Diagonal Invariants of the symmetric group and products of linear forms

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**Products of linear forms**—“Decomposable forms”

\( K \): algebraically closed field.

“Decomposable forms” of degree \( n \) in \( K[t_0, t_1, \ldots, t_r] \)

= totally factorizable forms

= products of \( n \) linear forms.

They are a closed algebraic subvariety in the space of forms of degree \( n \). What is its ideal?
Two nice works on the problem


Die einformigen und die elementaren Functionen. Bekanntlich lässt sich jede einformige Function als ganze Function der Elementarfunctionen umgekehrt darstellen. Die hiebei resultirenden Recursionformeln heissen nach ihrem Entdecker Newton'sche und sind für r Gruppen von je zwei Variabelnpaaren angegeben durch ...

(up to permutation)

The monotonous and the elementary Functionen. Admitting light can be represented each monotonous Function as whole Function of the Elementarfunctionen in reverse. Hiebei the resultirenden Recursionformeln is called after its discoverer Newton' and is for r groups indicated by two pairs of variable each through...
Defining maps for the subvariety of decomposable forms

\( V : K \)-vector space of dimension \( r + 1 \).

The subvariety of decomposable forms is the image of:

\[
\pi : (V^*)^n / \mathcal{S}_n \longrightarrow S^n V^*
\]

\((\ell_1, \ell_2, \ldots, \ell_n) \text{ mod } \mathcal{S}_n \longmapsto \ell_1 \cdot \ell_2 \cdots \ell_n\)

It is the affine cone over the image of:

\[
\mathbb{P}_\pi : \text{Symm}^n \mathbb{P}^r = (\mathbb{P}V^*)^n / \mathcal{S}_n \longrightarrow \mathbb{P}(S^n V^*)
\]

\((H_1, H_2, \ldots, H_n) \text{ mod } \mathcal{S}_n \longmapsto H_1 + H_2 + \cdots + H_n\)

The map \( \mathbb{P}_\pi \) is injective and its image, \( \text{Chow}(n, 0, \mathbb{P}^r) \), is the Chow variety of the 0-cycles of degree \( n \) in \( \mathbb{P}^r \).
Associated maps of graded algebra

\[ S^n V = \text{symmetric power over } V \text{ (quotient of } \otimes^n V). \]

\[ T^n_{\text{sym}} V = \text{symmetric tensors over } V \text{ (subspace of } \otimes^n V). \]

The map\[ P \pi : \text{Symm}^n P^r = (P V^*)^n / \mathfrak{S}_n \rightarrow P(S^n V^*) \]
gives rise to a map between the homogeneous coordinate rings:

\[ \pi^* : \bigoplus_{d=0}^{\infty} T^n_{\text{sym}} S^d V \leftarrow \bigoplus_{d=0}^{\infty} S^d T^n_{\text{sym}} V = S^* T^n_{\text{sym}} V \]
Questions

\[ P_\pi : \Sym^n \mathbb{P}^r \longrightarrow \mathbb{P}(S^n V^*) \]

\[ (H_1, H_2, \ldots, H_n) \mod \mathfrak{S}_n \longmapsto H_1 + H_2 + \cdots + H_n \]

Pb 1. Is \( P_\pi \) an isomorphism \( \Sym^m \mathbb{P}^r \cong \Chow(n, 0, \mathbb{P}^r) \)?

Pb 2. Compute, or describe, \( \ker \pi^* \) (defining equations for the subvariety of decomposable forms)
In coordinates

The map: 
\[ \mathbb{P}_\pi : \text{Symm}^n \mathbb{P}^r = (\mathbb{P}V^*)^n/\mathfrak{S}_n \quad \longrightarrow \quad \mathbb{P}(S^nV^*) \]
can be worked out in homogeneous coordinates.

Write \( \ell_i = a_{i0}t_0 + a_{i1}t_1 + \cdots + a_{ir}t_r \) and \( \prod_i \ell_i = \sum_\alpha \hat{e}_\alpha \ t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_r^{\alpha_r} \).

It appears as: 
\[ \hat{A} = \begin{bmatrix} a_{10} & a_{11} & \cdots & a_{1r} \\ a_{20} & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nr} \end{bmatrix} \quad \mapsto \quad (\hat{e}_\alpha(\hat{A}))_{|\alpha|=n} \]

The algebra \( \bigoplus_d T^n_{\text{Sym}}SV \) of \( \text{Symm}^n \mathbb{P}^r \) is \( \text{HDSym}^{r+1}_n(\mathbb{K}) \) (homogeneous diagonal invariants of the symmetric group): the polynomials in the entries of \( \hat{A} \), invariants under row permutations, homogeneous in the variables of each row.

The \( \hat{e}_\alpha \) (fundamental homogeneous invariants) are a linear basis for the piece of degree 1 of \( \text{HDSym}^{r+1}_n(\mathbb{K}) \).
In coordinates, locally

The map:
$$\mathbb{P}_\pi : \text{Symm}^n \mathbb{P}^r = \mathbb{P}(V^*)^n / \mathfrak{S}_n \longrightarrow \mathbb{P}(S^n V^*)$$
can be worked out in affine charts: $a_{i0} = 1$ for all $i$ and $\hat{e}_{n00\ldots0} = 1$.

Write $\ell_i = 1 + a_{i1}t_1 + \cdots + a_{ir}t_r$ (we set $t_0 = 1$) and $\prod_i \ell_i = 1 + \sum_\alpha e_\alpha t_1^{\alpha_1} \cdots t_r^{\alpha_r}$ ($1 \leq |\alpha| \leq n$).

It appears as:
$$\pi_{\text{aff}} : A = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ a_{21} & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nr} \end{bmatrix} \mapsto (e_\alpha(A))_{1 \leq |\alpha| \leq n}$$

The algebra of the affine chart of $\text{Symm}^n \mathbb{P}^r$ is $\text{DSym}_n^r(\mathbb{K})$ ($\text{diagonal invariants of the symmetric group}$): the polynomials in the entries of $A$, invariants under row permutations.

The $e_\alpha$ are the elementary polynomials. The algebra $\text{DSym}_n^r$ also contains analogues of the classical power sums and monomial functions and conversion algorithms.
The isomorphism problem

Pb 1. Is $\mathbb{P}_\pi$ an isomorphism $\text{Symm}^n \mathbb{P}^r \cong \text{Chow}(n, 0, \mathbb{P}^r)$?

Neeman (1989): if $\text{char } K = 0$ or $> n$ then $\mathbb{P}_\pi$ is an isomorphism.

Dalbec (1999): $\mathbb{P}_\pi : \text{Symm}^2 \mathbb{P}^2 \to \text{Chow}(2, \mathbb{P}^2)$ is an isomorphism regardless of $\text{char } K$.


- For $\text{char } K = 2$, $\mathbb{P}_\pi$ is an isomorphism iff $r = 1$ or $n = 1$ or $r = 2$ with $n \leq 3$.

- For $\text{char } K > 2$, $\mathbb{P}_\pi$ is an isomorphism iff $n > \text{char } K$ or $r = 1$. 

The isomorphism problem

Idea of the proof:

This can be established locally.
$\mathbb{P}_\pi$ is an isomorphism
iff $\pi_{aff}$ (the map between affine charts) is an embedding
iff the $e_\alpha$ generate $DSym^r_n(\mathbb{K})$.

- Use a finite generating set of $DSym^r_n$ (over the integers) like Fleischmann’s monomial functions or the Bergeron–Lamontagne basis.

- Use projections $DSym^r_n \rightarrow DSym^r_{n-1}$ and $DSym^r_n \rightarrow DSym^r_{n-1}$.

- Finish with a small number of small “brute force” computations.
From now on $K = \mathbb{C}$

**Pb. 2: Compute the ideal of $\text{Chow}(n, 0, \mathbb{P}^r)$**

It is $\ker \pi^* = \text{ideal of the algebraic relations between the}$

$\hat{e}_\alpha \in HDSym_{n}^{r + 1}(\mathbb{C})$

It is easier to get $\ker \pi_{\text{aff}}^* = \text{ideal of the algebraic relations between the}$

$e_\alpha \in DSym_n^r(\mathbb{C})$ because:

- A nice algorithm to produce all relations in given multidegree is known (Junker + Dalbec).
- The (multigraded) Hilbert series of $DSym_n^r$ is known (it is the generating function for *vector partitions*). (Gessel–Garsia 1979, Bergeron–Lamontagne 2005)

All is in favour of *Hilbert–driven Gröbner basis computations*.
Compute the ideal of \( \text{Chow}(n, 0, \mathbb{P}^r) \)

Two types of monomial orderings are interesting:

1. **total degree order.**

   Useful: A Gröbner basis for \( \ker \pi^*_\text{aff} \) (relations between elementary polynomials) provides a Gröbner basis for \( \ker \pi^* \) by mere homogenization of its elements.

2. **“easy” order, enjoying the structure of \( \text{DSym}_n^r \).**

   \( \text{DSym}_n^r \) is a free module of rank \((n!)^{r-1}\) over a subalgebra \( \cong \otimes^r \text{Sym}_n \). This provides small Gröbner bases and smaller (non Gröbner) generating sets.

   \# small set of generators / \# gb for nice order / \# gb for total degree

<table>
<thead>
<tr>
<th>Symm$^n_{\mathbb{P}^r}$</th>
<th>( r = 2 )</th>
<th>( r = 3 )</th>
<th>( r = 4 )</th>
<th>( n = 5 )</th>
<th>( n = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td>1/1/1</td>
<td>5/5/35</td>
<td>15/23/1139</td>
<td>35/102/?</td>
<td>70/518/?</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>6/6/12</td>
<td>43/53/1779</td>
<td>177/743/?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 4 )</td>
<td></td>
<td></td>
<td>177/743/?</td>
<td></td>
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</tbody>
</table>

Dalbec (1999) conjectured that the ideal of \( \text{Chow}(3, 0, \mathbb{P}^3) \) is generated in degree 4: true.
Foulkes–Howe conjecture

Foulkes’ (open) plethysm conjecture: (1950)

\(h_d \circ h_n - h_n \circ h_d\) is Schur–positive for \(d \geq n\).

Consider

\[
\pi^* = \bigoplus_d \pi_d^* : \bigoplus_{d=0}^{\infty} T^n_{\text{Sym}} S^d V \leftarrow \bigoplus_{d=0}^{\infty} S^d T^n_{\text{Sym}} V
\]

\(GL(V)\)–characters for the pieces of degree \(d\):

\(h_n \circ h_d\) and \(h_d \circ h_n\)
Foulkes–Howe conjecture

Foulkes’ (open) plethysm conjecture: (1950)

\[ h_d \circ h_n - h_n \circ h_d \] is Schur–positive for all \( d \geq n \).

Consider

\[ \pi^* = \bigoplus_d \pi^*_d : \bigoplus_{d=0}^{\infty} T^n_{\text{Sym}} S^d V \leftarrow \bigoplus_{d=0}^{\infty} S^d T^n_{\text{Sym}} V \]

\( \text{GL}(V) \)–characters for the pieces of degree \( d \):

\[ h_n \circ h_d \quad \text{and} \quad h_d \circ h_n \]

Howe’s (stronger) conjecture(s) (1987)

FH (i) \( \pi^*_d \) is injective for all \( d \leq n \).

FH (ii) \( \pi^*_d \) is surjective for all \( d \geq n \).
Foulkes–Howe conjecture

Foulkes’ (open) plethysm conjecture: (1950)
$h_d \circ h_n - h_n \circ h_d$ is Schur–positive for all $d \geq n$.

Consider
$$\pi^* = \bigoplus_d \pi_d^* : \bigoplus_{d=0}^{\infty} T^n_{Sym} S^d V \leftarrow \bigoplus_{d=0}^{\infty} S^d T^n_{Sym} V$$

$GL(V)$–characters for the pieces of degree $d$:

$h_n \circ h_d$ and $h_d \circ h_n$

Howe’s (stronger) conjecture(s) (1987):

FH (i) $\pi_d^*$ is injective for all $d \leq n$.
($\Leftrightarrow \pi_n^*$ is bijective)
($\Leftrightarrow$ No form of degree $\leq n$ vanishes on $Chow(n, 0, \mathbb{P}^r)$ ($\Leftrightarrow$ The degree $n$ piece of $HDSym^{r+1}_n$ is generated by the fundamental homogeneous invariants $\hat{e}_\alpha$.)
($\Leftrightarrow$ $Chow(n, 0, \mathbb{P}^r)$ has no equation of degree $\leq n$)
Foulkes–Howe conjecture

M. Brion (1997): FH conjecture (ii) is true for $d >> n$ (with explicit lower bound depending on $n$ and $r$)

E.B. : FH conjectures (i) and (ii) are true for $n = 3$.

**Foulkes–Howe conjecture**

M. Brion (1997): FH conjecture (ii) is true for \( d >> n \) (with explicit lower bound depending on \( n \) and \( r \))

E.B.: FH conjectures (i) and (ii) are true for \( n = 3 \).

E.B. (2002), J. Jacob (2004): FH conjecture (i) is true for \( n = 4 \).

Müller+Neunhöffer (2005): FH conjectures (i) and (ii) are false for \( n = 5 \).
Foulkes–Howe conjecture

M. Brion (1997): FH conjecture (ii) is true for $d \gg n$ (with explicit lower bound depending on $n$ and $r$)

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Anyway . . . when is $\pi_d^*$ injective? surjective?
Showing that FH (i) holds for fixed \( n \): Toy example \( n = 2 \)

To show: that the fundamental homogeneous invariants generate the degree \( n \) piece of \( HDSym_n^{r+1}(\mathbb{C}) \).

Ex: \( n = 2 \), \[ A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \]

The monomial functions = orbit sums of monomials \( \Sigma(a_1^{\alpha_1}a_2^{\alpha_2}b_1^{\beta_1}b_2^{\beta_2}) \) (under row permutations of the matrix) are a linear basis for \( DSym \).

The decomposition

\[ \Sigma(a_1^2b_1b_2) = e_{11}e_{20} \]

is obtained by applying the \textit{polarization operator}

\[ \frac{1}{2}(a_1 \frac{\partial}{\partial a_2} + b_1 \frac{\partial}{\partial b_2}) \]

to the \textit{key identity}:
\[ \Sigma(a_1^2b_2^2) = a_1^2b_2^2 + a_2^2b_1^2 = e_{11}^2 - 2e_{02}e_{02} \]

The key identity is also invariant under Column permutations!
Doubly symmetric functions

Checking FH (i) for fixed $n$ can be reduced to computations in the subspace of $\mathbb{C}[A]S_n \times S_n$: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ of the elements homogeneous of degree $n$ w.r.t. the variables of each row and homogeneous of degree $n$ w.r.t. the variables of each column.

Ex: $n = 3$, linear bases are indexed by:

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}
$$

Even the enumeration of these objects is difficult!
Conference: Diagonally symmetric polynomials and applications.

October 15-19, 2007

Castro-Urdiales, Spain.

http://www.congreso.us.es/dsym/

Invited speakers:
François Bergeron,
Riccardo Biagioli,
Ira Gessel,
Francesco Vaccarino.
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