π_p, the value of π in ℓ_p

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The two-dimensional space ℓ_p is the set of points in the plane, with the distance between two points (x, y) and (x', y') defined by $(|x - x'|^p + |y - y'|^p)^{1/p}$, $1 \le p \le \infty$. The distance from (x, y) to the origin is then $(|x|^p + |y|^p)^{1/p}$. The equation of the unit circle C_p, i.e., the circle with its center at the origin and radius 1, is

$$(|x|^p + |y|^p)^{1/p} = 1. \tag{1}$$

Figure 1 shows C_p for $p = 1$, $3/2$, 2, 3, and ∞. Equation (1) is unchanged when x is replaced by $-x$, when y is replaced by $-y$, and when x and y are interchanged. Therefore C_p is symmetric about the y-axis, about the x-axis, and about the line $x = y$.

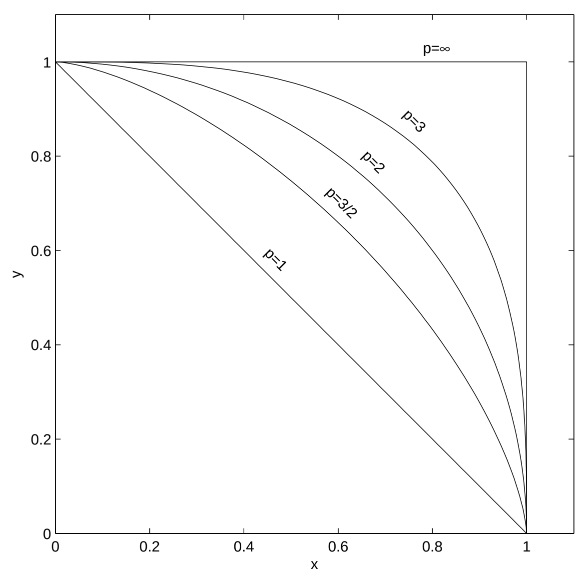

Figure 1. The unit circle C_p in the first quadrant, defined by (1), for $p = 1, 3/2, 2, 3, \infty$.

It is natural to define π_p as the ratio of the circumference of C_p (in the p-metric) to two times its radius (also in the p-metric), which is its "diameter," 2. This definition has been well studied, see for example [2], [1], and [3]. The circumference is the integral of the element of arclength $ds = (|dx|^p + |dy|^p)^{1/p}$ around C_p. Thus

$$\pi_p = \frac{1}{2}\int_{C_p} (|dx|^p + |dy|^p)^{1/p} = \frac{1}{2}\int_{C_p} \left(1 + \left|\frac{dy}{dx}\right|^p\right)^{1/p} |dx| \,. \tag{2}$$

Because of the symmetry of C_p, its circumference is equal to four times its arclength in the first quadrant, or eight times its arclength in the first quadrant between the lines $x = 0$ and $x = y$. When $x = y$, (1) shows that $x = 2^{-1/p}$, so the integral in (2) is 8 times the integral from 0 to $2^{-1/p}$. By calculating dy/dx from (1), we can rewrite (2) as

$$\pi_p = 4\int_0^{2^{-1/p}} \left(1 + \left|x^{-p} - 1\right|^{1-p}\right)^{1/p} dx. \tag{3}$$

For $p = 1$, (3) yields $\pi_1 = 4(2^{1/p})(2^{-1/p}) = 4$. For $p = \infty$, the integrand is 1 and the upper limit is 1, so $\pi_\infty = 4$. At $p = 2$, $\pi_2 = \pi$. If geometry had been developed using the ℓ_p distance instead of the ℓ_2 distance, π_p would have replaced π_2, which is just the familiar π.

Figure 2 shows a graph of π_p as a function of p, obtained by numerical integration of (3). The graph suggests that as p increases from $p = 1$, π_p decreases monotonically from its maximum value $\pi_1 = 4$ to its minimum value $\pi_2 = \pi$, and then increases monotonically to $\pi_\infty = 4$. In fact this is the case, and was proved by Adler and Tanton in [1].

Thus for each p in $1 \le p \le 2$, there is a q in $2 \le q \le \infty$ such that

$$\pi_p = \pi_q. \tag{4}$$

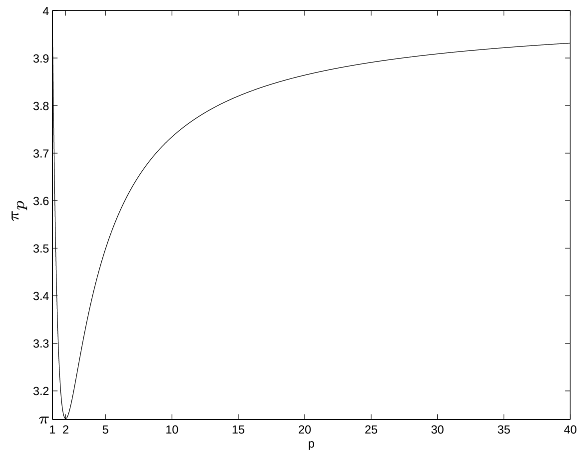

Figure 2. The graph suggests that $\pi_p = \pi_q$ for $1/p + 1/q = 1$

To find q we recall that the Hölder inequality involves two exponents p and q related by

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{5}$$

The numerical results shown in Figure 2 lead one to conjecture that (4) will hold when (5) does. Indeed, Adler and Tanton earlier asked precisely this question as the concluding remark to [1]. (We learned of Adler and Tanton's work only after writing this note.)

In fact, when (5) holds, the domains bounded by C_p and C_q are polar (or dual) to one another. (If K is a convex set in $\mathbf{R}^2$, then its polar is defined by $\{y \in \mathbf{R}^2 : x{\cdot}y \leq 1 \;\; \forall x \in K\}$.) Then a result of Schäffer [4] (see also Thompson [5, p. 118, Cor. 4.3.9, and p. 202, Cor. 6.3.2]) shows that (4) holds. This argument from Minkowski geometry suggested to us that there should be a direct elementary explanation.

We shall now give another proof that (4) holds when (5) does, by showing that then the integral (3) for π_p equals that for π_q. We begin by writing the equation for the arc of C_p in the first quadrant in terms of a parameter $t \in [0, \infty]$, setting $x = f_1(t)$ and $y = f_2(t)$. Then the length L_p of that arc can be written as

$$L_p = \int_0^\infty \left(|f_1'|^p + |f_2'|^p\right)^{1/p} \, dt. \tag{6}$$

We choose the parameter t such that $t^{q/p}$ is the slope of the line from the origin to the point $(f_1(t), f_2(t))$ on C_p, so that $t^{q/p} = f_2(t)/f_1(t)$. From this equation and (1) we find that

$$f_1(t) = (t^q + 1)^{-1/p}, \quad f_2(t) = \left(t^{-q} + 1\right)^{-1/p}. \tag{7}$$

We parameterize C_q in the same way, setting $x = g_1(t)$ and $y = g_2(t)$, with

$$g_1(t) = (t^p + 1)^{-1/q}, \quad g_2(t) = \left(t^{-p} + 1\right)^{-1/q}. \tag{8}$$

Now we define the function $F(t) = -f_1 g_2 + f_2 g_1$. At the ends of the two arcs, $t = 0$ and $t = \infty$, we have $f_1 = g_1$ and $f_2 = g_2$. Therefore $F(0) = 0$ and $F(\infty) = 0$, so $\int\limits_0^\infty F'(t) dt = 0$. This equation can be rewritten as follows, by differentiating the definition of $F(t)$ to get $F'(t)$ and then transposing:

$$\int_0^\infty (-f_1' g_2 + f_2' g_1) \, dt = \int_0^\infty (-g_1' f_2 + g_2' f_1) \, dt. \tag{9}$$

Of course, this is just integration by parts, but it is enlightening to interpret $F(t)$ as essentially a cross product. We now show that the integrand on the left side of (9) can be rewritten as

$$-f_1' g_2 + f_2' g_1 = \left(|f_1'|^p + |f_2'|^p\right)^{1/p}. \tag{10}$$

To prove (10) we first transform the left side as follows:

$$-f_1' g_2 + f_2' g_1 \quad = \quad -\left(-\frac{1}{p}\right)(t^q + 1)^{-\frac{p+1}{p}} q t^{q-1} (t^{-p} + 1)^{-1/q}$$

$$
\begin{aligned}
&+\left(-\frac{1}{p}\right)(t^{-q}+1)^{-\frac{p+1}{p}}(-q)t^{-(q+1)}(t^p+1)^{-1/q} \\
=\;& \frac{q}{p}(t^q+1)^{-\frac{p+1}{p}}(t^p+1)^{-1/q}(t^{q-1+p/q}+t^{q(p+1)/p-(q+1)}) \\
=\;& \frac{q}{p}(t^q+1)^{-\frac{p+1}{p}}(t^p+1)^{1/p-1}t^{\frac{1}{p-1}-1}(t^p+1) \\
=\;& \frac{q}{p}(t^q+1)^{-\frac{p+1}{p}}(t^p+1)^{1/p}t^{\frac{1}{p-1}-1}. \qquad (11)
\end{aligned}
$$

Then we transform the right side:

$$
\begin{aligned}
\left(|f_1'|^p+|f_2'|^p\right)^{1/p} &= \left[\left(\frac{1}{p}(t^q+1)^{-\frac{p+1}{p}}qt^{q-1}\right)^p+\left(\frac{1}{p}\left(t^{-q}+1\right)^{-\frac{p+1}{p}}qt^{-(q+1)}\right)^p\right]^{1/p} \\
&= \frac{q}{p}(t^q+1)^{-\frac{p+1}{p}}\left(t^{p(q-1)}+t^{q(p+1)-p(q+1)}\right)^{1/p} \\
&= \frac{q}{p}(t^q+1)^{-\frac{p+1}{p}}\left(t^q+t^{q-p}\right)^{1/p} \\
&= \frac{q}{p}(t^q+1)^{-\frac{p+1}{p}}(t^p+1)^{1/p}t^{\frac{1}{p-1}-1}. \qquad (12)
\end{aligned}
$$

The last forms of (11) and (12) are the same, which proves (10).

The integral from 0 to ∞ of the right side of (10) is just L_p, as (6) shows. Therefore the integral of the left side, which is also the left side of (9), is also L_p. A symmetrical argument shows that the right side of (9) is L_q, so (9) shows that $L_p = L_q$. This proves that (4) is true when p and q are related by (5).

This argument is geometrically motivated, not just formal manipulation. Suppose $q > p$. As t goes from 0 to 1, the point on C_p is "behind" the point on C_q. At $t = 1$ the p-point passes the q-point. The cumulative lengths at "time" t are *not* the same. The difference is twice the area of the triangle spanned by the origin, the p-point and the q-point (see the cross product comment above). This difference vanishes at $t = \infty$, when the two points coincide.

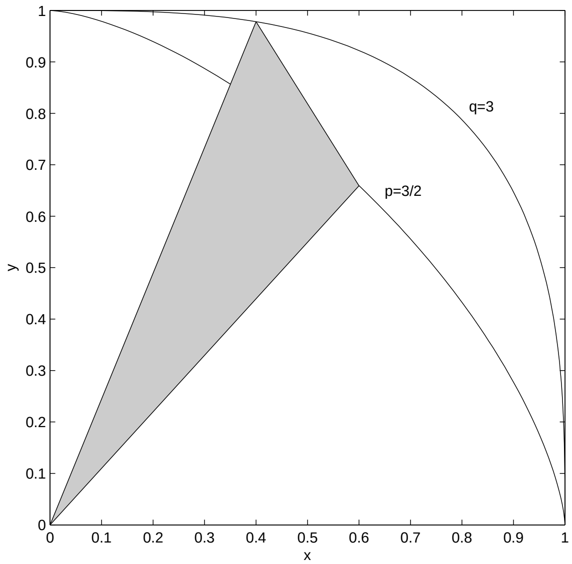

Figure 3. At time t, the q-point and the p-point are at different angles from the x-axis. The signed difference in cumulative lengths is the signed area of the region shown.

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