about the "real-world" relevance of olympiad problems? I bet that nobody ever did. Do you know why? Because those youngsters who participate in olympiads are too intelligent to buy cheap gimmicks.

More than once Vakil stresses that mathematics is beautiful. In his preface he writes: "Math is a uniquely aesthetic discipline; mathematicians use words like beauty, depth, elegance, and power to describe excellent ideas" (p. 10). When starting to speak about combinatorics, he writes: "More important (to me, at least) is its aesthetic appeal" (p. 45). Before presenting the proof of irrationality of $\sqrt{2}$, he writes: "It is also extremely beautiful—its elegance lies in its simplicity" (p. 121). Is Vakil alone in stressing the beauty of mathematics? By far not. He refers to G. H. Hardy, who said similar things in his *A Mathematician's Apology* (p. 120).

Here is the last problem in the book:

> On a remote Norwegian mountain top, there is a huge checkerboard, 1000 squares wide and 1000 squares long, surrounded by steep cliffs to the north, south, east, and west. Each square is marked with an arrow pointing in one of the eight compass directions, so (with the possible exception of some squares on the edges) each square has an arrow pointing to one of its eight nearest neighbors. The arrows on squares sharing an edge differ by at most 45°. A lemming is placed randomly on one of the squares, and it jumps from square to square following the arrows. Prove that the poor creature will eventually plunge from a cliff to its death.

Is this a "real-world" problem? Certainly not. There is no such checkerboard in the mountains of Norway. According to some educational theories, students should not be interested in this problem, but... they are. In fact this problem was invented by a secondary-school student Kevin Purbhoo. Is Kevin abnormal? If he is, I am also. Bobrov's book [**1**], which mixes mathematics with fantasy, accompanied all my childhood, and I liked its fantastic element most! Vakil writes about this problem: "Although Kevin did not know at the time, this problem anticipates several subtle and important results in topology... ." This is typical of Vakil's book: recreational mathematics leads to deep and important ideas. It deserves many readers.

There is a good-old editorial "Yes, Virginia, there is a Santa Claus"—an answer to a girl named Virginia who asked the editor of the New York *Sun* in 1897, "Is there a Santa Claus?" It seems to me that some modern educators have lost the pathos of spirituality expressed there, which makes humans human. It is in human nature to be interested in abstractions. The human ability to think without an immediate material gratification is at the base of civilization. In this context, Vakil's book might be called "Yes, Virginia, Math is Beautiful."

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