On the $H$ polynomials of reductive monoids

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What is a reductive monoid?
Naively speaking, it is the Zariski closure of a reductive group. More precisely, suppose $\rho : G_0 \to GL(V)$ is a (rat.) representation. Then

$$M(\rho) := \overline{\mathbb{C}^* \cdot \rho(G_0)} \subseteq End(V),$$

is a reductive monoid.

We will denote by $G$ the group of invertible elements in $M(\rho)$. 
Some questions about reductive monoids:
• The unit group $G$ (hence any Borel subgroup) of the monoid acts on $M$. What can be said about the orbits?
• How do you classify them?
• What is the representation theory?

For the answers and other useful stuff see the text book: Linear Algebraic Monoids by Lex Renner. Also, there is an excellent exposé by Lois Solomon, called An introduction to reductive monoids.
The following two examples are from the Solomon's article.

**Example 1.** Let $V = \mathbb{C}^4 \otimes \mathbb{C}^4$, and consider $\rho : SL_4 \to GL(V)$ defined by

$$\rho(g)(v \otimes v') = gv \otimes gv'.$$

Then, $\mathbb{C}^* \cdot \rho(SL_4) = \{g \otimes g | g \in GL_4\}$ and hence

$$M(\rho) = \mathbb{C}^* \cdot \rho(SL_4) = \{a \otimes a | a \in M_4\} \cong M_4.$$
Example 2. Now, consider $\sigma : SL_4 \to GL(V)$ defined by

$$\sigma(g)(v \otimes v') = gv \otimes (g^{-1})^t v'.$$

Then the unit group of $M(\sigma)$ is very similar to that of $M(\rho)$, however, these monoids are different in a fundamental way.

The difference can be read off from the idempotents $E(M(\rho)) = \{ e \in M(\rho) | e^2 = e \}$. 
$T \subseteq G$ is a maximal torus, then $M(\rho)$ contains the affine toric variety $\overline{T}$. Therefore $E(\overline{T}) \subseteq E(M(\rho))$.

$E(M(\rho))$ is a poset: $e \leq f \iff e = fe$. We consider $E(\overline{T})$ with the induced partial order.
Theorem - an eye opener: Let $T \subseteq T_n$ be a subtorus of the diagonal invertible $n \times n$ matrices. Let $\chi_1, ..., \chi_n \in X(T)$ be the restrictions of the coordinate functions on $T$ to $T$. Let

$$\mathcal{L} = \{\lambda \in X_* | \langle \chi_i, \lambda \rangle \geq 0, \text{ for } 1 \leq i \leq n\}$$

be the associated polyhedral cone.

Then the face lattice of $\mathcal{L}$ is anti-isomorphic to the lattice of idempotents $E(T)$.
If furthermore $G$ is semisimple and $0 \in M$, then we can replace the face lattice of the cone with the face lattice of a polytope.

In this spirit;

- $E(\overline{T}) \subseteq M(\rho)$ of the example 1 is isomorphic to the face lattice of the standard 4-simplex.
- $E(\overline{T}) \subseteq M(\sigma)$ is the face lattice of the cuboctahedron.
Definition. The cross section lattice $\Lambda \subseteq E(T)$ is the sublattice

$$\Lambda := \{e \in E(T) | Be \subseteq eB\}.$$ 

Theorem (Putcha) Let $M$ be a reductive monoid with the unit group $G$. Then

$$M = \bigsqcup_{e \in \Lambda} GeG$$
Let $R = \overline{N_G(T)}/T$, where $\overline{N_G(T)}$ is the Zariski closure in $M$.

**Theorem** (Renner)

- $R$ is a finite monoid.
- The group of units of $R$ is the Weyl group $W$, and $R = WE(R)$,
- $E(R) = E(T)$,
- For $e \in \Delta$, $GeG = \bigcup_{r \in WeW} BrB$,
- $M = \bigcup_{r \in R} BrB$,
- Bruhat-Chevalley order on $W$ extends to $R$. 


Remark. For $r \in R$,

$$BrB \cong \mathbb{C}^{\ell(r) - rk(r)} \times (\mathbb{C}^*)^{rk(r)},$$

where $rk(r) = \text{dim}(Tr)$ and $\ell(r) = \text{dim}(BrB)$.

Definition. (Renner)

$$H(M, q) = \sum_{r \in R} q^{\ell(r) - rk(r)}(q - 1)^{rk(r)}.$$
Remarks.

• This definition works for any variety with finitely many $B \times B$ orbit.
• $H(M, q) = \sum_{e \in \Lambda} H(GeG, q)$.
• If $M$ is quasi-smooth, then Renner shows that $(H(M, q) - 1)/(q - 1)$ is the intersection homology Poincare polynomial of $M \setminus \{0\}/\mathbb{C}^*$.  

Question. Would it be interesting to study

$$H_M(q, t) = \sum_{r \in R} q^{\ell(r) - rk(r)} t^{rk(r)}$$

Answer: Of course!
Theorem (Can, Renner) Let $M$ be a reductive monoid, and let $e \in \Lambda$. Then there exists a $B \times B$ equivariant fibration

$$G(e) \to GeG \to G/P \times G/P^-,\]

where $P$ is a maximal parabolic subgroup and $G(e)$ is a unit group of a submonoid of $M$. 

Corollary: $H_{GeG}(q, t) = H_{G/P}(q, t)^2 H_{G}(e)(q, t)$. 

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**Theorem.** (C., R.) Let $M = M_n$ be the monoid of $n \times n$ matrices. Then, the $H$-polynomial $H_M(q, t)$ is equal to

$$H_M(q, t) = \sum_{k=0}^{n} [k]_q! \left[ \begin{array}{c} n \\ k \\ q \end{array} \right]^2 q^k (2) t^k.$$ 

**Something hilarious:** replace $t$ by $q - 1$, then everything cancels to $q^{n^2}$. 
Classic Laguere polynomials:

\[ L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^k}{(\alpha + 1)_k k!}, \]

where \((a)_k = a(a + 1)(a + 2) \cdots (a + k - 1)\), and \(\alpha \in \mathbb{C}\). These polynomials satisfy the orthogonality relation

\[ \int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)x^{(\alpha)}e^{-x}dx = \delta_{mn}\Gamma(\alpha+n+1)/n!, \]
Moak’s $q$-analogue of the Laguere polynomials is defined as

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^k (1-q)^k (q^n+\alpha+1 x)^k}{(q^{\alpha+1}; q)_k (q;q)_k},$$

where $(q^a; q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1})$. These also satisfy certain orthogonality relations similar to classical case.
Theorem. (C., R.)

\[ H_{M_n}(q, t) = t^n q^{-(\binom{n}{2})} [n]_q! L_n^{(0)} \left( \frac{-1}{tq^{n-1}} ; q \right) \]

Corollary.

The length generating function \( \sum_{r \in R_{M_n}} q^{\ell(r)} \) is given by

\[ H_{M_n}(q, q) = q^{n-\binom{n}{2}} [n]_q! L_n^{(0)} (-q^{n}; q) \]
\( q \)-Rook polynomials

\[
R_k(\mathcal{F}; q) = \sum_{C} q^{\text{inv}(C, \mathcal{F})}
\]

Here \( \mathcal{F} \) is a right justified Ferrers board in an \( n \times n \) grid of squares, and the sum is over all placements \( C \) of \( k \) nonattacking rooks on the squares of \( \mathcal{F} \).
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Theorem. (C., R.) Let $\mathcal{F}$ be a Ferrers board of shape $\lambda$. And let $M^k_{\lambda, n} \subseteq M_n$ be the set of all rank $k$ matrices of shape $\lambda$. Then the $(q, t)$–$H$ polynomial of $M^k_{\lambda, n}$ is given by

$$H_{M^k_{\lambda, n}}(q, t) = t^k q^{\mid \lambda \mid - k} R_k(\mathcal{F}; \frac{1}{q}).$$

Note: $\bigcup_{k=0}^n M^k_{\lambda, n}$ is an affine subspace of $M_n$ of dimension $\mid \lambda \mid$. 
Definition. Let $Sp_{n} = \{g \in GL_{n}|g^tJg = J\}$ be the symplectic group, where $n = 2l$, $J = \begin{pmatrix} 0 & E_{l} \\ -E_{l} & 0 \end{pmatrix} \in M_{n}$, where $E_{l}$ is the $l \times l$ anti-diagonal $(1, ..., 1)$. Set $G = \mathbb{C}^* \cdot G_{0} \subseteq GL_{n}$. Then the symplectic monoid $MSp_{n}$ is defined as the Zariski closure of $G$ in $M_{n}$. 
Theorem. (C., R.) The $(q,t)$-H polynomial of the symplectic monoid $MSp_n$ is

$$H_{MSp_n}(q,t) = 1 + \sum_{k=1}^{l} q^{(l-k)^2} t^{l-k} + \frac{[2l]!!^2}{[2l-2k]!![k]!^2} + q^{l^2} t^{l+1} [2l]_q!!$$
We have been thinking about/work in progress:

* We can define $H$ polynomials for matrix Schubert varieties. In fact, we can do it for any interval in the poset $R$ (w.r.t. Bruhat-Chevalley order). So, we have been thinking about the relationship between intersection homology Poincare polynomial and the $H$-polynomials.

* A conjecture of Garsia and Remmel says that $q$-Rook polynomials are unimodal for any $\lambda$ and $k$. Remember Stanley’s proof of unimodality for $h$-polynomials. So, we have
been thinking about applying Hard-Lefchetz theorem.

- Other families of orthogonal polynomials specializing to the \((q, t)\)-\(H\) polynomial of \(MSp_n\) or of \(MSO_n\).