

# MURPHY'S LAW IN ALGEBRAIC GEOMETRY: BADLY-BEHAVED DEFORMATION SPACES

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ABSTRACT. We consider the question: "How bad can the deformation space of an object be?" The answer seems to be: "Unless there is some a priori reason otherwise, the deformation space may be as bad as possible." We show this for a number of important moduli spaces.

More precisely, every singularity of finite type over  $\mathbb{Z}$  (up to smooth parameters) appears on: the Hilbert scheme of curves in projective space; and the moduli spaces of smooth projective general-type surfaces (or higher-dimensional varieties), plane curves with nodes and cusps, stable sheaves, isolated threefold singularities, and more. The objects themselves are not pathological, and are in fact as nice as can be: the curves are smooth, the surfaces have very ample canonical bundle, the stable sheaves are torsion-free of rank 1, the singularities are normal and Cohen-Macaulay, etc. This justifies Mumford's philosophy that even moduli spaces of well-behaved objects should be arbitrarily bad unless there is an a priori reason otherwise.

Thus one can construct a smooth curve in projective space whose deformation space has any given number of components, each with any given singularity type, with any given non-reduced behavior along various associated subschemes. Similarly one can give a surface over  $\mathbb{F}_p$  that lifts to  $\mathbb{Z}/p^7$  but not  $\mathbb{Z}/p^8$ . (Of course the results hold in the holomorphic category as well.)

It is usually difficult to compute deformation spaces directly from obstruction theories. We circumvent this by relating them to more tractable deformation spaces via smooth morphisms. The essential starting point is Mnëv's Universality Theorem.

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*The best-laid schemes o' mice an' men  
 Gang aft agley  
 An' lea'e us nought but grief an' pain  
 For promis'd joy!*  
 — Robert Burns, "To a Mouse"

## 1. INTRODUCTION

Define an equivalence relation on pointed schemes generated by: If  $(X, p) \rightarrow (Y, q)$  is a smooth morphism, then  $(X, p) \sim (Y, q)$ . We call the equivalence classes *singularity types*, and will call pointed schemes *singularities* (even if the point is regular). We say that *Murphy's Law* holds for a moduli space if every singularity type of finite type over  $\mathbb{Z}$  appears on that moduli space. Although our methods are algebraic, our arguments all work in the holomorphic category.

**1.1. Main Theorem.** — *The following moduli spaces satisfy Murphy's Law.*

- M1a.** *the Hilbert scheme of nonsingular curves in projective space*
- M1b.** *the moduli space of maps of smooth curves to projective space (and hence Kontsevich's moduli space of maps)*
- M1c.**  $\mathcal{G}_d^r$  [HM, p. 5], *the space of curves with the data of a linear system of degree  $d$  and projective dimension  $r$*
- M2a.** *the versal deformation spaces of smooth surfaces (with very ample canonical bundle)*
- M2b.** *the fine moduli stack of smooth surfaces with very ample canonical bundle and reduced automorphism group*
- M2c.** *the coarse moduli space of smooth surfaces with very ample canonical bundle*
- M2d.** *the Hilbert scheme of nonsingular surfaces in  $\mathbb{P}^5$ , and the Hilbert scheme of surfaces in  $\mathbb{P}^4$*
- M3a–c.** *more generally, the versal deformation spaces and fine and coarse moduli spaces of smooth  $n$ -folds ( $n > 1$ ) with very ample canonical bundle and reduced automorphism group (as in 2a–c)*
- M4.** *the Chow variety of nonsingular curves in projective space, and of nonsingular surfaces in  $\mathbb{P}^5$ , allowing only seminormal singularities in the definition of Murphy's Law (recall that the Chow variety is seminormal [Kol1, Theorem 3.21])*
- M5a.** *branched covers of  $\mathbb{P}^2$  with only simple branching (nodes and cusps), in characteristic not 2 or 3*
- M5b.** *the "Severi variety" of plane curves with a fixed numbers of nodes and cusps, in characteristic not 2 or 3*
- M6.** *the moduli space of stable sheaves [Si]*
- M7.** *the versal deformation spaces of isolated normal Cohen-Macaulay threefold singularities*

The meaning of Murphy's Law for versal deformation spaces is the obvious one. We should say a few words on why certain moduli spaces exist. **1b:** Although one usually discusses Kontsevich's moduli space of stable maps in characteristic 0, one may as well define the moduli space of maps from nodal curves to projective space, with reduced automorphism group, over  $\text{Spec } \mathbb{Z}$ ; this is a Deligne-Mumford stack, by a standard generalization of the construction of [FuP]. (It is not proper!) **2b** and **3b:** [A, p. 182–3] shows

existence for surfaces, and the argument applies verbatim in higher dimension. The stack is Deligne-Mumford, locally of finite type. **2c** and **3c**: [Kol2, Theorem 1.8] shows that there is an algebraic space coarsely representing these moduli functors. (For surfaces **2c**, there is even a coarse moduli (algebraic) space of canonical models of surfaces of general type [Kol2, Theorem 1.7].)

To obtain results over other bases (such as algebraically closed fields such as  $\mathbb{C}$ ), note that most moduli spaces above behave well with respect to base change; hence any singularity obtained by base change from a finite type singularity over  $\mathbb{Z}$  may appear. Clearly, no other singularity may appear. Indeed, any moduli (pseudo-)functor admitting a smooth cover by a scheme locally of finite type over  $\text{Spec } \mathbb{Z}$  necessarily only has singularities of this sort. (For example, the singularity

$$(1) \quad xy(y-x)(y-\pi x) = 0$$

in  $\mathbb{C}^2$  may not appear as such a deformation space.) This leads to some natural questions, such as: does there exist an isolated complex singularity whose deformation space is equivalent to (1)? What if the singularity is required to be algebraic? Does there exist a compact complex manifold whose deformation space has such singularity type? What if the manifold were required to be projective? Before this project, we would have believed the answer could be “yes”, but paradoxically Murphy’s Law leads us now to expect that the answer is “no”. It would be very interesting to have any example of a non-pathological object (e.g. isolated complex algebraic singularity, complex projective manifold, or even non-algebraic examples) with deformation space not equivalent to one of finite type over  $\text{Spec } \mathbb{Z}$ .

**1.2. Philosophy.** To be explicit about why these results may be surprising: one can construct a smooth curve in projective space whose deformation space has any given number of components, each with any given singularity type, with any given non-reduced behavior along various associated subschemes. Similarly, one can give a smooth surface of general type in characteristic  $p$  that lifts to  $\mathbb{Z}/p^7$  but not to  $\mathbb{Z}/p^8$ .

We next give some philosophical comments, which motivated this result. The history sketched in Section 2 also provided motivation.

The moral of Theorem 1.1 is as follows. We know that some moduli spaces of interest are “well-behaved” (e.g. equidimensional, having at worst finite quotient singularities, etc.), often because they are constructed as Geometric Invariant Theory quotients of smooth spaces, e.g. the moduli space of curves, the moduli space of vector bundles on a curve, the moduli space of branched covers of  $\mathbb{P}^1$  (the Hurwitz scheme, or space of admissible or twisted covers), the Picard variety, the Hilbert scheme of divisors on projective space, the Severi variety of plane curves with a prescribed number of nodes, the moduli space of abelian varieties (notably [NO]), etc. In other cases, there has been some effort to try to bound how “bad” the singularities can get. Theorem 1.1 in essence states that these spaces can be arbitrarily singular, and gives a means of constructing an example where any given behavior happens.

Murphy's law suggests that unless there is some natural reason for the space to be well-behaved, it will be arbitrarily badly behaved. For example, arithmetically Cohen-Macaulay surfaces in  $\mathbb{P}^4$  are always unobstructed [El]; but surfaces in general in  $\mathbb{P}^4$  can have arbitrarily bad deformations (by **M2d**). Other examples are given in the following table.

<i>Well-behaved moduli space</i>	<i>Badly-behaved moduli space</i>
curves branched covers of $\mathbb{P}^1$ (e.g. [HM, Theorem 1.53]) surfaces in $\mathbb{P}^3$ Picard variety over the moduli space of curves Severi variety of nodal plane curves (e.g. [HM, Theorem 1.49])	surfaces (by <b>M2b–c</b> ) branched covers of $\mathbb{P}^2$ (by <b>M5a</b> ) surfaces in $\mathbb{P}^4$ (by <b>M2d</b> ) its subscheme $\mathcal{G}_d^r$ (by <b>M1c</b> ) Severi variety of nodal and cuspidal plane curves (by <b>M5b</b> )

Furthermore, our experience and intuition tells us that pathologies of moduli spaces occur on the boundary, and that moduli spaces of "good" objects are also "good". Murphy's Law shows that this intuition is incorrect; we should expect pathologies even where the objects being parameterized seem harmless. Kodaira says "The theory of deformation was at first an experimental science" [Kod2, p. 259]. This result shows that our intuition is flawed because it is based on experimental knowledge of a very small part of the moduli spaces we are interested in; it supports Mumford's philosophy that pathologies are the rule rather than the exception. Alternatively, from the point of view of A. Vershik, this result states that the "universality" philosophy (e.g. [Ve, Section 7]) applies widely in algebraic geometry.

As a side comment, Theorem 1.1 indicates that one cannot hope to desingularize the moduli space of surfaces, or any other moduli space satisfying Murphy's Law, by adding additional structure; this would imply a resolution of singularities. (Hence the program for desingularization of the space of stable maps informally proposed by some authors cannot succeed. However, see [VZ] for success in genus 1.)

**1.3. Notation.** Let  $\text{Def}$  denote the versal or Kuranishi deformation space (not the space of first-order deformations). The object being deformed will be clear from the context.

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## 2. HISTORY, AND FURTHER QUESTIONS

**2.1. Hilbert schemes.** The motivation for both the equivalence relation  $\sim$  and the terminology “Murphy’s Law” comes from the folklore conjecture that the Hilbert scheme “satisfies Murphy’s Law”.

**2.2. Law** [HM, p. 18]. — *There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.*

I am not sure of the origin of this philosophy, but it seems reasonable to ascribe it to Mumford. This traditional statement of Murphy’s Law is admittedly informal and imprecise (see the MathReview [Lax]). Clearly not every singularity appears on the Hilbert scheme of projective space. For example, the only zero-dimensional Hilbert schemes are reduced points. Allowing “smooth equivalence classes” of singularities seems the mildest way of rescuing the law.

In his famous paper [Mu], Mumford described a component of the Hilbert scheme of space curves that is everywhere nonreduced. Other examples of nonreduced components of the Hilbert scheme have since been given [GP, Kl, E, M-DP]. Other pathologies relating to the number of components of the Hilbert scheme of *smooth* space curves were given by Ellia, Hirschowitz, and Mezzetti [EHM], and by Fantechi and Pardini [FP1]. (The results of the latter will be essential to our argument.)

Raynaud’s example (see Section 2.3) gives a component of a Hilbert scheme of smooth surfaces which exists in characteristic  $p$ , but does not lift to characteristic 0 (by the standard methods of Section 4.6). Mohan Kumar, Peterson, and Rao [MPR] give a component of the Hilbert scheme of smooth surfaces in  $\mathbb{P}^4$  which exists in characteristic 2 but does not lift. See [EHa, Section 3] for more on problems of lifting curves out of characteristic  $p$ .

Although the Hilbert scheme of projective spaces was suspected to behave badly, other moduli spaces were believed (or hoped) to be better-behaved. We now discuss these.

**2.3. Surfaces and higher-dimensional varieties.** (See [Ca2] for an excellent overview of the subject.) The first example of an obstructed smooth variety was due to Mumford, obtained by blowing up his curve in  $\mathbb{P}^3$  [Mu, p. 643-4]. The first example of an obstructed surface is due to Kas [Kas]. Other examples were later given by Burns and Wahl [BW], and later many others. Horikawa [Ho], Miranda [Mi], and Catanese [Ca3] gave examples of generically nonreduced components of the moduli space of surfaces; in each case the surfaces did not have ample canonical bundle, and this appeared to be a common explanation of this pathology [Ca3, p. 294]. Catanese conjectured that if  $S$  is a surface of general type with  $q = 0$  and  $K_S$  ample, then the moduli space  $\mathfrak{M}(S)$  is smooth on an open dense set ([Ca2, p. 34, 69], [Ca3, p. 294]). Theorem 1.1 **M2b-c** gives a counterexample to this conjecture, and as Catanese pointed out, even to the stronger conjecture where  $K_S$  is very ample. Manetti gave an earlier counterexample in his thesis [Man1, Corollary 3.4]; the added advantage of **M2** is that every (finite type) nonreduced structure is shown to occur.

Catanese showed that the moduli space of complex surfaces in a given homeomorphism class can have arbitrarily many components of different dimension [Ca1, Theorem A], and asked if this were still true for those in a given diffeomorphism class [Ca1, p. 485]; Theorem 1.1 **M2b–c** answers this in the affirmative. A prior answer was recently given by Catanese and B. Wajnryb [CaW]. The added benefit of **M2** is that all possibilities are shown to occur.

Serre gave the first example of a projective variety that could not be lifted to characteristic 0 [Ser]. Raynaud gave the first example of such a surface [Ray]; W. Lang gave more [Lang].

**2.4. Plane curves with nodes and cusps.** If  $C$  is a reduced complex plane curve, the classical question of “completeness of the characteristic linear series” asks (in modern language) if an appropriate equisingular moduli space is smooth. Severi proved this is true if  $C$  has only nodes ([Sev], see also [Z, Section VIII.4]), and asserted this if  $C$  has nodes and cusps [W2]. (See [Z, p. 116–7 and Section VIII] for motivation for the study of nodal and cuspidal plane curves.) It was later realized that Severi’s assertion was unjustified. Enriques tried repeatedly to show that such curves were unobstructed [Ca2, p. 51]; Zariski also raised this question [Z, p. 221]. The first counterexample was given by Wahl [W1, Section 3.6], and another was given by Luengo [Lu]. Theorem 1.1 **M5b** shows that Severi was in some sense “maximally wrong”.

**2.5. Stable coherent sheaves.** The moduli space of stable coherent sheaves is due to Simpson [Si]. Our example is in fact a torsion-free sheaf on  $\mathbb{P}^5$ ; the theory of the moduli of torsion-free sheaves was developed earlier by Maruyama [Mar], building on Gieseker’s work in the surface case [Gi].

**2.6. Singularities.** The theory of deformations of singularities is too large to summarize here. We point out however that it was already established by Burns and Wahl [BW] that such deformation spaces can be bad, although not this pathological.

**2.7. Further questions.** Theorem 1.1, and the philosophy and history given above, beg further questions. Do deformations of surface singularities (say isolated and Cohen–Macaulay) satisfy Murphy’s Law? How about the Hilbert scheme of curves in  $\mathbb{P}^3$ ? The Hilbert scheme of points on a smooth threefold? The moduli of vector bundles on smooth surfaces? Can the extra dimensions allowed in the definition of *type* be excised, i.e. can “smooth” be replaced by “étale” in the definition of type? (As observed above, this is not possible for the Hilbert scheme.) Catanese asks if Murphy’s law for surfaces is still true if we require not only that the surface has very ample canonical bundle, but also that the canonical embedding is cut out by quadrics. *Conjecture:* for any given  $p$ , the surfaces whose canonical divisor induces an embedding satisfying property  $N_p$  satisfy Murphy’s Law. One might hope that the constructions given in Section 4 suffice by taking the divisor class  $A$  (Sections 4.2 and 4.3) to be sufficiently ample. The case  $p = 1$  would give an affirmative answer to Catanese’s question.

### 3. THE STARTING POINT: MNËV'S UNIVERSALITY THEOREM

We will prove Theorem 1.1 by drawing connections among various moduli spaces, taking as a starting point a remarkable result of Mnëv. Define an *incidence scheme of points and lines in  $\mathbb{P}^2$* , a locally closed subscheme of  $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n = \{p_1, \dots, p_m, l_1, \dots, l_n\}$  parameterizing  $m \geq 4$  marked points and  $n$  marked lines, as follows.

- $p_1 = [1; 0; 0], p_2 = [0; 1; 0], p_3 = [0; 0; 1], p_4 = [1; 1; 1]$ .
- We are given some specified incidences: For each pair  $(p_i, l_j)$ , either  $p_i$  is required to lie on  $l_j$ , or  $p_i$  is required not to lie on  $l_j$ .
- The marked points are required to be distinct, and the marked lines are required to be distinct.
- Given any two marked lines, there is a marked point required to be on both of them.
- Each marked line contains at least three marked points.

**3.1. Theorem (Mnëv).** — *Every singularity type appears on some incidence scheme.*

This is a special case of Mnëv's Universality Theorem [Mn1, Mn2]. A short proof is given by Lafforgue in [Laf, Théorème 1.14]. Lafforgue's construction does not necessarily satisfy the first, fourth and fifth requirements of an incidence scheme, but they can be satisfied by adding more points. (The only subtlety in adding these extra points is verifying that in the configuration constructed by Lafforgue, no three lines pass through the same point unless required to by the construction.) *Caution:* Other expositions of Mnëv's theorem do not prove the result scheme-theoretically, only “variety-theoretically,” as this is all that is needed for most purposes.

For the rest of the paper fix a singularity type. By Mnëv's Theorem 3.1, there is an incidence scheme exhibiting this singularity type at a certain configuration  $\{p_1, \dots, p_m, l_1, \dots, l_n\}$ . Consider the surface  $S$  that is the blow-up of  $\mathbb{P}^2$  at the points  $p_i$ . Let  $C$  be the proper transform of the union of the  $l_j$ , so  $C$  is a smooth curve (a union of  $\mathbb{P}^1$ 's). This induces a morphism from the incidence scheme to the moduli space of surfaces with marked smooth divisors.

**3.2. Proposition.** — *This morphism is étale at  $(\mathbb{P}^2, \{p_i\}, \{l_j\}) \mapsto (S, C)$ .*

Thus the singularity at  $(\mathbb{P}^2, \{p_i\}, \{l_j\})$  has the same type as the moduli space of surfaces with marked smooth divisor at  $(S, C)$ .

*Proof.* We will produce an étale-local inverse near  $(S, C)$ . Consider a deformation of  $(S, C)$ :

$$(2) \quad \begin{array}{ccc} (S, C) & \hookrightarrow & (\mathcal{S}, \mathcal{C}) \\ \downarrow & & \downarrow \\ pt & \hookrightarrow & B. \end{array}$$

Pull back to an étale neighborhood of  $pt$  so that the components of  $C$  are labeled. The Hilbert scheme of  $(-1)$ -curves is étale over the base. (I am not aware of the first reference for this well-known fact. It follows for example from the exact sequence of [Ran] —

see e.g. the proof of [Ran, Theorem 3.2] — which specializes to give a natural bijection between the deformations, respectively obstructions, of  $S$  and  $\mathbb{P}^1 \rightarrow S$ . The proof in the holomorphic category is due to Kodaira [Kod1, Theorem 3].)

Let  $E_i$  be the  $(-1)$ -curve corresponding to  $p_i$ . Pull back to an étale neighborhood so that the points of the Hilbert scheme corresponding to  $E_i$  extend to sections (so there are divisors  $\mathcal{E}_i$  on the total space of the family that are  $(-1)$ -curves on the fibers). By abuse of notation, we use the same notation (2) for the resulting family. By Castelnuovo's criterion,  $S$  can be blown down along the  $\mathcal{E}_i$  so that the resulting surface is smooth, with marked sections extending  $\{p_i\}$ . (Again, “Castelnuovo's criterion over an Artin local scheme” is presumably well-known to experts, but I am unaware of a reference. It follows by applying the “usual” Castelnuovo criterion over the closed point, and then using Theorem 5.1 to show that the blow-down “deforms”. This is just the old idea proved by Horikawa in the smooth holomorphic category [Ho]. Alternatively, the proof of the usual Castelnuovo's criterion, for example [Ha, Theorem V.5.7], can be extended.)

The central fiber is then  $\mathbb{P}^2$ , so (as  $\mathbb{P}^2$  is rigid) the family is locally trivial. The marked points  $p_1, \dots, p_4$  give a canonical isomorphism with  $\mathbb{P}^2$ . (We may need to restrict to a smaller neighborhood to ensure that these points are in general position.) As the components  $\{C_j\}$  of  $C$  necessarily meet various  $E_i$ , their images  $\{l_j\}$  necessarily pass through the necessary  $p_i$ .  $\square$

#### 4. ABELIAN COVERS: PROOF OF M2

We use this intermediate moduli space of surfaces with marked divisors to prove **M2**, by connecting such marked surfaces to abelian covers. We use the theory of abelian covers developed by Catanese, Pardini, Fantechi, and Manetti [Ca1, P, FP1, Man2]. (Bidouble covers were introduced by Catanese. Pardini developed the general theory of abelian covers. Key deformation-theoretic results were established by Fantechi-Pardini and Manetti.) Let  $G = (\mathbb{Z}/p)^3$ , where  $p = 2$  or  $3$  is prime to the characteristic of the residue field of the singularity. Let  $G^\vee$  be the dual group, or equivalently the group of characters. Let  $\langle \cdot, \cdot \rangle : G \times G^\vee \rightarrow \mathbb{Z}/p$  be the pairing (after choice of root of unity  $\zeta$ ), which we extend to  $\langle \cdot, \cdot \rangle : G \times G^\vee \rightarrow \mathbb{Z}$  by requiring  $\langle \sigma, \chi \rangle \in \{0, \dots, p-1\}$ . Suppose we have two maps  $D : G \rightarrow \text{Div}(S)$ ,  $L : G^\vee \rightarrow \text{Pic}(S)$ . We say  $(D, L)$  satisfies the *cover condition* [P, Proposition 2.1] if  $(D, L)$  satisfies  $D_0 = 0$  and

$$pL_\chi = \sum_{\sigma} \langle \sigma, \chi \rangle D_\sigma$$

for all  $\sigma, \chi$ . (Equality is taken in  $\text{Pic}(S)$ .)

**4.1. Proposition (Pardini).** — Suppose  $(D, L)$  satisfies the cover condition, and suppose the  $D_\sigma$  are nonsingular curves, no three meeting in a point, such that if  $D_\sigma$  and  $D_{\sigma'}$  meet then they are transverse and  $\sigma$  and  $\sigma'$  are linearly independent in  $G$ . Then:

- (i) There is a corresponding  $G$ -cover  $\pi : \tilde{S} \rightarrow S$  with branch divisor  $D = \cup D_\sigma$ .

- (ii)  $\tilde{S}$  is nonsingular.
- (iii)  $\pi_* \mathcal{O}_{\tilde{S}} = \bigoplus_{\chi} \mathcal{O}_S(-L_{\chi})$ .
- (iv)  $\pi_* \mathcal{K}_{\tilde{S}} \cong \bigoplus_{\chi} \mathcal{K}_S(L_{\chi})$ . The Galois group  $G$  acts on the left side in the obvious way; it acts on the  $\chi$ -summand on the right by the character  $\chi$ .

Note for future reference that the branch divisor  $D_{\sigma}$  corresponds to the subgroup of  $G$  generated by  $\sigma$ . (Note also that (iii) and (iv) are consistent with Serre duality on  $\tilde{S}$ .)

*Proof.* (i) is [P, Proposition 2.1], (ii) is [P, Proposition 3.1], and (iii) is a consequence of Pardini's construction [P, (1.1)]. Pardini points out that (iv) is a special case of duality for finite flat morphisms, see [Ha] Exercises III.6.10 and Ex. III.7.2. (It also follows by a straightforward local calculation. See [Ca1, p. 495] for the analogous proof for bouble covers. The generalization to abelian covers is analogous to Pardini's proof of (iii).)  $\square$

The next two examples apply to  $(S, C)$  produced at the end of Section 3. If the character of the residue field is 2 (respectively 3), then only Example 4.3 (respectively 4.2) applies; otherwise both apply.

**4.2. Key example:**  $p = 2$ . Fix  $\sigma_0 \neq 0$  in  $G$ . Let  $A$  be a sufficiently ample bundle such that  $A \equiv C \pmod{2}$ . Let  $D_{\sigma_0} = C$ ,  $D_0 = 0$ , and let  $D_{\sigma}$  be a general section of  $A$  otherwise, such that  $D_{\sigma'}$  meets  $D_{\sigma''}$  transversely for all  $\sigma' \neq \sigma''$ . Let  $L_0 = 0$ ,  $L_{\chi} = 2A$  if  $\langle \sigma_0, \chi \rangle = 0$  and  $\chi \neq 0$ , and  $L_{\chi} = (3A + C)/2$  if  $\langle \sigma_0, \chi \rangle = 1$ . (As  $\text{Pic } S$  is torsion-free, there is no ambiguity in the phrase  $(3A + C)/2$ .) It is straightforward to verify that  $(D, L)$  satisfies the hypotheses of Proposition 4.1.

**4.3. Key example:**  $p = 3$ . Fix  $\sigma_0 \neq 0$  in  $G$ , and  $\chi_0 \in G^{\vee}$  such that  $\langle \sigma_0, \chi_0 \rangle = 1$ . Let  $A$  be a sufficiently ample bundle such that  $A \equiv C \pmod{3}$ . Let  $D_{\sigma_0} = C$ ,  $D_{\sigma}$  be a general section of  $A$  if  $\langle \sigma, \chi_0 \rangle = 1$  and  $\sigma \neq \sigma_0$ , and  $D_{\sigma} = 0$  otherwise. Let

- $L_{\chi} = (8A + C)/3$  if  $\langle \sigma_0, \chi \rangle = 1$
- $L_0 = 0$
- $L_{\chi} = 3A$  if  $\langle \sigma_0, \chi \rangle = 0$  and  $\chi \neq 0$
- $L_{-\chi_0} = (16A + 2C)/3$
- $L_{\chi} = (7A + 2C)/3$  if  $\langle \sigma_0, \chi \rangle = 2$  and  $\chi \neq -\chi_0$

It is straightforward to verify that  $(D, L)$  satisfies the hypotheses of Proposition 4.1 (note that if  $\sigma \neq 0$ , then at most one of  $\{D_{\sigma}, D_{-\sigma}\}$  is nonzero).

**4.4. Theorem.** — In Examples 4.2 and 4.3, if  $A$  is sufficiently ample, then  $K_{\tilde{S}}$  is very ample. In particular,  $\tilde{S}$  is of general type, and is its own canonical model.

It is not hard to show that  $K_{\tilde{S}}$  is *ample*:

$$2K_{\tilde{S}} = \pi^* \left( 2K_S + \sum D_{\sigma} \right) = \pi^*(2K_S + C + qA)$$

where  $q = 6$  if  $p = 2$  and  $q = 8$  if  $p = 3$ . If  $A$  is sufficiently ample, then  $2K_S + \sum D_\sigma$  is ample, hence (as  $\pi$  is finite)  $K_{\tilde{S}}$  is ample. I am grateful to F. Catanese for pointing out that  $K_{\tilde{S}}$  is *very* ample, and explaining how to show this. The argument below directly generalizes Catanese's argument [Ca1, p. 502] for bidouble covers.

*Proof of Theorem 4.4.* By Proposition 4.1(iv), as  $\pi$  is finite,

$$(3) \quad H^0(\tilde{S}, \mathcal{K}_{\tilde{S}}) \cong \bigoplus_{\chi} H^0(S, \mathcal{K}_S(L_\chi))$$

We will need to understand this isomorphism more precisely, in particular how summands on the right of (3) give global differentials on  $\tilde{S}$ . For example, the map  $H^0(S, \mathcal{K}_S) \hookrightarrow H^0(\tilde{S}, \mathcal{K}_{\tilde{S}})$  (corresponding to the summand  $\chi = 0$ ) is the pullback map; the pullback of a nonzero  $s \in H^0(S, K_S)$  vanishes on the pullback of the divisor of zeros of  $s$ , along with the ramification divisor  $R = \bigoplus_{\sigma} R_\sigma$  with multiplicity  $p - 1$ . Let  $z_\sigma \in H^0(\tilde{S}, R_\sigma)$  ( $\sigma \in G$ ) be a section with divisor  $R_\sigma$  (preserved by the Galois group  $G$ ). A local calculation gives

$$(4) \quad H^0(\tilde{S}, \mathcal{K}_{\tilde{S}}) \cong \bigoplus_{\chi} \left( \prod_{\sigma} z_\sigma^{p-1-\langle \sigma, \chi \rangle} \right) H^0(S, \mathcal{K}_S(L_\chi)).$$

(By the hypotheses of Proposition 4.1, no more than two  $R_\sigma$  pass through any point. We consider three cases. Case 0: This is clear away from points of  $R = \bigoplus R_\sigma$ . Case 1: To do this local calculation near points lying on precisely one  $R_\sigma$ , use the fact that there are local coordinates  $(x, y)$  such that  $\pi$  is given by  $(x, y) \rightarrow (x^p, y)$ . Case 2: Near points lying on precisely two  $R_\sigma$ , the morphism is given by  $(x, y) \rightarrow (x^p, y^p)$  for appropriate  $x$  and  $y$ .)

We first show that the canonical system  $|\mathcal{K}_{\tilde{S}}|$  is base point free. Given a point  $q \in \tilde{S}$ ,  $\pi(q)$  lies on at most two  $D_\sigma$ . Choose a  $\chi$  such that  $\langle \sigma, \chi \rangle = p - 1$  for all such  $\sigma$ ; such a  $\chi$  exists as  $G$  has dimension 3 over  $\mathbb{F}_p$ . Choose a section of  $\mathcal{K}_S(L_\chi)$  not vanishing at  $\pi(q)$  (possible by sufficient ampleness of  $L_\chi$ ). Then by (4) the corresponding section of  $\mathcal{K}_{\tilde{S}}$  does not vanish at  $q$ .

We next show that  $|\mathcal{K}_{\tilde{S}}|$  separates points. Because  $\mathcal{K}_S(L_\chi)$  separates points for any  $\chi \neq 0$  (by sufficient ampleness of  $L_\chi$ ),  $|\mathcal{K}_{\tilde{S}}|$  separates points separated by  $\pi$ . Suppose now that  $\pi(p_1) = \pi(p_2)$ . For each  $\chi \neq 0$ , choose a section  $s_\chi$  of  $\mathcal{K}_S(L_\chi)$  not vanishing at  $\pi(p_1)$ . The corresponding  $|G| - 1$  sections of  $\mathcal{K}_{\tilde{S}}$  give a map near  $p_1$  and  $p_2$  to  $\mathbb{P}^{|G|-2}$  (that factors through  $|\mathcal{K}_{\tilde{S}}|$ ). As described above, an element  $g$  of the Galois group  $G$  acts on the section corresponding to  $\chi$  by the character  $\chi$ , i.e. by multiplication by the root of unity  $\zeta^{\langle g, \chi \rangle}$ . Suppose that  $g(p_1) = p_2$ . If  $p_1$  and  $p_2$  are mapped to the same point of  $\mathbb{P}^{|G|-2}$ , then

$$\left( \left( \prod_{\sigma} z_\sigma^{p-1-\langle \sigma, \chi \rangle} \right) s_\chi \right)_{\chi \neq 0} \text{ and } \left( \zeta^{\langle g, \chi \rangle} \left( \prod_{\sigma} z_\sigma^{p-1-\langle \sigma, \chi \rangle} \right) s_\chi \right)_{\chi \neq 0}$$

are linearly dependent, so  $\langle g, \chi_1 \rangle = \langle g, \chi_2 \rangle$  for all  $\chi_i$  satisfying  $\prod_{\sigma} z_\sigma^{p-1-\langle \sigma, \chi_i \rangle} \neq 0$  and  $\chi_i \neq 0$ . We again have three cases. Case 0: If no  $z_\sigma$  is zero (i.e. the  $p_i$  lie in the étale locus of  $\pi$ ), this forces  $g$  to be the identity, so  $p_1 = p_2$ . Case 1: if exactly one  $z_\sigma$  is 0, then  $\sigma$  preserves  $p_1$ . Now  $\langle g, \chi_1 \rangle = \langle g, \chi_2 \rangle$  for all  $\chi_i$  with  $\langle \sigma, \chi_i \rangle = p - 1$ . This set is non-empty; by translating all such  $\chi_i$  by one fixed such  $\chi$ , we have that  $\langle g, \chi_1 \rangle = \langle g, \chi_2 \rangle$  for all  $\chi_i$  with  $\langle \sigma, \chi_i \rangle = 0$ , i.e.  $\langle g, \chi \rangle = 0$  for all  $\chi$  with  $\langle \sigma, \chi \rangle = 0$ . By linear algebra over  $\mathbb{F}_p$ ,  $g$  must lie in the subspace generated by  $\sigma$ , i.e.  $g$  is a multiple of  $\sigma$ . Thus  $p_2 = g(p_1) = p_1$ .

Case 2: suppose two  $z_\sigma$  vanish at  $p_1$ , say  $z_{\sigma_1}, z_{\sigma_2}$ . Then for all  $\chi_1, \chi_2$  in the non-empty set  $\{\chi : \langle \sigma_1, \chi \rangle = \langle \sigma_2, \chi \rangle = p - 1\}, \langle g, \chi_1 \rangle = \langle g, \chi_2 \rangle$ . By translating by one such  $\chi$  we have that for all  $\chi$  such that  $\langle \sigma_1, \chi \rangle = \langle \sigma_2, \chi \rangle = 0, \langle g, \chi \rangle = 0$ . Again, by linear algebra on  $(\mathbb{F}_p)^3$ ,  $g$  must lie in the subspace generated by  $\sigma_1$  and  $\sigma_2$ , so again  $p_2 = g(p_1) = p_1$ .

Thus the canonical system  $|\mathcal{K}_{\tilde{S}}|$  separates points. We conclude the proof by showing that it separates tangent vectors. Case 0: Near any point disjoint from the  $R_i$  (i.e. where  $\pi$  is étale), the sections of  $\mathcal{K}_{\tilde{S}}$  corresponding to  $\mathcal{K}_S(L_\chi)$  separate tangent vectors for any  $\chi \neq 0$ . Case 1: Suppose  $q$  lies in precisely one  $R_\sigma$ . It suffices to exhibit two sections  $s_1, s_2$  of  $\mathcal{K}_{\tilde{S}}$  vanishing at  $q$  to precisely first order, whose tangent directions are transverse. First choose  $\chi$  such that  $\langle \sigma, \chi \rangle = p - 1$ , and a section of  $\mathcal{K}_S(L_\chi)$  vanishing to order 1 at  $\pi(q)$ , whose zero-set is transverse to  $D_\sigma$  at  $\pi(q)$ ; then the corresponding section  $s_1$  of  $\mathcal{K}_{\tilde{S}}$  vanishes to order 1 at  $q$ , and its zero-set is transverse to  $R_\sigma$ . Second, choose  $\chi \neq 0$  such that  $\langle \sigma, \chi \rangle = p - 2$ , and a section of  $\mathcal{K}_S(L_\chi)$  not vanishing at  $q$ ; the corresponding section  $s_2$  of  $\mathcal{K}_{\tilde{S}}$  vanishes to order 1 at  $q$  and its zero-set is contains  $R_\sigma$ . Case 2: Suppose  $q$  lies on  $R_{\sigma_1} \cap R_{\sigma_2}$ . For  $i = 1, 2$ , choose  $\chi_i \neq 0$  such that  $\langle \sigma_i, \chi_i \rangle = p - 2$  and  $\langle \sigma_{3-i}, \chi_i \rangle = p - 1$  (possible as  $\sigma_1$  and  $\sigma_2$  are linearly independent in  $G \cong (\mathbb{F}_p)^3$ , by the hypotheses of Proposition 4.1). Then near  $q$ ,  $s_i$  vanishes precisely along  $R_{\sigma_i}$ .  $\square$

**4.5. Theorem.** — In Examples 4.2 and 4.3, if  $A$  is sufficiently ample, then:

- (a)  $\tilde{S}$  is regular:  $q(\tilde{S}) := h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ .
- (b) The deformations of  $\tilde{S}$  are the same as the deformations of  $(S, \{D_\sigma\})$ . In particular, the deformations of  $G$ -covers are also  $G$ -covers.
- (c) The deformation space of  $\tilde{S}$  has the same type as the deformation space of  $(S, C)$ .
- (d)  $\text{Aut}(\tilde{S}) \equiv G$  (the only automorphisms of  $\tilde{S}$  are those preserving the cover of  $S$ ).

Part (c) implies that the fine moduli stack of surfaces of general type satisfies Murphy's law. Part (d) implies that the fine moduli stack is locally a quotient of the moduli space of  $(S, \{D_\sigma\})$  by a trivial  $G$ -action (automorphism groups are semicontinuous in families, see for example [FP1, Corollary 4.5]), so the coarse moduli space also satisfies Murphy's Law. Thus the Proposition implies **M2a–c**.

*Proof.* (a) By the Leray spectral sequence,

$$h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = h^1(S, \pi_* \mathcal{O}_{\tilde{S}}) = \sum_x h^1(S, L_x^{-1}) = 0$$

using Serre vanishing (for  $\chi \neq 0$ ) and the regularity of any blow-up of  $\mathbb{P}^2$  (for  $\chi = 0$ ).

(b) For example 4.2 ( $p = 2$ ), the result follows from [Man2, Corollary 3.23]; we restate the three hypotheses of Manetti's result for the reader's convenience. (i)  $S$  is smooth of dimension  $\geq 2$ , and  $H^0(S, T_S) = 0$ . (The latter is true because  $S$  has no non-trivial infinitesimal automorphism. Reason: any such would descend to an infinitesimal automorphism of  $\mathbb{P}^2$  fixing the  $p_i$ , in particular  $p_1 = [1; 0; 0], \dots, p_4 = [1; 1; 1]$ .) (ii)  $H^0(S, T_S(-L_\chi)) = \text{Ext}_{\mathcal{O}_S}^1(\Omega_S^1, L_\chi^{-1}) = H^1(S, L_\chi^{-1}) = 0$  (true by Serre vanishing, and sufficient ampleness of  $A$ ).

(iii)  $H^0(S, D_\sigma - L_\chi) = 0$  for all  $\chi \neq 0$ ,  $\langle \sigma, \chi \rangle = 0$  (true by Serre vanishing). Hence (b) holds for Example 4.2.

The paper [Man2] deals with  $(\mathbb{Z}/2)^r$  covers. However, [Man2, Corollary 3.23] applies without change for  $(\mathbb{Z}/p)^r$ -covers. The only change in the proof arises in the proof of the prior result [Man2, Proposition 3.16]; the statement of this proposition remains the same, and the proof is changed in the obvious way. In particular, the fourth equation display should read

$$\Omega_{X/Y}^1 = \bigoplus_{\sigma} \frac{\mathcal{O}_X(-(p-1)R_\sigma)}{w_\sigma \mathcal{O}_X(-pR_\sigma)} = \bigoplus_{\sigma} \mathcal{O}_{R_\sigma}(-R_\sigma).$$

Then the hypotheses of [Man2, Corollary 3.23] follow as in the case  $p = 2$ , and we have proved (b) for Example 4.3 ( $p = 3$ ) as well.

(c) Choose  $A = C + npK_{\tilde{S}}$  for  $n \gg 0$ , so that its higher cohomology vanishes. Then  $\text{Def}(S, \{D_\sigma\}) \rightarrow \text{Def}(S, C)$  is a smooth morphism: in any deformation of  $S$  the divisor class  $[D_\sigma]$  extends (as  $C$  and  $K_{\tilde{S}}$  extend), and extends uniquely (by  $h^1(S, \mathcal{O}_S) = 0$ ), and the choice of divisor in the divisor class is a smooth choice.

(d) follows from [FP1, Theorem 4.6] (taking  $D_1$  of [FP1] to be any of the  $D_\sigma$  in class  $A$ ; Fantechi and Pardini's  $m_1$  is our  $p$ ).  $\square$

**4.6. Proof of M2d.** By taking six general sections of a sufficiently positive multiple of the canonical bundle (very ample with vanishing higher cohomology), and using this to embed  $\tilde{S}$  in  $\mathbb{P}^5$ , we see that the Hilbert scheme of nonsingular surfaces in  $\mathbb{P}^5$  satisfies Murphy's Law. (The choice of the sections, up to scalar, is a smooth choice over the fine moduli space of surfaces; we use here  $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ , by Theorem 4.5(a), so the line bundle over  $\tilde{S}$  extends uniquely over the universal surface over the moduli space.)

Using five general sections of the bundle to map  $\tilde{S}$  to  $\mathbb{P}^4$  yields a surface with only singularities in codimension 2; each consists of two nonsingular branches meeting transversely (a non-Cohen-Macaulay singularity). The deformations of such a singularity preserve the singularity. (This can be checked formally locally; the calculation can then be done tediously by hand using two transverse co-ordinate planes in  $\mathbb{A}^4$ .) Hence deformations of the singular surface in  $\mathbb{P}^4$  correspond to deformations of the nonsingular surface  $\tilde{S}$  along with the map to  $\mathbb{P}^4$ . We have thus proved **M2d**.

## 5. DEFORMATIONS OF PRODUCTS: PROOF OF M3

We use  $\tilde{S}$  to construct an example in any dimension  $d > 2$  of a  $d$ -fold with very ample canonical bundle with deformation space of the same type as that of  $\tilde{S}$ , and with automorphism group  $G$ ; **M3** then follows. As  $q_{\tilde{S}} = h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ , there are no nonconstant morphisms  $\tilde{S} \rightarrow C$  to a nonsingular curve of positive genus. Let  $C_1, \dots, C_{d-2}$  be general curves of genus 3, and let  $Y = \prod C_i$ . Now  $\text{Def} \prod C_i = \prod \text{Def } C_i$  (true on the level of first-order deformations by [vO, Theorem 2.2], cf. [HM, Exercise 3.33]; and  $\text{Def } C_i$  is unobstructed), so  $Y$  is unobstructed.

We will need the following improvement of a result of Ran. (In the smooth holomorphic case, this result is very similar to Horikawa's [Ho, Theorem 8.2].) The result is likely known to experts.

**5.1. Theorem.** — *Let  $f : X \rightarrow Y$  be a morphism with  $f_* \mathcal{O}_X = \mathcal{O}_Y$  and  $R^1 f_* \mathcal{O}_X = 0$ . Then  $\text{Def}(f : X \rightarrow Y) \rightarrow \text{Def } X$  is an isomorphism.*

(See [Ran, Definition 1.1] for a definition of  $\text{Def}(f : X \rightarrow Y)$ .) Ran's theorem [Ran, Theorem 3.3] is identical to this, except with the additional hypothesis that  $R^2 f_* \mathcal{O}_X = 0$ , and the weaker conclusion that  $\text{Def}(f : X \rightarrow Y) \rightarrow \text{Def } X$  is smooth. This proof is simply a refinement of his.

*Proof.* Consider the spectral sequence  $\text{Ext}_Y^p(\Omega_Y, R^q f_* \mathcal{O}_X) \Rightarrow \text{Ext}_f^i(\Omega_Y, \mathcal{O}_X)$  [Ran, (6)]; the  $E^2$  term is

$$\begin{array}{ccc} \text{Hom}_Y(\Omega_Y, R^2 f_* \mathcal{O}_X) & \text{Ext}_Y^1(\Omega_Y, R^2 f_* \mathcal{O}_X) & \text{Ext}_Y^2(\Omega_Y, R^2 f_* \mathcal{O}_X) \\ \\ 0 & 0 & 0 \\ \\ \text{Hom}_Y(\Omega_Y, \mathcal{O}_Y) & \text{Ext}_Y^1(\Omega_Y, \mathcal{O}_Y) & \text{Ext}_Y^2(\Omega_Y, \mathcal{O}_Y) \end{array}$$

and hereafter in the spectral sequence the entry  $\text{Ext}_Y^i(\Omega_Y, \mathcal{O}_Y)$  will not change for  $i = 0, 1, 2$ . Hence we conclude that

$$(5) \quad T_Y^i := \text{Ext}_Y^i(\Omega_Y, \mathcal{O}_Y) \longrightarrow \text{Ext}_f^i(\Omega_Y, \mathcal{O}_X)$$

is an isomorphism for  $i = 0$  and  $1$  and an injection for  $i = 2$ .

By Ran's exact sequence [Ran, (2.2)]

$$T_X^0 \oplus T_Y^0 \longrightarrow \text{Ext}_f^0(\Omega_Y, \mathcal{O}_X) \longrightarrow$$

$$T_f^1 \longrightarrow T_X^1 \oplus T_Y^1 \longrightarrow \text{Ext}_f^1(\Omega_Y, \mathcal{O}_X) \longrightarrow$$

$$T_f^2 \longrightarrow T_X^2 \oplus T_Y^2 \longrightarrow \text{Ext}_f^2(\Omega_Y, \mathcal{O}_X),$$

we have that  $T_f^1 \rightarrow T_X^1$  is an isomorphism and  $T_f^2 \rightarrow T_X^2$  is injective, which gives the desired result.  $\square$

**5.2. Proposition.** —

- (a)  $\text{Def}(\tilde{S} \times Y) \cong \text{Def } \tilde{S} \times \text{Def } Y$
- (b)  $\text{Aut}(\tilde{S} \times Y) \cong G$

Part (a) shows that the deformation space of  $\tilde{S} \times Y$  has the same type as that of  $\tilde{S}$ , and hence the fine moduli stack satisfies Murphy's Law. Part (b) shows that the  $d$ -fold has no "extra" automorphisms, and thus (as in the surface case) the coarse moduli space satisfies Murphy's Law. Hence **M3** follows.

*Proof.* (a) We first show that the natural morphism  $\text{Def } \tilde{S} \times \text{Def } Y \rightarrow \text{Def}(\tilde{S} \times Y \rightarrow Y)$  is an isomorphism. For convenience let  $X = \tilde{S} \times Y$ . Let

(6)

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad} & \mathcal{Y} \\ & \searrow & \swarrow \\ & \text{Def}(X \rightarrow Y) & \end{array}$$

be the universal morphism over  $\text{Def}(X \rightarrow Y)$ ; all morphisms in (6) are flat. By the flatness of the horizontal morphism  $\mathcal{X} \rightarrow \mathcal{Y}$  of (6) we have a natural morphism  $\mathcal{Y} \rightarrow \text{Def } \tilde{S}$ . This morphism descends to  $\text{Def}(X \rightarrow Y) \rightarrow \text{Def } \tilde{S}$ . (Reason: Interpret

(7)

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad} & \text{Def } \tilde{S} \\ \downarrow & & \\ \text{Def}(X \rightarrow Y) & & \end{array}$$

as a family of maps from irreducible projective varieties to  $\text{Def } \tilde{S}$  over a formal local scheme  $\text{Def}(X \rightarrow Y)$ , that is constant on the central fiber. As constant maps deform to constant maps, (7) must factor through  $\text{Def}(X \rightarrow Y) \rightarrow \text{Def } \tilde{S}$ .)

Hence we have a natural morphism  $\text{Def}(\tilde{S} \times Y \rightarrow Y) \rightarrow \text{Def } \tilde{S} \times \text{Def } Y$ ; there is of course a natural morphism in the other direction. By observing how the universal families behave under these morphisms, we see that the morphisms are isomorphisms (and mutual inverses).

Finally,  $\text{Def}(\tilde{S} \times Y \rightarrow Y) \rightarrow \text{Def}(\tilde{S} \times Y)$  is an isomorphism by Theorem 5.1. The last hypothesis of Theorem 5.1 follows from Proposition 4.5(a).

(b) Suppose  $X$  and  $Y$  are varieties where (i)  $Y$  is connected and has no nontrivial automorphisms, (ii)  $X$  has discrete automorphism group, and (iii) the only morphisms from  $X$  to  $Y$  are constant morphisms (i.e.  $\text{Hom}(X, Y) = Y$ ). Then  $\text{Aut } X \rightarrow \text{Aut}(X \times Y)$  is an isomorphism. (*Proof:* Define the *fibers* of  $X \times Y$  to be the fibers of the projection to  $Y$ , so they are canonically isomorphic to  $X$ . Consider any automorphism  $a : X \times Y \rightarrow X \times Y$ . Each fiber of the source must map to a fiber of the target by (iii). Hence  $a$  induces an automorphism of  $Y$ , necessarily the identity by (i). Compose  $a$  with an automorphism of  $X$  so that  $a$  is the identity on one fiber. Then  $a$  is the identity on all fibers by (ii) and the connectedness of  $Y$ .)

As the  $C_i$  are chosen generally, the only morphisms  $C_i \rightarrow C_j$  ( $i \neq j$ ) are constant maps. Recall that there are no nonconstant maps  $\tilde{S} \rightarrow C_i$ , and  $C_i$  has no automorphisms. Hence by induction on the factors of  $\tilde{S} \times (\prod C_i)$ ,  $\text{Aut}(\tilde{S} \times Y) \cong \text{Aut } \tilde{S}$ .  $\square$

Pardini points out that in the first paragraph of the proof of (b), the right way to see that every automorphism of  $X \times Y$  sends a fiber to the fiber is that the projection  $\tilde{S} \times C_1 \times \cdots \times C_k \rightarrow C_1 \times \cdots \times C_k$  is (essentially) the Albanese map.

## 6. BRANCHED COVERS OF $\mathbb{P}^2$ AND THE PROOF OF M5

Take three sections of a sufficiently positive bundle on  $\tilde{S}$ . As in the proof of **M2d** (Section 4.6), the line bundle over  $\tilde{S}$  étale-locally extends uniquely over the universal surface over the moduli space, and the choice of three sections (up to non-zero scalar) is a smooth one. Hence we have **M5a**. J. Wahl provides the connection to **M5b**:

**6.1. Theorem** (Wahl [W1, p. 530]). — *Let  $Y \rightarrow \mathbb{P}^2$  be a finite surjective morphism,  $Y$  a nonsingular surface, whose branch curve  $C$  is reduced with only nodes and cusps as singularities. Then via taking branch curves, there is a one-to-one correspondence between infinitesimal deformations of the morphism  $Y \rightarrow \mathbb{P}^2$  and infinitesimal deformations of  $C$  in  $\mathbb{P}^2$  which preserve the formal nature of the singularities.*

Wahl's paper assumes that the characteristic is 0, but his proof of this result uses only that the characteristic is not 2 or 3. To reassure the reader, we point out the places where characteristic 0 is used before Wahl's proof of Theorem 6.1 concludes on p. 558. Proposition 1.3.1 and equation (1.5.3) are not used in the proof. Theorem 2.2.8 and its rephrasing (Theorem 2.2.11) give a normal form for stable singularities, and use only that the characteristic is not 2 or 3. (One might conjecture that an appropriate formulation is true in characteristic 2 and 3, but I have not attempted to prove this.) Part **M5b** then follows from the next result.

**6.2. Proposition.** — *If  $\tilde{S}$  is any smooth projective surface over an infinite base field of characteristic not 2 or 3, and  $\mathcal{L}'$  is an ample invertible sheaf, then for  $n \gg 0$ , three general sections of  $\mathcal{L}'^{\otimes n}$  give a morphism to  $\mathbb{P}^2$  with reduced branch curve with only nodes and cusps as singularities.*

In characteristic 0 the result is classical (presumably nineteenth century); the proof is by taking  $n$  large enough that  $\mathcal{L}'^{\otimes n}$  is very ample, and then taking a generic projection. Because we need the result in positive characteristic as well, we use a slightly different approach, although as usual we show the result by showing that “nothing worse can happen,” by excluding possibilities on a case-by-case basis.

*Proof.* We will make repeated use of the following useful fact [Fu, Example 12.1.11] without comment: Let  $E$  be a vector bundle on a variety  $X$  over an infinite field, generated by a finite-dimensional vector space  $W$  of sections. Let  $V$  be a subvariety of (the total space of)  $E$  of pure codimension  $m$ . Then for a general element of  $W$ , the pullback of  $V$  to  $X$  has pure codimension  $m$ .

We show that the branch curve is as desired for maps to  $\mathbb{P}^2$  given by three general sections of  $\mathcal{L}$ , for all invertible sheaves  $\mathcal{L}$ :

- separating 3-jets (i.e.  $J^3(\mathcal{L})$  is generated by global sections of  $\mathcal{L}$ ),

- separating 2-jets at pairs of distinct points (i.e.  $\pi_1^*J^2(\mathcal{L}) \oplus \pi_2^*J^2(\mathcal{L})$  on  $\tilde{S} \times \tilde{S} - \Delta$  is generated by global sections of  $\mathcal{L}$  on  $\tilde{S}$ ),
- and separating 1-jets at triples of distinct points.

A sufficiently high power of  $\mathcal{L}'$  certainly has these properties.

We first show that *the ramification locus is codimension 1*. The rank 9 bundle ( $3 \times 3$  matrix bundle)

$$E_1 = \text{Hom}(\mathcal{O}_{\tilde{S}}^{\oplus 3}, J^1(\mathcal{L})) \cong J^1(\mathcal{L})^{\oplus 3}$$

is generated by global sections of  $\mathcal{L}$ . The rank  $\leq 2$  (determinant 0) locus of  $E_1$  is codimension 1. Thus the ramification locus on  $\tilde{S}$ , i.e. where the induced section of  $E_1$  is rank  $\leq 2$ , indeed has pure codimension 1. The locus where the section of  $E_1$  meets the rank  $\leq 1$  locus is codimension 4 on  $\tilde{S}$ , i.e. empty. Hence at each point  $p$  of the ramification curve, the section of  $E_1$  has rank exactly 2, so there is a section  $x$  in our net vanishing at the point, but not vanishing to second order ( $x$  is a local coordinate on  $\tilde{S}$ ). There is another section  $z$  corresponding to the kernel  $\mathcal{J} := \ker(\mathcal{O}_{\tilde{S}}^{\oplus 3} \rightarrow J^1(\mathcal{L}))$  that vanishes to order (at least) 2 at  $p$ . (Note that  $x$  and  $z$  are lines on  $\mathbb{P}^2$  and hence local coordinates there.) The section  $z$  is unique up to multiplication by nonzero scalar. The section  $x$  is unique up to multiplication by nonzero scalar, and addition by some multiple of  $z$ .

Consider next the  $6 \times 3$  matrix bundle  $E_2 = \text{Hom}(\mathcal{O}_{\tilde{S}}^{\oplus 3}, J^2(\mathcal{L}))$ . The “rank 2 locus of  $E_1$ ” makes sense for  $E_2$  (and indeed any jet bundle surjecting onto  $E_1$ ). On this subvariety of  $E_2$  there is a map of invertible sheaves  $\alpha : \mathcal{J} \rightarrow (\Omega_{\tilde{S}}^2 \otimes \mathcal{L}) / (x\Omega_{\tilde{S}}^1)$  locally induced by

$$(8) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{\tilde{S}}^{\oplus 3} & & \\ & & \downarrow \alpha & & \downarrow & & \\ 0 & \longrightarrow & \frac{\Omega_{\tilde{S}}^2 \otimes \mathcal{L}}{x\Omega_{\tilde{S}}^1} & \longrightarrow & \frac{J^2(\mathcal{L})}{xJ^1(\mathcal{O}_{\tilde{S}})} & \longrightarrow & \frac{J^1(\mathcal{L})}{x\mathcal{O}_{\tilde{S}}} \longrightarrow 0. \end{array}$$

Note that both  $(\Omega_{\tilde{S}}^2 \otimes \mathcal{L}) / (x\Omega_{\tilde{S}}^1)$  and  $\alpha$  are independent of  $x$ . (Near a point  $p$  of the ramification curve the bottom exact sequence has a straightforward interpretation. If  $y$  is a local coordinate of  $\tilde{S}$  at  $p$  transverse to  $x$ , then once a trivialization of  $\mathcal{L}$  near  $p$  is chosen, the left term corresponds to the coefficient of  $y^2$ , and the right term corresponds to coefficients of 1 and  $y$ .) Let  $V$  be the subvariety of  $E_2$  where  $\alpha$  has rank 0 (inside the rank 2 locus of  $E_1$ ); then  $\text{codim } V = 2$ , so the corresponding locus on  $\tilde{S}$  has codimension 2 as well. For the purposes of this proof, call such points of the ramification curve *twisty* points. Thus for a non-twisty ramification point  $p$ , there is a section not vanishing at  $p$ , another section  $x$  vanishing to first order at  $p$ , and a third section  $z$  (corresponding to  $\mathcal{J}$ ) vanishing to order *exactly* 2 at  $p$  such that  $(x, z)$  has length 2 on  $\tilde{S}$ : we conclude that *the ramification is simple*, and the ramification curve is smooth at  $p$ . (Note that  $x$  and  $z$  are local coordinates on  $\mathbb{P}^2$ , and there are local coordinates  $x$  and  $y$  on  $\tilde{S}$  such that formally locally  $z - y^2 \in (y^3, xy, x^2)$ .) We remark that we have defined a codimension 2 subvariety of  $E_2$  (the “twisty subvariety”), and of any jet bundle surjecting onto  $E_2$ .

Consider now the locus in the  $9 \times 3$  matrix bundle

$$E_3 = \text{Hom} \left( \mathcal{O}_{\tilde{S}}^{\oplus 3}, \pi_1^* J^1(\mathcal{L}) \oplus \pi_2^* J^1(\mathcal{L}) \oplus \pi_3^* J^1(\mathcal{L}) \right)$$

on  $\tilde{S} \times \tilde{S} \times \tilde{S} - \Delta$  where the image in the  $3 \times 3$  matrix bundle

$$\text{Hom} \left( \mathcal{O}_{\tilde{S}}^{\oplus 3}, \pi_1^* \mathcal{L} \oplus \pi_2^* \mathcal{L} \oplus \pi_3^* \mathcal{L} \right)$$

has rank 1 (“the three points of  $\tilde{S}$  map to the same point of  $\mathbb{P}^2$ ,” codimension 4 in  $E_3$ ) and where the image in each of the  $3 \times 3$  matrix bundles  $\pi_i^* E_1$  has rank 2 (“each point is on the ramification curve,” codimension 1 in  $E_3$ ). The pullback of this locus (via our three sections of  $\mathcal{L}$ ) has codimension  $4 + 1 + 1 + 1 = 7$  on  $\tilde{S} \times \tilde{S} \times \tilde{S} - \Delta$ , and hence is empty. Hence *there cannot be three points of the ramification curve mapping to the same point of  $\mathbb{P}^2$* .

Consider the locus in the  $9 \times 3$  matrix bundle

$$E_4 = \text{Hom} \left( \mathcal{O}_{\tilde{S}}^{\oplus 3}, \pi_1^* J^2(\mathcal{L}) \oplus \pi_2^* J^1(\mathcal{L}) \right)$$

on  $\tilde{S} \times \tilde{S} - \Delta$ , where we require: the image in the  $2 \times 3$  matrix bundle  $\text{Hom}(\mathcal{O}_{\tilde{S}}^{\oplus 3}, \pi_1^* \mathcal{L} \oplus \pi_2^* \mathcal{L})$  to be rank 1 (“the two points map to the same point of  $\mathbb{P}^2$ ,” a codimension 2 condition); the first point to be a twisty ramification point (shown earlier to be a codimension 2 condition on  $E_4$ ); and the second point to be a ramification point (a codimension 1 condition on  $E_4$  as shown above). These conditions are independent, so this locus in  $E_4$  has codimension 5, hence the pullback (by our section) is empty on  $\tilde{S} \times \tilde{S} - \Delta$ . Thus *if two points of the ramification curve map to the same point of  $\mathbb{P}^2$ , neither is twisty*.

Next, in the  $6 \times 3$  matrix bundle

$$E_5 = \text{Hom} \left( \mathcal{O}_{\tilde{S}}^{\oplus 3}, \pi_1^* J^1(\mathcal{L}) \oplus \pi_2^* J^1(\mathcal{L}) \right)$$

on  $\tilde{S} \times \tilde{S} - \Delta$ , we consider the locus where: the image in the  $2 \times 3$  matrix bundle  $\text{Hom}(\mathcal{O}_{\tilde{S}}^{\oplus 3}, \pi_1^* \mathcal{L} \oplus \pi_2^* \mathcal{L})$  has rank 1 (“the two points map to the same point in  $\mathbb{P}^2$ ,” codimension 2); and  $E_5$  itself has rank  $\leq 2$  (“the two points are ramification points, and the branch curves in  $\mathbb{P}^2$  share a tangent line,” contributing an additional codimension of 3 — not 4, because of the dependence of these conditions with the previous ones). The total codimension of this locus is 5, hence (the pullback of) this locus is empty on  $\tilde{S} \times \tilde{S} - \Delta$ , so *if two ramification points map to the same point of  $\mathbb{P}^2$ , the tangent vectors of the branch curve in  $\mathbb{P}^2$  are transverse*. In particular, *the branch curve is reduced, and has only nodes away from the twisty points*.

Finally we show that the twisty ramification points give cusps of the branch curve. Similar to (8), on the ramification curve we locally have a morphism  $\beta$  from  $\mathcal{J} = \ker(\mathcal{O}_{\tilde{S}}^{\oplus 3} \rightarrow J^1(\mathcal{L}))$  to a rank 2 bundle  $Q$  satisfying defined by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{\tilde{S}}^{\oplus 3} & & \\ & & \downarrow \beta & & \downarrow & & \\ 0 & \longrightarrow & Q & \longrightarrow & \frac{J^3(\mathcal{L})}{xJ^2(\mathcal{O}_{\tilde{S}})} & \longrightarrow & \frac{J^1(\mathcal{L})}{x\mathcal{O}_{\tilde{S}}} \longrightarrow 0. \end{array}$$

satisfying

$$\begin{array}{ccccccc}
& & \mathcal{J} & & & & \\
& & \downarrow \beta & \searrow \alpha & & & \\
0 & \longrightarrow & \frac{\Omega_{\tilde{S}}^3 \otimes \mathcal{L}}{x\Omega_{\tilde{S}}^2} & \longrightarrow & Q & \longrightarrow & \frac{\Omega_{\tilde{S}}^2 \otimes \mathcal{L}}{x\Omega_{\tilde{S}}^1} \longrightarrow 0.
\end{array}$$

(Near a point  $p$  of the ramification curve, the local interpretation is similar to that of (8). The term  $(\Omega_{\tilde{S}}^3 \otimes \mathcal{L})/x\Omega_{\tilde{S}}^2$  corresponds to the coefficient of  $y^3$ , and the term  $(\Omega_{\tilde{S}}^2 \otimes \mathcal{L})/x\Omega_{\tilde{S}}^1$  corresponds to the coefficient of  $y^2$ .) Hence on the (codimension 2) twisty locus, where  $\alpha = 0$ , there is a morphism of invertible sheaves  $\gamma : \mathcal{J} \rightarrow (\Omega_{\tilde{S}}^3 \otimes \mathcal{L})/(x\Omega_{\tilde{S}}^2)$ ; and both  $(\Omega_{\tilde{S}}^3 \otimes \mathcal{L})/(x\Omega_{\tilde{S}}^2)$  and  $\gamma$  are independent of our choice of  $x$ . Consider the subvariety of  $\text{Hom}(\mathcal{O}_{\tilde{S}}^{\oplus 3}, J^3(\mathcal{L}))$  (in the codimension 2 twisty locus) such that the rank of  $\gamma$  is 0. This is a codimension 3 subvariety of  $\text{Hom}(\mathcal{O}_{\tilde{S}}^{\oplus 3}, J^3(\mathcal{L}))$ , and hence empty on  $\tilde{S}$ . Thus formally locally, at a twisty ramification point, “the  $y^3$ -coefficient is non-zero,” so the morphism is given by

$$(x, y) \mapsto (x, y^3 f_1(x, y) + y^2(xg_1(x)) + y(xh_1(x)) + x^2 i_1(x)) = (x, z)$$

where  $f_1(0, 0) \neq 0$ . Consider again the  $6 \times 3$  matrix bundle  $E_2 = \text{Hom}(\mathcal{O}_{\tilde{S}}^{\oplus 3}, J^2(\mathcal{L}))$ . The rank 2 locus of  $E_2$  is codimension 4, and hence our section misses it: every non-zero section of  $\mathcal{L}$  restricts to something nonzero in  $J^2(\mathcal{L})$  at every point. Hence  $z \notin (x, y)^3$ , so either  $i_1(0) \neq 0$  or  $h_1(0) \neq 0$ . By replacing  $x$  by  $x + y$  if necessary, we may assume that  $h_1(0) \neq 0$ .

By replacing  $y$  by a scalar multiple, we may assume  $f_1(0, 0) = 1$ . By replacing  $y$  by  $yf_1(x, y)^{1/3}$ , and rearranging, the morphism may be rewritten as

$$(x, y) \mapsto (x, y^3(1 + xf_2(x, y)) + y^2(xg_2(x)) + y(xh_2(x)) + x^2i_2(x)) = (x, z)$$

where again  $h_2(x) \neq 0$ . (The  $xf_2(x, y)$  arose because of contributions of  $x(y^2g_1(x) + yh_1(x))$  in the change of variables.) By replacing  $y$  by  $y(1 + xf_2(x, y))^{1/3}$  we obtain

$$(x, y) \mapsto (x, y^3(1 + x^2f_3(x, y)) + y^2(xg_3(x)) + y(xh_3(x)) + x^2i_3(x)) = (x, z).$$

By repeating this process inductively, and noting that the lower degree terms of  $g_n(x)$ ,  $h_n(x)$ , and  $i_n(x)$  stabilize, we obtain

$$(x, y) \mapsto (x, y^3 + y^2(xG_1(x)) + y(xH_1(x)) + x^2I_1(x)) = (x, z)$$

where  $H_1(0) \neq 0$ . Replacing  $y$  by  $(y + xG_1(x))/3$ , the morphism may be rewritten as

$$(x, y) \mapsto (x, y^3 + y(xH_2(x)) + x^2I_2(x)) = (x, z).$$

Replacing  $z$  by  $z - x^2I_2(x)$  and then replacing  $x$  by  $xH_2(x)$  (here finally using  $H_2(0) \neq 0$ ), we have shown that near a twisty point in formal local co-ordinates the morphism is given by  $(x, y) \mapsto (x, z) = (x, y^3 + xy)$ . Then the branch locus in the  $(x, z)$ -plane is given by  $4x^3 + 27z^2 = 0$ , i.e. it is a cusp.  $\square$

## 7. FROM SURFACES TO THE REST OF THEOREM 1.1

**7.1. Proof of M1.** We are fortunate that Fantechi and Pardini have proved precisely the result that we need for the proof of **M1**. If  $X \subset \mathbb{P}^n$  is a subscheme, let  $\text{Hilb}(X)$  be the (connected component of the) Hilbert scheme containing  $[X]$ .

**7.2. Theorem.** — (a) (Fantechi-Pardini [FP2, Proposition 4.2]) Let  $\tilde{S} \subset \mathbb{P}^n$  be a smooth, regular, projectively normal surface. Let  $H$  be a smooth hypersurface of degree  $l$  in  $\mathbb{P}^n$  meeting  $\tilde{S}$  transversely along a curve  $C$ , and let  $U \subset \text{Hilb}(\tilde{S}) \times \text{Hilb}(H)$  be the open set of pairs  $(\tilde{S}', H')$  such that  $\tilde{S}'$  and  $H'$  are smooth and transverse and  $\tilde{S}'$  is projectively normal. If  $l \gg 0$ , then the morphism  $U \rightarrow \text{Hilb}(C)$  (induced by the intersection) is smooth.

(b) Furthermore,  $C$  is embedded by a complete linear system.

We note that Fantechi and Pardini's proof of (a) invokes Kodaira vanishing to show that if  $F$  is a hypersurface of degree  $l$  then  $H^1(F, N_{F/\mathbb{P}^n}) = 0$ , but this may be easily checked directly, so their result is not characteristic-dependent.

*Proof of (b).* If  $\mathcal{I}_{C/\tilde{S}}$  is the ideal sheaf of  $C$  in  $\tilde{S}$ , we have the exact sequence

$$0 \rightarrow \mathcal{I}_{C/\tilde{S}}(1) \rightarrow \mathcal{O}_{\tilde{S}}(1) \rightarrow \mathcal{O}_C(1) \rightarrow 0.$$

As  $\mathcal{I}_{C/\tilde{S}} \cong \mathcal{O}_{\tilde{S}}(-l)$ ,  $H^1(\tilde{S}, \mathcal{I}_{C/\tilde{S}}(1)) = 0$  by Serre vanishing. Thus  $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(1)) \rightarrow H^0(C, \mathcal{O}_C(1))$  is surjective. As  $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(1))$  is also surjective ( $\tilde{S}$  is embedded by a complete linear system), the result follows.  $\square$

We now prove **M1**. Choose a sufficiently ample line bundle on  $\tilde{S}$ , so that the corresponding embedding  $\tilde{S} \hookrightarrow \mathbb{P}^n$  (by the complete linear system) is projectively normal, and so that the line bundle has no higher cohomology. The deformation space of  $\tilde{S} \hookrightarrow \mathbb{P}^n$  is smooth over the deformation space of  $\tilde{S}$ , as described in Section 4.6. Then Theorem 7.2(a) gives **M1a**. Deformations of a smooth curve in  $\mathbb{P}^n$  are the same as deformations of the corresponding immersion, yielding **M1b**. Theorem 7.2(b) gives **M1c**.

**7.3. Proof of M4.** Near a seminormal point of the Hilbert scheme, there is a morphism from the Hilbert scheme to the Chow variety [Kol1, Theorem 6.3]. If the point of the Hilbert scheme parametrizes an object that is geometrically reduced, normal, and of pure dimension, then this morphism is a local isomorphism [Kol1, Corollary 6.6.1]. Hence **M4** follows from **M1a** and **M2d**.

**7.4. Proof of M6.** (I am grateful to R. Thomas for explaining how to think about this problem, and for greatly shortening the following argument.) The sheaf in question will be the ideal sheaf  $\mathcal{I}$  of  $\tilde{S}$  embedded in  $\mathbb{P}^5$  (from **M2d**). The next result implies **M6**.

**7.5. Proposition.** — If  $Y$  is a nonsingular variety with  $h^1(Y, \mathcal{O}_Y) = h^2(Y, \mathcal{O}_Y) = 0$ , and  $X \hookrightarrow Y$  is a local complete intersection, then the deformation space of  $X \hookrightarrow Y$  is canonically isomorphic to the deformation space of the ideal sheaf  $\mathcal{I}$  of  $X$ .

*Warning:* This result does not hold for general  $X \hookrightarrow Y$ .

*Proof.* As usual, we describe an isomorphism of first-order deformations and an injection of obstructions. The first-order deformations and obstructions for  $X \hookrightarrow Y$  are  $H^0(X, N_{X/Y})$  and  $H^1(X, N_{X/Y})$  respectively. The first-order deformations and obstructions of the torsion-free sheaf  $\mathcal{I}$  are  $\text{Ext}^1(\mathcal{I}, \mathcal{I})$  and  $\text{Ext}^2(\mathcal{I}, \mathcal{I})$  respectively. The  $E^2$  term of the local-to-global spectral sequence for  $\text{Ext}^*(\mathcal{I}, \mathcal{I})$  is:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ H^0(Y, \text{Ext}^1(\mathcal{I}, \mathcal{I})) & \xrightarrow{\quad} & H^1(Y, \text{Ext}^1(\mathcal{I}, \mathcal{I})) & & H^2(Y, \text{Ext}^1(\mathcal{I}, \mathcal{I})) & \cdots \\ & & \searrow & & & & \\ H^0(Y, \mathcal{H}\text{om}(\mathcal{I}, \mathcal{I})) & & H^1(Y, \mathcal{H}\text{om}(\mathcal{I}, \mathcal{I})) & \xrightarrow{\quad} & H^2(Y, \mathcal{H}\text{om}(\mathcal{I}, \mathcal{I})) & \cdots \end{array}$$

A straightforward and well-known argument yields  $\text{Ext}^q(\mathcal{I}, \mathcal{I}) \cong \wedge^q N_{X/Y}$ , so  $E^2$  may be written as

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ H^0(X, N_{X/Y}) & & H^1(X, N_{X/Y}) & & H^2(X, N_{X/Y}) & \cdots \\ k & & 0 & & 0 & \cdots \end{array}$$

using  $h^i(Y, \mathcal{O}_Y) = 0$  for  $i = 1, 2$ . Thus we have  $H^0(X, N_{X/Y}) \cong \text{Ext}^1(\mathcal{I}, \mathcal{I})$  and  $H^1(X, N_{X/Y}) \hookrightarrow \text{Ext}^2(\mathcal{I}, \mathcal{I})$ , concluding the proof.  $\square$

**7.6. Proof of M7.** We obtain the threefold singularity by embedding our surface in projective space by a complete linear system arising from a sufficiently positive line bundle (as in the proof of **M2d**). The deformations of the cone over the surface are the same as the deformations of the surface in projective space, by the following theorem of Schlessinger.

**7.7. Theorem (Schlessinger [Sch, Theorem 2]).** — Let  $\tilde{S} \subset \mathbb{P}^n$  be a projectively normal variety (over a field) of dimension  $\geq 2$ , such that

$$h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(v)) = h^1(\tilde{S}, T_{\tilde{S}}(v)) = 0$$

for  $v > 0$ . Then the versal deformation spaces of  $\tilde{S}$  in  $\mathbb{P}^n$  and the singularity  $C_{\tilde{S}}$  (the cone over  $\tilde{S}$ ) are isomorphic.

(Although Schlessinger works in the complex analytic category, his proof is purely algebraic, and characteristic-independent.) This singularity is Cohen-Macaulay by the following result, concluding the proof of **M7**.

**7.8. Proposition.** — Suppose  $\tilde{S}$  is a Cohen-Macaulay scheme (over a field),  $h^i(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$  for  $i = 1, \dots, \dim \tilde{S} - 1$  and  $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 1$ . Then the embedding of  $\tilde{S}$  by a sufficiently ample line bundle is arithmetically Cohen-Macaulay.

This result follows from a statement of Hartshorne and Ogus [HaO, p. 429 #3]. See [GW, p. 207–8] or [CuH, Lemma 1.1(2)] for a proof. The hypotheses follow from the regularity of  $\tilde{S}$ , Theorem 4.5(a). (It turns out that in characteristic 0,  $2K_{\tilde{S}}$  is ample enough, using Kodaira vanishing.)

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