THE SPACE OF ORDERED POINTS ON THE PROJECTIVE LINE IS ALMOST ALWAYS CUT OUT BY QUADRICS

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ABSTRACT. A central question in geometric invariant theory is that of determining the relations among invariants. Geometric invariant theory quotients come with a natural ample line bundle, and hence often a natural projective embedding. This question translates to determining the equations of the moduli space in this embedding. This note deals with one of the most classical quotients, the space of n ordered points on the projective line. We show that under any linearization, this quotient is cut out set-theoretically by a particularly simple set of quadric relations, with the single exception of the Segre cubic threefold (the space of six points with equal linearization). The stable locus is cut out scheme-theoretically by these quadrics. Unlike many facts in geometric invariant theory, these results are field-independent, and indeed work over the integers. Even in the exceptional case of six points, we show how this $\mathfrak{S}_n$-equivariant perspective clarifies the moduli maps, and the connections to other spaces, such as the space of six points in $\mathbb{P}^3$ (the Igusa quartic threefold) and $\mathbb{P}^2$. The key tool is a possibly new description of the outer automorphism of $\mathfrak{S}_6$. *(Caution: the material in this paper and* [HMSV] *will be re-organized, to produce two different papers.)*

CONTENTS

1. INTRODUCTION

In this note, we study the space of n points on the projective line, up to automorphisms of the line. In characteristic 0, the best descriptions of this are as Geometric Invariant Theory quotients $(\mathbb{P}^1)^n//SL(2)$. Indeed, this is one of the most classical examples of a GIT quotient, and is one of the first examples given in any course (see [MS, §2], [MFK, §3], [N, §4.5], [D, Ch. 11], [DO, Ch. I], ...). The construction of the quotient depends on a linearization, in the form of a choice of weights of the points $\mathbf{w} = (w_1, \dots, w_n)$ (the *weight*

Date: Sunday, November 6, 2005.
1991 *Mathematics Subject Classification.* Primary 14L24, Secondary 14D22, 14H10.

vector). We will denote the resulting quotient $M_{\mathbf{w}}$. A particularly interesting case is of course when all points are treated equally, when $\mathbf{w} = (1, \dots, 1) = 1^n$. (Dolgachev calls this the *democratic* case.) At risk of confusion, we denote this important case M_{1^n} by M_n for convenience. n (ordered) points of $\mathbb{P}^1$ are $\mathbf{w}$-*stable* (resp. $\mathbf{w}$-*semistable*) if the sum of the weights of points that coincide is less than (resp. no more than) half the total weight. The dependence on $\mathbf{w}$ will be clear from the context, so the prefix $\mathbf{w}$- will be omitted. The n points are *strictly semistable* if they are semistable but not stable. Then $M_{\mathbf{w}}$ is a projective variety, and GIT gives a natural projective embedding. The stable locus of $M_{\mathbf{w}}$ is a fine moduli space for the stable points of $(\mathbb{P}^1)^n$. The strictly semistable locus of $M_{\mathbf{w}}$ is a finite set of points, which are the only singular points of $M_{\mathbf{w}}$. The question we wish to address is: what are the equations of $M_{\mathbf{w}}$?

We prefer to work over the integers, so we now define the moduli problem of stable n-tuples of points in $\mathbb{P}^1$. For any scheme B, *families of stable n-tuples of points in* $\mathbb{P}^1$ over B are defined to be n morphisms to $\mathbb{P}^1$ such that no more than half the weight is concentrated at a single point of $\mathbb{P}^1$. More precisely a family is a morphism $(\phi_1, \dots, \phi_n) : B \times \{1, \dots, n\} \to \mathbb{P}^1$ such that for any $I \subset \{1, \dots, n\}$ such that $\sum_{i \in I} w_i \geq \sum_{i=1}^n w_i/2$, we have $\cap_{i \in I} \phi^{-1}(p) = \emptyset$ for all p. Then there is a fine moduli space for this moduli problem, quasiprojective over $\mathbb{Z}$, which indeed has a natural ample line bundle, the one suggested by GIT. (This is well-known, but in any case will fall out of our analysis.) In parallel with GIT, we define stable and semistable points. Hence this space has extrinsic projective geometry. The question we will address in this context is: what are its equations?

The main moral of this note is taken from Chevalley's construction of Chevalley groups: when understanding a vector space defined geometrically, choosing a basis may obscure its structure. Instead, it is better to work with an equivariant generating set, and equivariant linear relations. A prototypical example is the standard representation of $\mathfrak{S}_n$, which is best understood as the permutation representation on the vector space generated by e_1, ..., e_n subject to the relation $e_1 + \cdots + e_n = 0$. As an example, we give yet another short proof of Kempe's theorem (Theorem 3.3), which has been called the "deepest result" of classical invariant theory [HMSV, p. 1]. Another application is the computation in §3.14 of the degree of all $M_{\mathbf{w}}$.

Our question is related to another central problem in invariant theory: the invariants of binary forms, or equivalently n unordered points on $\mathbb{P}^1$, or equivalently equations for $M_n/\mathfrak{S}_n$. These generators and relations are more difficult, and the relations are certainly not just quadratic. Mumford describes Shioda's solution for $n = 8$ [Sh] as "an extraordinary tour de force" [MFK, p. 77]. One might dream that the case $n = 10$ might be tractable by computer, given the relations for M_{10} described here.

This paper is a companion to [HMSV]. Both approaches seek an analogue of the Second Fundamental Theorem of Invariant Theory (for the Grassmannian) in this situation [D, p. 24], to describe explicitly the ideal of relations. We wish to emphasize the differences. Their main result is very strong: the ring of invariants is always cut out by relations of degree at most 4. (This shows that breaking symmetry can also be very advantageous!) We are able to reduce the degree of the relations, and to use relations which are particularly natural and $\mathfrak{S}_n$-equivariant, at the cost of not knowing whether they generate the ideal of relations.

We find it remarkable that we still don't understand this ideal of relations, given the centrality of this moduli space, and the fact that it may well be cut out by simple quadrics. The combination of the two results raises some natural and important questions.

Question. Do the simple binomial quadratic relations cut out the moduli space scheme-theoretically, even at the strictly semi-stable points (ignoring the exceptional case of the Segre cubic threefold)? In the democratic case, this is true when the number of points is at most 8 (§3.8).

Question. Might it even be true that these quadrics generate the ideal of relations among the invariants?

[HMSV] suggests one approach: one could hope to show that the explicit generators given there lie in the ideal given by these quadrics.

Even special cases are striking, and are simple to state but computationally too complex to verify even by computer. For example, we will describe particularly attractive relations for M_{10} of degree degrees 3, 5, 7 (§3.11); do these lie in the ideal of our simple quadrics? We will describe a quadric relation for M_{12} (§3.7). Is this a linear combination of our simple binomial quadrics?

These questions lead to more speculation. The shape of the proof of the main theorem, Theorem 3.13, suggests an explanation for the existence of an exception: when the number of points gets "large enough", the relations are all inherited from "smaller" moduli spaces, so at some point (in our case, when $n = 8$) the relations "stabilize".

Speculative question. We know that M_n satisfies Green's property N_0 (projectively normality — Kempe's Theorem 3.3), and the questions above suggest that M_n might satisfy property N_1 (projectively normal and cut out by quadrics) for $n > 6$. Might it be true that each p, M_n satisfies N_p for $p \gg 0$? (Property N_2 means that the scheme satisfies N_1, and the syzygies among the quadrics are linear. These properties measure the "niceness" of the ideal of relations.)

Speculative question. Might similar results hold true for other moduli spaces of a similar flavor, e.g. $\overline{\mathcal{M}}_{0,n}$ or even $\overline{\mathcal{M}}_{g,n}$? Are equations for moduli spaces inherited from equations of smaller moduli spaces for $n \gg 0$? For further motivation, see the striking work of [KT] on equations cutting out $\overline{\mathcal{M}}_{0,n}$. Also, an earlier example of quadratic equations inherited from smaller moduli spaces appeared in [BP], in the case of Cox rings of del Pezzo surfaces. I thank B. Hassett for pointing this out.

1.1. Outline of paper. In §2, we give some gentle geometric motivation for the questions we wish to address. In §3, we describe the variables, relations, and main results. Our natural invariants or variables are described in terms of directed graphs, rather than the usual tableaux. We describe some straightforward (and mostly well-known) relations among these invariants: sign relations, Plücker relations, binomial relations, and others. We show that this perspective makes Kempe's Theorem 3.3 straightforward. We state our main theorem, Theorem 3.13, and briefly discuss the examples of 5 and 8 points (the smallest "typical" odd and even cases). We then use these invariants to compute the degrees of all such quotients. In §4, we discuss the exceptional (and exceptionally classical) case of 6 points. We use explicit invariant theory and a new description of the outer automorphism to describe simply all related maps between related moduli spaces. Finally, we prove the main theorem in §5. The main part of the proof deals with the democratic case. The general case is a straightforward extension.

Many of these results are undoubtedly known to experts, but we were unable to find them in the literature.

Acknowledgments. The fourth author thanks Lucia Caporaso and Igor Dolgachev for introducing him to the subject; and Diane Maclagan, Brendan Hassett, and Harm Derksen for useful suggestions.

2. Motivation: Understanding cross-ratio properly, and generalizing it appropriately

In some sense, our problem is that of generalizing the cross-ratio map from 4 points to an arbitrary number of points. The classical *cross-ratio* map may be interpreted as a rational map $(\mathbb{P}^1)^4 \dashrightarrow \mathbb{P}^1$, defined on the locus where no three of the four points coincide. This is the GIT quotient $M_4 = (\mathbb{P}^1)^4//SL_2$. The classical definition of the map is

$$(p_1, p_2, p_3, p_4) \mapsto \frac{(p_3 - p_2)(p_1 - p_4)}{(p_1 - p_2)(p_3 - p_4)},$$

or more precisely,

$$([u_1; v_1], [u_2; v_2], [u_3; v_3], [u_4; v_4]) \mapsto \frac{(u_3v_2 - u_2v_3)(u_1v_4 - u_4v_1)}{(u_1v_2 - u_2v_1)(u_3v_4 - u_4v_3)}.$$

We will use the first notation for simplicity. With this definition, the action of $\mathfrak{S}_4$ (induced by permutations of the 4 points) is not clear. For example, it is not immediate that the Klein 4-group acts trivially.

We believe a better description of the cross-ratio is as a rational map $(\mathbb{P}^1)^4 \dashrightarrow \mathbb{P}^2$ given by

$$(p_1, p_2, p_3, p_4) \mapsto [(p_2 - p_3)(p_1 - p_4); (p_1 - p_2)(p_3 - p_4); (p_1 - p_3)(p_4 - p_2)]. \tag{1}$$

The reader may immediately verify that the sum of the variables on the right is 0, so the image is still isomorphic to $\mathbb{P}^1$, but now the action of $\mathfrak{S}_4$ is quite visible. (By taking more variables that are the negative of these, it can be made more visible still.) The Klein 4-group clearly acts trivially on this morphism. This is a first clue that keeping "extra" variables with linear relations can make the moduli morphism more transparent.

The GIT quotient of five points on the projective line has a similarly beautiful interpretation: it is the degree 5 del Pezzo surface in $\mathbb{P}^5$, cut out by 5 quadrics. Ideally the quadric equations giving the rational map $(\mathbb{P}^1)^5 \dashrightarrow \mathbb{P}^5$, as well as the $\mathfrak{S}_5$ action on both $\mathbb{P}^5$ and the quadrics, should be clear.

The GIT quotient of six points was also classically known to have a very beautiful structure. It may be interpreted as a threefold in $\mathbb{P}^5$ cut out by the equations [DO, p. 17]

$$z_1 + \cdots + z_6 = z_1^3 + \cdots + z_6^3 = 0. \tag{2}$$

This is the *Segre cubic relation* [D, Example 11.6], and this quotient is known as the *Segre cubic threefold*, which we denote $\mathcal{S}_3$. Once again, having more variables, with linear relations, makes the algebraic structure clearer.

There is an obvious $\mathfrak{S}_6$-action on both $(\mathbb{P}^1)^6$ and the variables $z_1, \ldots, z_6$. One might hope that these actions are conjugate, which would imply some bijection between the six points and the six variables, and indeed this would be the case if 6 were replaced by any other number. However, remarkably, they are related by the outer automorphism of $\mathfrak{S}_6$.

The cross-ratio of a certain four of the six points is given by $[z_1, \ldots, z_6] \mapsto -(z_1+z_2)/(z_3+z_4)$. This is a hint that the "correct" description (in terms of the cross-ratio description of (1)) is $[z_1; \ldots; z_6] \mapsto [z_1+z_2; z_3+z_4; z_5+z_6]$. This is also a hint that the outer automorphism is relevant.

Ideally the moduli map $(\mathbb{P}^1)^6 \dashrightarrow \mathbb{P}^5$, the Segre cubic relation, the projections to M_4, and the presence of the outer automorphism of $\mathfrak{S}_6$ should be transparent.

Finally, this picture should extend to a larger number of points.

These (and other) questions will be addressed in the remainder of this note.

3. THE ALGEBRA BEHIND n POINTS ON THE LINE, GRAPHICALLY

See [HMSV] for background and notational conventions. However, we hope that this note is self-contained for those familiar with Geometric Invariant theory.

We will use the following convenient alternate description of the generators (as a group) of the ring of invariants. By *graph* we will mean a *directed* graph on n vertices labeled 1 through n. Graphs may have multiple edges, but may have no loops. The *multidegree* of a graph Γ is the n-tuple of valences of the graph, denoted $\mathbf{deg}\,\Gamma$. The bold font is a reminder that this is a vector. We consider each graph as a set of edges. For each edge e of Γ, let $h(e)$ be the head vertex of e and $t(e)$ be the tail. We use multiplicative notation for the "union" of two graphs: if Γ and Δ are two graphs on the same set of vertices, the union is denoted by $\Gamma \cdot \Delta$ (so for example $\mathbf{deg}\,\Gamma + \mathbf{deg}\,\Delta = \mathbf{deg}\,\Gamma \cdot \Delta$), see Figure 1. (We will occasionally use additive and subtractive notation when we wish to "subtract" graphs. We apologize for this awkwardness.)

For each graph Γ, define $X_\Gamma \in H^0((\mathbb{P}^1)^n, \mathcal{O}_{(\mathbb{P}^1)^n}(\mathbf{deg}\,\Gamma))$ by

$$X_\Gamma = \prod_{\text{edge } e \text{ of } \Gamma} (p_{h(e)} - p_{t(e)}) = \prod_{\text{edge } e \text{ of } \Gamma} (u_{h(e)}v_{t(e)} - u_{t(e)}v_{h(e)}) \tag{3}$$

(cf. §2). If S is a non-empty set of graphs of the same degree, $[X_\Gamma]_{\Gamma \in S}$ denotes a point in projective space $\mathbb{P}^{|S|-1}$ (assuming some such X_Γ is nonzero of course). For any such S, the map $(\mathbb{P}^1)^n \dashrightarrow [X_\Gamma]_{\Gamma \in S}$ is easily seen to be invariant under $SL(2)$: replacing p_i by $p_i + a$

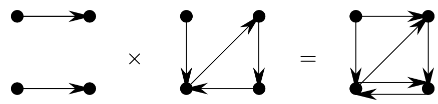

FIGURE 1. Multiplying (directed) graphs

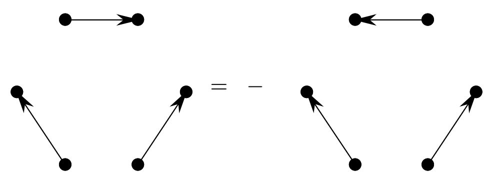

FIGURE 2. An example of the sign relation

preserves X_Γ; replacing p_i by ap_i changes each X_Γ by the same factor; and replacing p_i by $1/p_i$ also changes each X_Γ by the same factor.

The First Fundamental Theorem of Invariant Theory [D, Thm. 2.1] states that, given a weight $\mathbf{w}$, the ring of invariants of $(\mathbb{P}^1)^n//SL_2$ is generated (as a group) by the X_Γ where $\mathbf{deg}\,\Gamma$ is a multiple of $\mathbf{w}$. The translation to the tableaux of [HMSV] and [D] is as follows. Choose any ordering of the edges $e_1, \dots, e_{|\Gamma|}$ of Γ. Then X_Γ corresponds to any $2 \times |\Gamma|$ tableau where the top row of the ith column is $h(e_i)$ and the bottom row is $t(e_i)$. We will soon see many advantages of this graphical description.

We now describe three types of relations among the X_Γ, which will all be straightforward: the sign relations, the Plücker (or straightening) relations, the simple binomial relations, and the Segre cubic relation.

3.1. *The sign (linear) relations.* The *sign relation* $X_{\Gamma \cdot \vec{xy}} = -X_{\Gamma \cdot \vec{yx}}$ (Figure 2) is immediate, given the definition (3). This relation appears in [HMSV, §4.4]. Because of the sign relation, we may omit arrowheads in identities where it is clear how to consistently add them (see for example Figures 9 and 11, where even the vertices are implicit).

3.2. *The Plücker (linear) relations.* The identity of Figure 3 may be verified by direct calculation. (Indeed, this is the same cross-ratio calculation described after (1).) Then clearly if Γ is any graph on n vertices, and Δ_1, Δ_2, Δ_3 are three graphs on the same vertices given by identifying the four vertices of Figure 3 with the some four of the n vertices of Γ, then

$$X_{\Gamma\cdot\Delta_1} + X_{\Gamma\cdot\Delta_2} + X_{\Gamma\cdot\Delta_3} = 0. \tag{4}$$

These relations are called *Plücker relations* (or *straightening rules*). See Figure 4 for an example. We will sometimes refer to this relation as the Plücker relation for $\Gamma \cdot \Delta_1$ with respect to the vertices of Δ_1. This relation appears in [HMSV, §4.4].

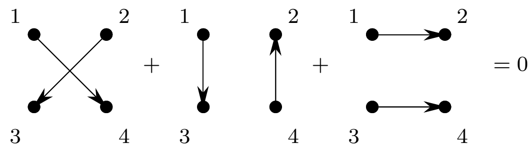

FIGURE 3. The Plücker relation for $n = 4$ (and $\mathbf{w} = (1, 1, 1, 1)$)

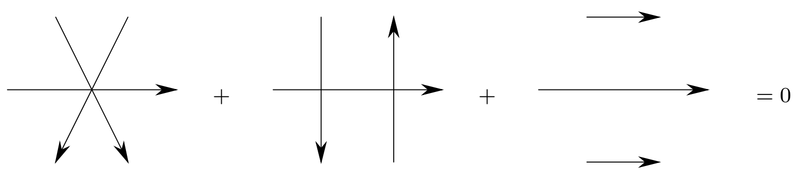

FIGURE 4. An example of a Plücker relation

Using the Plücker relations, one can reduce the number of generators to a smaller set, which we will do shortly (Proposition 3.4). However, a central thesis of this note is that this is the wrong thing to do too soon; not only does it obscure the $\mathfrak{S}_n$ symmetry of this generating set, it also makes certain facts opaque. As an example, we give yet another proof of Kempe's theorem. The proof will also serve as preparation for the proof of the main theorem, Theorem 3.13.

3.3. *Kempe's Theorem* [HMSV, Thm. 4.6]. — *The lowest degree invariants generate the ring of invariants.*

Note that the lowest-degree invariants are of weight $\epsilon_{\mathbf{w}}\mathbf{w}$, where $\epsilon_{\mathbf{w}} = 1$ if $|\mathbf{w}|$ is even, and 2 if $|\mathbf{w}|$ is odd.

Proof. We begin in the case when $\mathbf{w} = (1, \dots, 1)$ where n is even. Recall Hall's Marriage Theorem: given a finite set of men M and women W, and some men and women are compatible (a subset of $M \times W$), and it is desired to pair the women and men compatibly, then it is necessary and sufficient that for each subset S of women, the number of men compatible with at least one of them is at least $|S|$.

Given a graph Γ of multidegree $(d, \dots, d)$, we show that we can find an expression $\Gamma = \sum \pm \Delta_i \cdot \Xi_i$ where $\mathbf{deg}\, \Delta_i = (1, \dots, 1)$. Construct an $n + n$ bipartite graph as follows. The vertices of each half of the graph are numbered 1 through n. For each edge ij in Γ, we construct an edge connecting woman i to man j, and another from man i to woman j. Then each vertex of the bipartite graph has the same valence d, so any set of w women must connect to at least w men. By Hall's Marriage Theorem, we can find a matching. These n edges determine n edges in our original graph, such that each vertex has valence precisely 2 (considering only edges of this subgraph). (It is possible that an edge is counted twice; this won't matter to the subsequent argument.) We have an even number

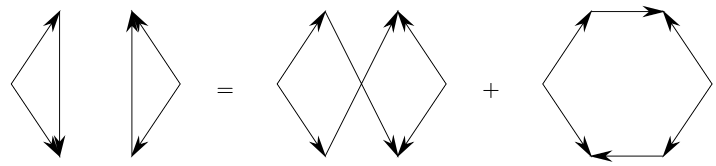

FIGURE 5. Turning two odd cycles into an even cycle using the Plücker relation (4)

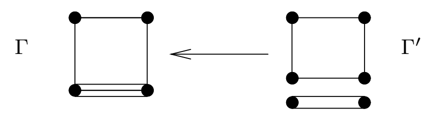

FIGURE 6. Constructing Γ' from Γ (example with $\mathbf{w} = (1, 1, 2, 2)$, $d = 2$)

of odd cycles. Pair up the odd cycles, and use the Plücker relations (4) to express our 2-regular graph in terms of other graphs, each of which has n marked edges all forming even cycles (as in Figure 5). By taking every other edge in each even cycle, we obtain a matching.

We next treat the general case. If $|\mathbf{w}|$ is odd, it suffices to consider the case $2\mathbf{w}$, so by replacing $\mathbf{w}$ by $2\mathbf{w}$ if necessary, we may assume $\epsilon_{\mathbf{w}} = 1$. The key idea is that $M_{\mathbf{w}}$ is a linear section of $M_{|\mathbf{w}|}$. Suppose $\deg \Gamma = d\mathbf{w}$. Construct an auxiliary graph Γ' on $|\mathbf{w}|$ vertices, and a map of graphs $\pi : \Gamma' \to \Gamma$ such that (i) the preimage of vertex i of Γ consists of w_i vertices of Γ', (ii) π gives a bijection of edges, and (iii) each vertex of Γ' has valence d, i.e. Γ' is d-regular. (See Figure 6 for an illustrative example. There may be choice in defining Γ'). Then apply the algorithm of the previous paragraph to Γ'. By taking the image under π, we have our desired result for Γ. □

Choosing a planar representation makes termination of certain algorithms straightforward as well, as illustrated by the following argument. Consider the vertices of the graph to be the vertices of a regular n-gon, numbered (clockwise) 1 through n. A graph is said to be *non-crossing* if no two edges cross. Two edges sharing one or two vertices are considered not to cross. A variable X_Γ is said to be non-crossing if Γ is.

3.4. *Proposition. — For each* $\mathbf{w}$*, the non-crossing variables of degree* $\mathbf{w}$ *generate* $\langle X_\Gamma \rangle_{\deg \Gamma = \mathbf{w}}$ *(as an abelian group).*

This is essentially the straightening algorithm (e.g. [D, §2.4]) in this situation.

Proof. We explain how to express X_Γ in terms of non-crossing variables. If Γ has a crossing, choose one crossing $wx \cdot yz$ (say $\Gamma = wx \cdot yz \cdot \Gamma'$), and use the Plücker relation (4) involving $wxyz$ to express Γ in terms of two other graphs $wy \cdot xz \cdot \Gamma'$ and $wz \cdot xy \cdot \Gamma'$. Repeat this if

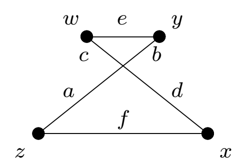

FIGURE 7. The triangle inequality implies termination of straightening: $b + c > e$, $a + d > f$

possible. We now show that this process terminates, i.e. that this algorithm will express X_Γ in terms of non-crossing variables. Both of these graphs have lower sum of edge-lengths than Γ (see Figure 7, using the triangle inequality on the two triangles with side lengths a, d, f and b, c, e). As there are finite number of graphs of weight $\mathbf{w}$, and hence a finite number of possible sums of edge-lengths, the process must terminate. □

A straightforward Gröbner argument using weights given by the planar embedding of the n points shows that the non-crossing variables in fact form a basis for $\langle X_\Gamma \rangle_{\deg \Gamma = \mathbf{w}}$. I thank Diane Maclagan for pointing this out.

3.5. *Aside: ambient space for n even.* In the case where n is even and $\mathbf{w} = (1, \ldots, 1) = 1^n$, it is well-known that the number of matchings (1-regular graphs) on n vertices is the Catalan number $C_n = \binom{2n}{n}/(n+1)$ [St, p. 223, Exercise 19(n)]. This gives a simple explicit description of the irreducible representation of $\mathfrak{S}_n$ on the underlying C_n-dimensional vector space, which is the irreducible representation corresponding to the partition $2 + \cdots + 2$. Again, we emphasize that this representation is most symmetrically understood as the quotient of a $(2n-1)!!$-dimensional vector space by the Plücker relations (4).

3.6. *Aside: ambient space for n odd.* The ambient space of M_n for n odd can be similarly understood. For example, the list of non-crossing 2-regular graphs on 5 vertices in Figure 8 shows that M_5 is non-degenerately contained in $\mathbb{P}^5$. We now describe the analogue of the Catalan numbers for 2-regular graphs. We will not need these results later.

We have that $h^0(M_n, \mathcal{O}(2))$ is the number of non-crossing 2-regular graphs on n vertices. The generating function $T(x)$ for this number is

$$T(x) = \frac{1}{2x}\left(1 - \sqrt{\frac{1-3x}{1+x}}\right) = 1 + x^2 + x^3 + 3x^4 + 5x^5 + 15x^6 + \cdots . \tag{5}$$

We sketch an argument. First show inductively that for $n > 0$

$$T_n = \sum_{i+j+k=n-2} T_i (T_j + T_{j-1}) T_k$$

where i, j, k are non-negative, and $T_0 = 1$. (This may be done as follows. Divide the set of non-crossing 2-regular graphs into cases. Consider the case where the edges from vertex n go to vertices I and J, where there are i vertices between n and I cyclically, k vertices

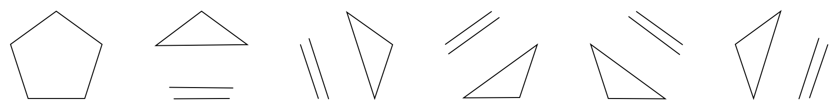

FIGURE 8. The six non-crossing 2-regular graphs on 5 vertices show that M_5 is non-degenerately contained in $\mathbb{P}^5$

between J and n, and $j-1$ vertices between I and J. Then there are T_i ways of choosing a 2-regular graph on the vertices between n and I, and T_j ways for those between n and J. If IJ is an edge, there are T_{j-1} ways of choosing a 2-regular graph on the vertices between I and J. If IJ is not an edge, then by conflating I and J, and considering the graph involving the vertices between I and J inclusive, there are T_j ways of choosing a 2-regular graph in this case.) After some manipulation, this translates to the relation of generating functions

$$(x(1+x)T^2 - (1+x)T + 1)(xT+1) = 0.$$

As $xT + 1 \neq 0$, $T(x)$ must satisfy the quadratic, which yields (5).

3.7. *Binomial (quadratic) relations.* We next describe some obvious binomial relations. If $\mathbf{deg}\,\Gamma_1 = \mathbf{deg}\,\Gamma_2$ and $\mathbf{deg}\,\Delta_1 = \mathbf{deg}\,\Delta_2$, then clearly $X_{\Gamma_1\cdot\Delta_1}X_{\Gamma_2\cdot\Delta_2} = X_{\Gamma_1\cdot\Delta_2}X_{\Gamma_2\cdot\Delta_1}$. We call these the *binomial relations*. A special case are the *simple binomial relations* when $\mathbf{deg}\,\Delta_i = (1,1,1,1,0,\dots,0) = 1^4 0^{n-4}$, or some permutation thereof. Examples are shown in Figures 9 and 11.

In the even democratic case, the smallest binomial relations that are not simple binomial relations appear for $n = 12$, and

$$\mathbf{deg}\,\Gamma_i = \mathbf{deg}\,\Delta_i = (1,1,1,1,1,1,0,0,0,0,0,0).$$

In the introduction, we asked if these quadratics are linear combinations of the simple binomial relations.

By the Plücker relations, the binomial relations are generated by those where the Γ_i and Δ_i are non-crossing, and similarly for the simple binomial relations. (This restriction can be useful to reduce the number of equations, but as always, symmetry-breaking obscures other algebraic structures.) Note that even though we may restrict to the case where Γ_i and Δ_j are non-crossing, we may not restrict to the case where $\Gamma_i \cdot \Delta_j$ are non-crossing, as the following examples with $n = 5$ and $n = 8$ show.

As an example, consider the case $n = 5$ (with the smallest democratic linearization $(2,2,2,2,2)$). One of the simple binomial relations is shown in Figure 9. The building blocks Γ_i and Δ_j are shown Figure 10. In fact, these quadric relations cut out M_5 in $\mathbb{P}^5$, as can be checked directly, or as follows from Theorem 3.13. The $\mathfrak{S}_5$-representation on the quadrics is visible.

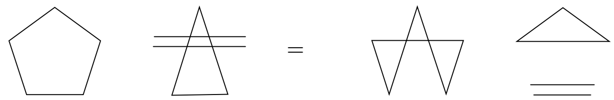

FIGURE 9. A simple binomial relation for $n = 5$

FIGURE 10. The building blocks of Figure 9

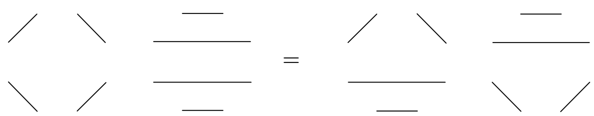

FIGURE 11. One of the simple binomial relations for $n = 8$ points

3.8. As a second example, consider $n = 8$ (and the democratic linearization $(1, \ldots, 1)$). Because there are $\binom{8}{4}/2 = 35$ ways of partitioning the 8 vertices into two subsets of size 4, and each such partition gives one simple binomial relation (where the Γ_i and Δ_j are non-crossing, see comments two paragraphs previous), we have 35 quadric relations on M_8, shown in Figure 11.

The space of quadric relations forms an irreducible 14-dimensional $\mathfrak{S}_8$-representation, which we show by representation theory. If V is the vector space of quadratic relations, we have the exact sequence

$$0 \to V \to \operatorname{Sym}^2 H^0(M_8, \mathcal{O}(1)) \to H^0(M_8, \mathcal{O}(2)) \to 0$$

of representations. By counting non-crossing graphs (or invoking §3.6), we can calculate $h^0(M_8, \mathcal{O}(2)) = 91$, and we have already calculated $h^0(M_8, \mathcal{O}(1)) = C_4 = 14$, from which $\dim V = 14$. As the representation $H^0(M_8, \mathcal{O}(1))$ is identified, we can calculate the representation $\operatorname{Sym}^2 H^0(M_8, \mathcal{O}(1))$, and observe that the only 14-dimensional subrepresentations it contains are irreducible. (Simpler still is to compute the character of the 196-dimensional representation $H^0(M_8, \mathcal{O}(1))^{\otimes 2}$, and decompose it into irreducible representations, using Maple for example, finding that it decomposes into representations of dimension $1+14+14+20+35+56+56$; $\operatorname{Sym}^2 H^0(M_8, \mathcal{O}(1))$ is of course a subrepresentation of this.)

As our quadric relations are nontrivial, and form an $\mathfrak{S}_8$-representation, we have given generators of the quadric relations. Necessarily they span the same vector space of the

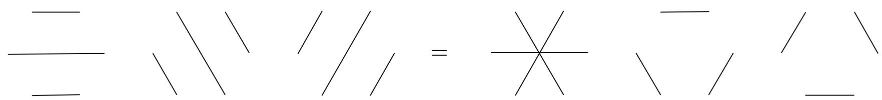

FIGURE 12. The Segre cubic relation (graphical version)

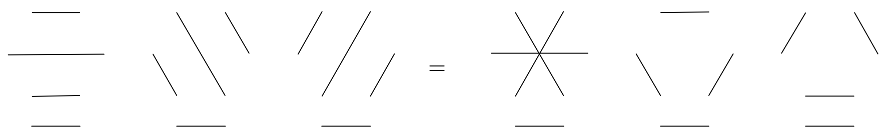

FIGURE 13. The Segre cubic relation for $n = 8$

14 relations given in [HMSV, §9.4.4]. Our relations have the advantage that the $\mathfrak{S}_8$-action is clear, but the major disadvantage that it is not a priori clear that the vector space they span has dimension 14. We suspect there is an $\mathfrak{S}_8$-equivariant description of the linear relations among the generators, but we have been unable to find one.

3.9. *The Segre cubic relation.* Other relations are also clear from this graphical perspective. For example, Figure 12 shows an obvious relation for M_6. As this is a nontrivial cubic relation (this can be verified by writing it in terms of a non-crossing basis), it must be the Segre cubic relation. Interestingly, although the relation is not $\mathfrak{S}_6$-invariant, it becomes so modulo the Plücker relations (4). Note that there are no (nontrivial) binomial relations for M_6 (which is cut out by this cubic), so the Segre relation cannot be in the ideal generated by the binomial relations.

3.10. *Remark: Segre cubic relations for $n \geq 8$.* There are analogous cubic relations for $n \geq 8$, by simply adding other vertices. The $n = 8$ case is given in Figure 13. For $n \geq 8$, these Segre cubic relations lie in the ideal generated by the simple binomial relations. We will use this in the proof of Theorem 3.13. This follows from the case $n = 8$, which can be verified in a couple of ways. [HMSV] shows that the ideal cutting out M_8 is generated by the fourteen quadrics of [HMSV, §9.4.4], which by §3.8 is the ideal generated by the simple binomial relations, and the cubic lies in this ideal. One can also show that the Segre relation lay in the ideal generated by the fourteen quadrics explicitly.

3.11. *Other relations.* There are other relations, that we will not discuss further. For example, consider the democratic case for n even. Then $\mathfrak{S}_n$ acts on the set of graphs. Choose any graph Γ. Then

$$\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) X^i_{\sigma(\Gamma)} = 0$$

is a relation for i odd and $1 < i < n-1$. Reason: substituting for X's in terms of p's (or more correctly the u's and v's) using (3) to obtain an expression E, and observing that $\mathfrak{S}_n$

acts oddly on E, we see that we must obtain a multiple of the Vandermonde, which has degree $(n-1,\dots,n-1) > \deg E$. Hence $E = 0$. It is not clear that this is a nontrivial relation, but it appears to be so in small cases. In particular, the case $n = 6$, $i = 3$ is the Segre cubic relation. In the introduction, we asked if these relations for $n = 10$ lie in the ideal generated by the simple binomial quadric relations.

3.12. Statement of main result. We are now ready to state the main result of this note. Recall that for any $\mathbf{w}$, we know that the ring of invariants is generated by X_Γ where $\mathbf{deg}\,\Gamma = \epsilon_{\mathbf{w}}\mathbf{w}$, and $\epsilon_{\mathbf{w}} = 1$ or 2 (recall the parity of $\epsilon_{\mathbf{w}}$ is opposite that of $|\mathbf{w}|$).

3.13. *Main Theorem. — Suppose* $\mathbf{w} \neq (1,1,1,1,1,1)$.

(a) *Suppose we are working over a field of characteristic* 0. *Then the GIT quotient* $(\mathbb{P}^1)^n//SL_2$ *(with its natural projective embedding) is cut out set-theoretically by the Plücker and simple binomial relations. The stable locus is cut out scheme-theoretically by these relations.*

(b) *Suppose we are working over* $\mathbb{Z}$. *The moduli space of stable* n*-tuples of points on* $\mathbb{P}^1$ *(defined in the introduction) is quasiprojective over* $\mathbb{Z}$. *Under its natural embedding, its closure is cut out by the Plücker and simple binomial relations.*

The proof is given in Section 5. As preliminary evidence, the case $n = 5$ may be checked by hand. The exceptional case $\mathbf{w} = (1,1,1,1,1,1)$ is the Segre cubic threefold, as described in §2.

3.14. Degree of the GIT quotient. As an application of these coordinates, we compute the degree of all $M_{\mathbf{w}}$. For example, we will use this to verify that the degree is 1 when $|\mathbf{w}| = 6$ and $\mathbf{w} \neq (1,\dots,1)$, although this can also be done directly.

We would like to intersect the moduli space $M_{\mathbf{w}}$ with $n-3$ coordinate hyperplanes $X_\Gamma = 0$ and count the number of points, but these hyperplanes will essentially never intersect properly. Instead, we note that each hyperplane $X_\Gamma = 0$ is reducible, and consists of a finite number of components of the form $M_{\mathbf{w}'}$ where the number of points $\#\mathbf{w}'$ is $n-1$. We can compute the multiplicity with which each of these components appears. The algorithm is then complete, given the base case $n = 4$. Here, more precisely, is the algorithm.

(a) (trivial case) If $n = 3$, the moduli space is a point, so the degree is 1.

(b) (base case) If $\mathbf{w} = (d,d,d,d)$, then $\deg M_{\mathbf{w}} = d$, as the moduli space is isomorphic to $\mathbb{P}^1$, embedded by the d-uple Veronese. (This may be seen by direct calculation, or by noting that a basepointfree subset of those variables of degree (d,d,d,d) are "dth powers" of variables of degree $(1,1,1,1)$.)

(c) If $n > 4$ and $\mathbf{w}$ satisfies $w_j + w_k \leq \sum w_i/2$ for all j, k, we choose any Γ of weight $\mathbf{w}$. We can understand the components of $X_\Gamma = 0$ by considering the morphism $\pi : (\mathbb{P}^1)^n - U_{\mathbf{w}} \to [X_\Gamma]_{\mathbf{deg}\,\Gamma = \mathbf{w}}$, where $U_{\mathbf{w}}$ is the unstable locus. Directly from the formula for X_Γ, we see that for each pair of vertices j, k with an edge joining them, such that $w_j + w_k < \sum w_i/2$, there is a component that can be interpreted as $M_{\mathbf{w}'}$, where $\mathbf{w}'$ is the same as $\mathbf{w}$ except that w_j and w_k are removed, and $w_j + w_k$ is added (call this w_0 for convenience). We interpret this as removing vertices j and k, and replacing them with vertex 0. This component corresponds to the divisor

$$(u_j v_k - u_k v_j)^{m_{jk}} = 0 \tag{6}$$

on the source of π, where m_{jk} is the number of edges joining vertices j and k. If Δ is the reduced version of this divisor, $u_j v_k - u_k v_j = 0$, then the correspondence between between $\Delta \to M_{\mathbf{w}}$ and $(\mathbb{P}^1)^{n-1} - U_{\mathbf{w}'} \to M_{\mathbf{w}'}$ is as follows. For each Γ' of degree $\mathbf{w}'$, we lift $X_{\Gamma'}$ to any X_Γ where Γ is a graph on $\{1, \dots, n\}$ of degree $\mathbf{w}$ whose "image" in $\{1, \dots, n\} \cup \{0\} \setminus \{j, k\}$ is Γ'. (In other words, to w_j of the w_0 edges meeting vertex 0 in Γ', we associate edges meeting vertex j in Γ, and similarly with j replaced by k.) If Γ'' is any other lift, then $X_\Gamma = \pm X_{\Gamma''}$ on Δ, because using the Plücker relations, $X_\Gamma \pm X_{\Gamma''}$ can be expressed as a combination of variables containing edge jk, which all vanish on Δ.

From (6), the multiplicity with which this component appears is m_{ij}, the number of edges joining vertices j and k.

If $w_j + w_k = \sum w_i/2$, then $M_{\mathbf{w}'}$ is a strictly semistable point, and of dimension 0 smaller than $\dim M_{\mathbf{w}} - 1$, and hence is not a component. (Our base case is $n = 4$, not 3, for this reason.)

(d) If $n \geq 4$ and there are j and k such that $w_j + w_k > \sum w_i/2$, then the rational map $(\mathbb{P}^1)^4 \dashrightarrow M_{\mathbf{w}}$ has a base locus. Any graph X_Γ of degree $\mathbf{w}$ necessarily contains a copy of edge jk, so $(u_j v_k - u_k v_j)$ is a factor of any of the X_Γ. Hence $M_{\mathbf{w}}$ is naturally isomorphic to $M_{\mathbf{w}-e_j-e_k}$, so we replace $\mathbf{w}$ by $\mathbf{w} - e_j - e_k$, and repeat the process. Note that if $n = 4$, then the final resulting quadruple must be of the form (d, d, d, d).

For example, $\deg M_4 = 1$, $\deg M_6 = 3$, $\deg M_8 = 40$, and $\deg M_{10} = 1225$ were computed by hand. (This appears to be sequence A012250 on Sloane's *On-line encyclopedia of integer sequences* [Sl].) The calculations $\deg M_6 = 3$ and $\deg M_{2,2,2,2,2} = 5$ are shown in Figure 14. and 15 respectively. At each stage, $\mathbf{w}$ is shown, as well as the Γ used to calculate the next stage. In these examples, there is essentially only one such $\mathbf{w}'$ at each stage, but in general there will be many. The vertical arrows correspond to identifying components of X_Γ (step (c)). The first arrow in Figure 14 is labeled $\times 3$ to point out the reader that the next stage can be obtained in three ways. The degrees are obtained inductively from the bottom up. (The reader is encouraged to show that $\deg M_8 = 40$, and that this algorithm indeed gives $\deg M_{d\mathbf{w}} = d^{n-3} \deg M_{\mathbf{w}}$.)

4. Aside: six points in projective line and the mystic pentagons

In this section, we show further advantages of using $\mathfrak{S}_n$-equivariant co-ordinates, in the case $n = 6$, by describing the connections to the Segre cubic threefold $\mathcal{S}_3$ and the Igusa quartic threefold $\mathcal{I}_4$, and their relation to the outer automorphism of $\mathfrak{S}_6$. These moduli spaces are of special interest because they are essentially moduli spaces of abelian surfaces or genus 2 curves with complete level 2 structure (e.g. [I, vdG]). The explicit formulas are quite attractive, and were obtained by representation theory. As they may

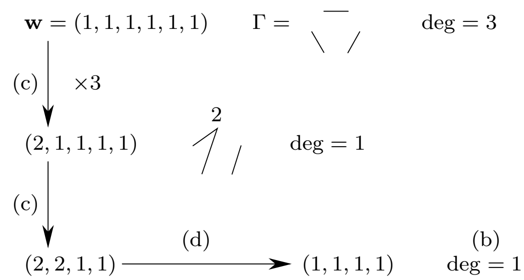

FIGURE 14. Computing $\deg M_6 = 3$ (recall that M_6 is the Segre cubic threefold $\mathcal{S}_3$)

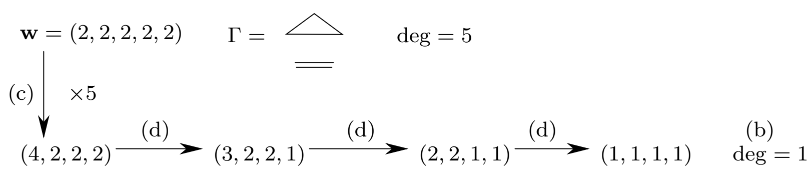

FIGURE 15. Computing $\deg M_{(2,2,2,2,2)} = 5$ using an inconvenient choice of Γ (recall that $M_{(2,2,2,2,2)}$ is a degree 5 del Pezzo surface)

be verified by direct computation, their demonstrations are omitted. En route, we give a new description of the outer automorphism of $\mathfrak{S}_6$.

4.1. Warm-up: The mystic pentagons, representations of $\mathfrak{S}_5$ and $\mathfrak{S}_6$, and the outer automorphism. There are twelve ways to two-color the edges of a complete graph on five vertices, such that the edges of each color form a five-cycle: the six ways **a**–**f** shown in Figure 16, and their "opposites" $\overline{\mathbf{a}}$–$\overline{\mathbf{f}}$ (with the colors interchanged). Each element of $\mathfrak{S}_5$ induces a permutation of the six mystic pentagon pairs via its action on the vertices. (I have chosen representatives of each pair symmetrically — note that each edge appears in each color precisely three times with this choice. This has the advantage that any odd permutation in $\mathfrak{S}_5$ permutes the six pentagons and exchanges the colors. However, this does not matter too much.)

4.2. *Aside.* This gives a convenient way of understanding the two 5-dimensional irreducible representations of $\mathfrak{S}_5$. The permutation representation induced by this $\mathfrak{S}_5$ action on the mystic pentagon pairs splits into an irreducible 5-dimensional representation F_5 and a trivial representation. The other irreducible 5-dimensional $\mathfrak{S}_5$-representation F_5' is obtained by tensoring F_5 with the sign representation, but is better-understood from the mystic pentagons as follows. To each pentagon $\mathbf{x}$, we associate a variable $z_{\mathbf{x}}$, with the convention that $z_{\mathbf{x}} = -z_{\overline{\mathbf{x}}}$. These variables span a six-dimensional $\mathfrak{S}_5$-representation, which is the direct sum of the sign representation and the representation F_5'. This convenient

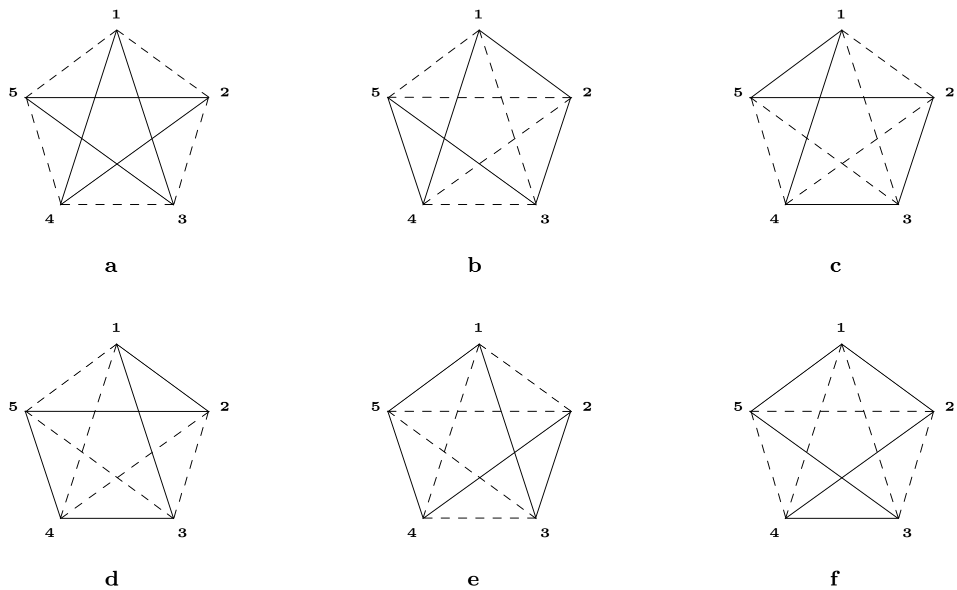

FIGURE 16. The six mystic pentagons

decomposition of the 12-dimensional permutation representation into $\mathbf{1} \oplus F_5 \oplus sgn \oplus F_5'$ will have a useful analog for $\mathfrak{S}_6$ in §4.6.

4.3. First new description of the outer automorphism of $\mathfrak{S}_6$. The permutation action of $\mathfrak{S}_5$ on the six mystic pentagon pairs gives an injection $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$. The $\mathfrak{S}_6$-action on the six cosets of this subgroup is the outer automorphism of $\mathfrak{S}_6$. This may be interpreted as an $\mathfrak{S}_6$-action on the six mystic pentagon-pairs.

We can make this description more symmetric, not privileging the element $6 \in \{1, \dots, 6\}$, as follows. We add a sixth vertex to each of these pentagons, labeled 6. Instead of coloring edges, we color triangles. A triangle $6ab$ is colored the same as edge ab. The triangle cde ($6 \neq c, d, e$) is colored the opposite of the "complementary" edge ab. The new description may be summarized as follows.

4.4. Second new description of the outer automorphism of $\mathfrak{S}_6$. Consider the $\binom{6}{3} = 20$ triangles on 6 labeled vertices. There are 6 ways of dividing the triangles into two sets of 10 so that (i) two disjoint triangles are opposite colors, and (ii) every tetrahedron has 2 triangles of each color. The $\mathfrak{S}_6$-action on this set is the outer automorphism of $\mathfrak{S}_6$.

4.5. *Relation to the usual description of the outer automorphism of* $\mathfrak{S}_6$. The usual description of the outer automorphism is as follows [C]. A *syntheme* is a matching of the numbers $1, \dots,$ 6, i.e. an unordered partition of $\{1, \dots, 6\}$ into three sets of size two. A *pentad* is a set of five synthemes whose whose union is the set of all 15 pairs. Then there are precisely 6 pentads, and the action of $\mathfrak{S}_6$ on this set is via the outer automorphism of $\mathfrak{S}_6$. We explain how to get a pentad from one of the six mystic pentagon-pairs. Each mystic pentagon determines a bijection between the white edges and the black edges, where edge ab corresponds with edge cd if ab and cd don't share a vertex. If $e = \{1, \dots, 5\} - \{a, b, c, d\}$, then to each such pair we obtain the syntheme $ab/cd/e6$, and there are clearly five such synthemes, no two of which share an edge, which hence form a pentad. Hence for example mystic pentagon **a** yields the pentad

$$\{12/35/56, 23/14/56, 34/25/16, 45/13/26, 15/24/36\}.$$

4.6. *The mystic pentagons and the* 5*-dimensional representations of* $\mathfrak{S}_6$*.* There are four irreducible 5-dimensional representation of $\mathfrak{S}_6$. One is the standard representation (which we here denote B_5), obtained by subtracting the trivial representation from the usual permutation representation. A second is obtained by tensoring with the sign representation sgn: $B_5' := B_5 \otimes sgn$. A third is analogous to the standard representation, obtained by subtracting the trivial representation from the (outer) permutation representation of $\mathfrak{S}_6$ on the six mystic pentagon-pairs. One might denote this the *outer automorphism representation*. The fourth 5-dimensional $\mathfrak{S}_6$-representation is $O_5' := O_5 \otimes sgn$. One might term this the *signed outer automorphism representation*. We interpret it as follows. To each pentagon $\mathbf{x}$, we associate a variable $z_{\mathbf{x}}$, with the convention that $z_{\mathbf{x}} = -z_{\overline{\mathbf{x}}}$. These variables span a six-dimensional $\mathfrak{S}_6$-representation, which is the direct sum of the sign representation and the representation O_5'. This description directly generalizes the description above (§4.2) of the 5-dimensional irreducible representations of $\mathfrak{S}_5$, and indeed this shows that F_5 and F_5' are obtained by restriction from O_5 and O_5'.

We will see that the hyperplane sections of both the Segre cubic $\mathcal{S}_3$ and the Igusa quartic $\mathcal{I}_4$ are isomorphic to O_5', and thus a natural way of describing them will be in terms of the mystic pentagons.

4.7. *The Segre cubic* $\mathcal{S}_3$*.* As described in §2, the Segre cubic threefold $\mathcal{S}_3$ is the (democratic) GIT quotient of 6 points on $\mathbb{P}^1$ with the "democratic" linearization $(1, \dots, 1)$, and is given by (2)

$$z_1 + \cdots + z_6 = z_1^3 + \cdots + z_6^3 = 0.$$

We now describe the moduli map $(\mathbb{P}^1) \dashrightarrow \mathcal{S}_3$. If the points are given by $[p_i; 1]$ $(1 \leq i \leq 6)$, the moduli map is given (in terms of the second description of the outer automorphism, §4.4) by

$$z_{\mathbf{x}} = \sum_{\{a,\dots,f\}=\{1,\dots,6\}} \pm p_a p_b p_c$$

where the sign is 1 if triangle abc is black, and -1 if the triangle is white. Note that $\sum z_{\mathbf{x}} = 0$. Hence the $\mathfrak{S}_6$-representation on $H^0(\mathcal{S}_3, \mathcal{O}(1))$ is the signed outer automorphism representation O_5'.

4.8. *The Igusa quartic* $\mathcal{I}_4$*.* The Igusa quartic threefold $\mathcal{I}_4$ is the GIT quotient of six points on $\mathbb{P}^3$. To my knowledge, the presence of the outer automorphism was realized surprisingly recently, by van der Geer in 1982 [vdG, §5]:

$$w_{\mathbf{a}} + \cdots + w_{\mathbf{f}} = 0, \quad (w_{\mathbf{a}}^2 + \cdots + w_{\mathbf{f}}^2)^2 - 4(w_{\mathbf{a}}^4 + \cdots + w_{\mathbf{f}}^4) = 0.$$

(Igusa's original equation [I, p. 400] obscured the $\mathfrak{S}_6$-action. See also [DO, p. 122] and [vdG] for a description using $Sp(6, \mathbb{F}_2)$, which is isomorphic to $\mathfrak{S}_6$.) Via the Gale transform, this is birational to the space of six points on $\mathbb{P}^1$. The rational map $(\mathbb{P}^1)^6 \dashrightarrow \mathcal{I}_4$ is described as follows, using the first description of the outer automorphism, §4.3:

$$w_{\mathbf{x}} = \sum_{\{a,\dots,e\}=\{1,\dots,5\},\ \{\alpha,\beta,\gamma\}=\{0,1,2\}} N_{a,\dots,e,\alpha,\beta,\gamma}(p_6 p_a)^{\alpha}(p_b p_c)^{\beta}(p_d p_e)^{\gamma}$$

where $N = 2$ if the edge bc has the same sign as edge de, and $N = -1$ otherwise. (A quick inspection of the mystic pentagons shows that $\sum w_{\mathbf{x}} = 0$.) Hence the $\mathfrak{S}_6$-representation on $H^0(\mathcal{I}_4, \mathcal{O}(1))$ is O_5'.

This birationality arises by duality (the representation O_5' is self-dual), which should not involve the outer automorphism. Indeed, the duality map $\mathcal{S}_3 \dashrightarrow \mathcal{I}_4$ is given by

$$w_{\mathbf{x}} = z_{\mathbf{x}}^2 - \frac{1}{6}\sum_{\mathbf{y}=1}^{6} z_{\mathbf{y}}^2,$$

and the duality map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ is given by

$$z_{\mathbf{x}} = \left(\sum_{\mathbf{y}=1}^{6} w_{\mathbf{y}}^2\right) w_{\mathbf{x}} - 4w_{\mathbf{x}}^3 + \frac{2}{3}\sum_{\mathbf{y}=1}^{6} w_{\mathbf{y}}^3.$$

4.9. Relation to the graphs of §3. It remains to translate the explicit invariant theory in terms of the mystic pentagons to the matching diagrams of §3. The variables $z_{\mathbf{x}}$ of the Segre cubic threefold $\mathcal{S}_3$ are related to the matching diagrams in a straightforward way:

$$X_{\vec{13}\cdot\vec{26}\cdot\vec{45}} = (z_{\mathbf{a}} + z_{\mathbf{b}})/2$$

(and similarly after application of the $\mathfrak{S}_6$-action on both sides). Notice that under the outer automorphism, pairs are exchanged with matchings (= partitions into three pairs = synthemes), and that is precisely what we see here.

This can of course be easily inverted, using

$$z_{\mathbf{a}} = (z_{\mathbf{a}} + z_{\mathbf{b}})/2 + (z_{\mathbf{a}} + z_{\mathbf{c}})/2 - (z_{\mathbf{b}} + z_{\mathbf{c}})/2.$$

As the X-variables form a 15-dimensional vector space with many relations, there are many formulas for the z-variables in terms of the X-variables. It would be most attractive to have a description in terms of the pentads (as the X-variables correspond to synthemes), but we have been unable to do so.

4.10. *Six points in* $\mathbb{P}^2$. As a final aside, we describe the (democratic) invariant theory of six points in $\mathbb{P}^2$ explicitly, in terms of the mystic pentagons. This quotient is a double cover of $\mathbb{P}^4$ branched over the Igusa quartic $\mathcal{I}_4$. The Gale transform exchanges the sheets. The branch locus corresponds to when the 6 points lie on a conic (the "self-associated" sets); by choosing an isomorphism of this conic with $\mathbb{P}^1$, the rational map $(\mathbb{P}^1)^6 \dashrightarrow \mathcal{I}_4$ is precisely the moduli map described above.

We describe the moduli map $(\mathbb{P}^2)^6 \dashrightarrow \mathbb{P}^4$ in terms of the mystic pentagons §4.3.

$$w_{\mathbf{x}} = \sum_{\{a,\dots,f\}=\{1,\dots,6\}} N_{a,\dots,f}(x_a x_b)(y_c y_d)(z_e z_f).$$

Corresponding to each term are two edges (corresponding to the pairs ab, cd, ef not containing 6). Then $N = 2$ if the two edges have the same color, and -1 otherwise. Notice the similarity to the moduli map for the Igusa quartic above §4.8; this is not a coincidence, and we have chosen the variable names $w_{\mathbf{x}}$ for this reason.

The condition for six points to be on a conic is for their image on the Veronese to be coplanar, hence that the following expression is 0:

$$v := \det \begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1y_1 & y_1z_1 & z_1x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & y_6z_6 & z_6x_6 \end{pmatrix}.$$

Then the formula for the double fourfold that is the GIT quotient of 6 points on $\mathbb{P}^2$ is

$$\left(\sum w_{\mathbf{x}}^2\right)^2 - 4\sum w_{\mathbf{x}}^4 + 324v^2 = 0$$

and it is clear that it is branched over the Igusa quartic $\mathcal{I}_4$. (See [DO, p. 17, Example 3] for more information.)

5. Proof of Main Theorem 3.13

As in the proof of Kempe's theorem 3.3, we first deal with the "democratic" case n even, and $\mathbf{w} = (1,\dots,1) = 1^n$ (§5.2), and then the general case will follow, as relevant spaces will be linear sections of the corresponding spaces in some democratic case (§5.17). We first dispatch four exceptional cases.

5.1. Proof if $|\mathbf{w}| = 6$ and $\mathbf{w} \neq (1,\dots,1)$. In this case $\mathbf{w} = (3,2,1)$, $(2,2,2)$, $(2,2,1,1)$, or $(2,1,1,1,1)$. The first two cases are points, and the next two cases were verified to have degree 1 in §3.14 (see Figure 14).

5.2. Proof in the "democratic case" n even, $\mathbf{w} = (1,\dots,1) = 1^n$. The case $n = 2$ and $n = 4$ are immediate, and we verified the case $n = 8$ in §3.8, so we assume hereafter that $n \geq 10$. Let V_n be the subscheme of $\mathbb{P}^{N-1}$ (where $N = (2n-1)!!2^n = n!/(n/2)!2^n$) cut out by the Plücker and simple binomial relations. Then M_n is a closed subscheme of V_n, and Theorem 3.13 states that the two schemes have the same underlying set, and are isomorphic away from the strictly semistable points.

The reader will notice that we will use the simple binomial relations very little. In fact we just use the inductive structure of the moduli space: given a matching Δ on $n-k$ of n vertices ($4 \leq k < n$), and a point $[X_\Gamma]_\Gamma$ of V_n, then either these X_Γ with $\Delta \subset \Gamma$ are all zero, or $[X_\Gamma]_{\Delta\subset\Gamma}$ satisfies the Plücker and simple binomial relations for k, and hence is a point of V_k if $k \neq 6$. (The reader should think of this rational map $[X_\Gamma] \dashrightarrow [X_\Gamma]_{\Delta\subset\Gamma}$ as a forgetful map, remembering only the moduli of the k points.) In fact, even if $k = 6$ (and $n \geq 8$), the point must lie in M_6, as the simple binomial relations for $n > 6$ induce the Segre cubic relation (§3.10). The central idea of our proof is, ironically, to use the case $n = 6$, where the Theorem 3.13 doesn't apply.

We will call such Δ, where the X_Γ with $\Delta \subset \Gamma$ are not all zero and the corresponding point of M_6 is stable, a *stable* $(n-6)$*-matching*. One motivation for this definition is that given a stable configuration of n points on $\mathbb{P}^1$, there always exists a stable $(n-6)$-matching. (Hint: Construct Δ inductively as follows. We say two of the n points are in the same *clump* if they have the same image on $\mathbb{P}^1$. Choose any y in the largest clump, and any z in the second-largest clump; yz is our first edge of Δ. Then repeat this with the remaining vertices, stopping when there are six vertices left.) Caution: This is false with 6 replaced by 4!

The theorem (in the democratic case) will be a consequence of the following two statements.

(I) There is a natural bijection between points of V_n with no stable $(n-6)$-matching, and strictly semistable points of M_n.

(II) If B is any scheme, there is a bijection between morphisms $B \to V_N$ missing the "no stable $(n-6)$-matching" locus (i.e. missing the strictly semistable points of M_n, by (I)) and stable families of n points $B \times \{1, \dots, n\} \to \mathbb{P}^1$. (In other words, we are exhibiting an isomorphism of functors.)

The next result is one direction of the bijection of (I). The other direction is immediate.

5.3. *Claim. — If* $[X_\Gamma]_\Gamma$ *is a point of* V_n *(*$n \geq 10$*) having no stable* $(n-6)$*-matching, then* $[X_\Gamma]_\Gamma$ *is a strictly semistable stable point of* M_n*.*

Several of the steps will be used in the proof of (II). We give them names so they can be referred to later.

Proof. Our goal is to produce a partition of n into two subsets of size $n/2$, such that the point of M_n given by this partition is our point of V_n. Throughout this proof, partitions will be assumed to mean into two equal-sized subsets.

We work by induction. We will use the fact that the result is also true for $n = 6$ (tautologically) and $n = 8$ (as $V_8 = M_8$, see the first paragraph of §5.2).

Fix a matching Δ such that $X_\Delta \neq 0$. By the inductive hypothesis, each edge xy yields a strictly semistable point of M_{n-2}, and hence a partition of $\{1, \dots, n\} - \{x, y\}$, by considering all matchings containing xy. Thus for each $xy \in \Delta$, we get a partition of $\{1, \dots, n\} - \{x, y\}$. If wx, yz are two edges of Δ, then we get the same induced partition of $\{1, \dots, n\} - \{w, x, y, z\}$ (from the inductive hypothesis for $n-4$), so all of these partitions arise from a single partition $\{1, \dots, n\} = S_0 \coprod S_1$.

5.4. Δ *two-overlap argument.* As this partition is determined using any two edges of Δ, we would get the same partition if we began with any Δ' sharing two edges with Δ, such that $X_{\Delta'} \neq 0$.

Defining the map to $\mathbb{P}^1$. Define $\phi : S_0 \coprod S_1 = \{1, \dots, n\} \to \mathbb{P}^1$ by $S_0 \to 0$ and $S_1 \to 1$. For each matching Γ, define X'_Γ using these points of $\mathbb{P}^1$. Rescale (or normalize) all the X_Γ so $X'_\Delta = X_\Delta$. We will show that $X'_\Gamma = X_\Gamma$ for all Γ, which will prove Claim 5.3.

5.5. *One-overlap argument.* For any Γ sharing an edge xy with Δ, $X'_\Gamma = X_\Gamma$, for the following reason: $[X_\Xi]_{xy \in \Xi}$ lies in M_{n-2} by the inductive hypothesis, and this point of M_{n-2} corresponds to the map ϕ (as the partition $S_0 \coprod S_1$ was determined using this point of M_{n-2}), so $[X_\Xi]_{xy\in\Xi} = [X'_\Xi]_{xy\in\Xi}$, and the normalization $X'_\Delta = X_\Delta \neq 0$ ensures that $X'_\Xi = X_\Xi$ for all Ξ containing xy.

5.6. *Reduction to* Γ *with* $X'_\Gamma \neq 0$. It suffices to prove the result for those graphs Γ, all of whose edges connect S_0 and S_1 (i.e. no edge is contained in S_0 or S_1; equivalently, $X'_\Gamma \neq 0$). We show this by showing that *any* X_Γ is a linear combination of such graphs, by induction on the number i of edges of Γ contained in S_0 (= the number contained in S_1). The base case $i = 0$ is tautological. For the inductive step, choose an edge $wx \in \Gamma$ contained in S_0 and an edge yz contained in S_1. Then the Plücker relation using Γ and $wxyz$ (with appropriate signs depending on the directions of edges) is

$$\pm X_\Gamma \pm X_{\Gamma - wx - yz + wy + xz} \pm X_{\Gamma - wx - yz + wz + xy} = 0,$$

and both $\Gamma - wx - yz + wy + xz$ and $\Gamma - wx - yz + wz + xy$ have $i - 1$ edges contained in S_0, and the result follows.

5.7. *$pqrs$ argument, first version.* Finally, assume that $X'_\Gamma \neq 0$ and that Γ shares no edge with Δ. See Figure 17. Let qr be an edge of Γ (so $\phi(q) \neq \phi(r)$), and let pq and rs be edges of Δ containing q and r respectively (so $\phi(p) \neq \phi(q)$ and $\phi(r) \neq \phi(s)$). Then $\phi(p) \neq \phi(s)$, as ϕ takes on only two values. Let $\Delta' = \Delta - pq - rs + qr + ps$, so $X'_{\Delta'} \neq 0$ as $\phi(q) \neq \phi(r)$ and $\phi(p) \neq \phi(s)$. Then $X_{\Delta'} = X'_{\Delta'}$ by the one-overlap argument 5.5, as Δ' shares an edge with Δ (indeed all but two edges), so $X_{\Delta'} \neq 0$. Hence by the Δ two-overlap argument 5.4, Δ' defines the same partition $S_0 \coprod S_1$, and hence the same map $\phi : \{1, \dots, n\} \to \mathbb{P}^1$. Finally, Γ shares an edge with Δ', so $X'_\Gamma = X_\Gamma$ by the one-overlap argument 5.5.

We have thus completed the proof of Claim 5.3. □

Proof of (II). The result boils down to the following desideratum: Given any $(n-6)$-matching Δ on some $\{1, \dots, n\} - \{a, b, c, d, e, f\}$, there should be a bijection between:

(a) morphisms $\pi : B \to V_n$ contained in the open subset where Δ is a stable $(n-6)$-matching, and
(b) stable families of points $\phi : B \times \{1, \dots, n\} \to \mathbb{P}^1$ where $\phi|_{B \times \{a,\dots,f\}}$ is also a stable family, and for any edge xy of Δ, $\phi|_{B\times\{x\}}$ does not intersect $\phi|_{B\times\{y\}}$.

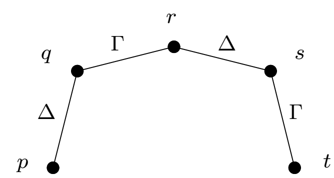

FIGURE 17. The $pqrs$ argument (vertex t is used in §5.12)

We have already described the map (b) ⇒ (a) in §3. We now describe the map (a) ⇒ (b), and verify that (a) ⇒ (b) ⇒ (a) is the identity. (It will then be clear that (b) ⇒ (a) ⇒ (b) is the identity: given a stable family of points parameterized by B, we get a map from B to an open subset of M_n, which is a fine moduli space, hence (b) ⇒ (a) is an injection. The result then follows from the fact that (a) ⇒ (b) ⇒ (a) is the identity.)

We work by induction on n. The case $n = 8$ was checked earlier (see the first paragraph of §5.2).

The map to $\mathbb{P}^1$. Given an element of (a), define a family of n points of $\mathbb{P}^1$ (an element of (b)) as follows. (i) $\phi : B \times \{a, \dots, f\} \to \mathbb{P}^1$ is given by the corresponding map $B \to M_6$. (ii) If yz is an edge of Δ, we define $B \times (\{1, \dots, n\} - \{y, z\}) \to \mathbb{P}^1$ extending (i) by considering the matchings containing yz, which by the inductive hypothesis give a point of M_{n-2}. (iii) The morphisms of (ii) agree "on the overlap", as given two edges wx and yz of Δ, we get $B \times (\{1, \dots, n\} - \{w, x, y, z\}) \to \mathbb{P}^1$ by considering the matchings containing $wx \cdot yz$, which by the inductive hypothesis give a map to M_{n-4}. Here we are using that $n \geq 10$; and if $n = 10$, we need the fact that the Segre cubic relation cutting out M_6 is induced by the quadrics cutting out M_n for $n \geq 8$ (Remark 3.10). Thus we get a well-defined morphism $\phi : B \times \{1, \dots, n\} \to \mathbb{P}^1$.

5.8. Δ *two-overlap argument, cf. §5.4.* If Δ' is another matching on $\{1, \dots, n\} - \{a, \dots, f\}$ sharing at least 2 edges with Δ, with $X_{\Delta' \cdot \Xi} \neq 0$ for some matching Ξ of $\{a, \dots, f\}$, we obtain the same ϕ, as ϕ can be recovered by considering only two edges of Δ when using (ii).

Defining X'. Define X'_Γ for all matchings Γ using ϕ and the moduli morphism of eqn. (3). The coordinates X_Γ are projective (i.e. the set of X_Γ is defined only up to scalars); scale them so that $X_{\Delta \cdot \Xi} = X'_{\Delta \cdot \Xi}$ for all matchings Ξ of $\{a, \dots, f\}$. Note that if xy is an edge of Δ, then $\phi(x) \neq \phi(y)$, as there exists a matching Ξ of $\{a, \dots, f\}$ such that $X'_{\Delta \cdot \Xi} \neq 0$.

The following result will confirm that (a) ⇒ (b) ⇒ (a) is the identity, concluding the proof of (II).

5.9. *Claim.* — We have the equality $X_\Gamma = X'_\Gamma$ for all Γ.

Proof. This proof will occupy us until the end of §5.15.

5.10. *One-overlap argument.* As in §5.5, the result holds for those Γ sharing an edge yz with Δ: by considering only those variables $X_{\Gamma'}$ containing the edge yz (including both X_Γ and X_Δ), we obtain a point of M_{n-2}. This point of M_{n-2} is the one given by ϕ (this was part of how ϕ was defined), so $[X_{\Gamma'}]_{yz\in\Gamma'} = [X'_{\Gamma'}]_{yz\in\Gamma'}$. By choosing a matching Ξ on $\{a,\ldots,f\}$ so that $X_{\Delta\cdot\Xi} \neq 0$, we have that $X_\Gamma X'_{\Delta\cdot\Xi} = X_{\Delta\cdot\Xi} X'_\Gamma$. Using $X_{\Delta\cdot\Xi} = X'_{\Delta\cdot\Xi} \neq 0$, we have $X_\Gamma = X'_\Gamma$, as desired.

We now deal with the remaining case, where Γ and Δ share no edge.

5.11. *Reduction to Γ with $X'_\Gamma \neq 0$ (cf. §5.6).* It suffices to prove the result for those graphs such that $X'_\Gamma \neq 0$, or equivalently that for each edge xy of Γ, $\phi(x) \neq \phi(y)$. We show this by showing that any X_Γ is a linear combination of such graphs, by induction on the number of edges xy of Γ with $\phi(x) = \phi(y)$. For the purposes of this paragraph, call these *bad edges*. The base case $i = 0$ is tautological. For the inductive step, choose a bad edge $wx \in \Gamma$ (with $\phi(w) = \phi(x)$), and another edge yz such that $\phi(y), \phi(z) \neq \phi(w)$. (Such an edge exists, as by stability, less than $n/2$ elements of $\{1,\ldots,n\}$ take the same value in $\mathbb{P}^1$.) Then the Plücker relation using Γ with respect to $wxyz$ is

$$\pm X_\Gamma \pm X_{\Gamma-wx-yz+wy+xz} \pm X_{\Gamma-wx-yz+wz+xy} = 0,$$

and both $\Gamma - wx - yz + wy + xz$ and $\Gamma - wx - yz + wz + xy$ have at most $i-1$ bad edges, and the result follows.

Recall that we are proceeding by induction. We first deal with the case $n \geq 14$, assuming the cases $n = 10$ and $n = 12$. We will then deal with these two stray cases. This logically backward, but the $n \geq 14$ case is cleaner, and the two other cases are similar but more ad hoc.

5.12. The case $n \geq 14$. *$pqrs$ argument, second version.* As $n \geq 14$, there is an edge qr of Γ not meeting $abcdef$. See Figure 17. By §5.11, we may assume $\phi(q) \neq \phi(r)$. Let pq and rs be the edges of Δ meeting q and r respectively (so $\phi(p) \neq \phi(q)$ and $\phi(r) \neq \phi(s)$. (i) (cf. the similar argument of §5.7) If $\phi(p) \neq \phi(s)$, then let $\Delta' = \Delta - pq - rs + qr + ps$; then Δ' defines the same family of n points as Δ by the two-overlap argument §5.8, and Γ and Δ' share an edge, so we are done by the one-overlap argument §5.10. (More precisely, this argument applies on the open subset of B where $\phi(p) \neq \phi(s)$.) (ii) If $\phi(p) = \phi(s)$, then $\phi(p) \neq \phi(r)$. (More precisely, this argument applies on the open set where $\phi(p) \neq \phi(r)$.) Let st be the edge of Γ containing s. (It is possible that $t = p$.) Let $\Gamma' = \Gamma - qr - st + rs + qt$ and $\Gamma'' = \Gamma - qr - st + qs + rt$ be the other two terms in the Plücker relation for Γ for $pqrs$. Then Γ' shares edge rs with Δ, so $X'_{\Gamma'} = X_\Gamma$ by the one-overlap argument §5.10, and by applying (i) to Γ'' (swapping the names of r and s), $X'_{\Gamma''} = X_{\Gamma''}$, so by the Plücker relation, $X'_\Gamma = X_\Gamma$ as desired.

5.13. The cases $n = 10$ and $n = 12$. We are assuming that Γ and Δ share no edges. If there is an edge of Γ not meeting $\{a,\ldots,f\}$, the $pqrs$-argument §5.12 applies, so assume otherwise. Divide $\{1,\ldots,n\}$ into two subsets $abcdef$ and $ghij$ (resp. $ghijkl$) if $n = 10$ (resp. $n = 12$), where the edges of Δ are gh, ij, and (if $n = 12$) kl. By renaming $abcdef$, we

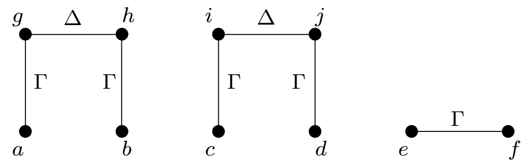

FIGURE 18. The problematic graphs for $n = 10$

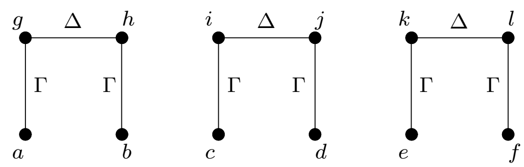

FIGURE 19. The problematic graphs for $n = 12$

may assume the edges of Γ are ag, bh, ci, dj, and either ef (if $n = 10$, see Figure 18) or ek and fl (if $n = 12$, see Figure 19).

5.14. Suppose that $\phi(a) \neq \phi(b)$. Note that we will only use that $ag, bh \in \Gamma$, $gh \in \Delta$, and $\phi(a) \neq \phi(b)$ — we will use this argument again below. There is a matching Ξ of $cdef$ so that if $xy \in \Xi$, then $\phi(x) \neq \phi(y)$. (This is a statement about stable configurations of 6 points on $\mathbb{P}^1$: if we have a stable set of 6 points on $\mathbb{P}^1$, then no three of them are the same point. Hence for any four of them $cdef$, we can find a matching of this sort.) Let $\Delta' = \Xi \cdot ab \cdot \Delta$. Then by the simple binomial relations (our first invocation!) $X_{\Delta'} X_\Gamma = X_{\Delta'-ab-gh+ag+bh} X_{\Gamma+ab+gh-ag-bh}$ and $X'_{\Delta'} X'_\Gamma = X'_{\Delta'-ab-gh+ag+bh} X'_{\Gamma+ab+gh-ag-bh}$. However, by the one-overlap argument §5.10, $X_{\Delta'} = X'_{\Delta'} \neq 0$ (Δ' and Δ share edge ij), $X_{\Delta'-ab-gh+ag+bh} = X'_{\Delta'-ab-gh+ag+bh}$ ($\Delta' - ab - gh + ag + bh$ and Δ share edge ij), and $X_{\Gamma+ab+gh-ag-bh} = X'_{\Gamma+ab+gh-ag-bh}$ ($\Gamma + ab + gh - ag - bh$ and Δ share edge gh), so we are done.

We are left with the case $\phi(a) = \phi(b)$.

5.15. Suppose now that $n = 10$. As $\phi(a) = \phi(b)$, $\phi(b)$ is distinct from $\phi(e)$ and $\phi(f)$ (as $\phi(a)$, $\dots$, $\phi(f)$ are a stable set of six points on $\mathbb{P}^1$). By the Plücker relations for Γ (using $agef$),

$$\pm X_\Gamma \pm X_{\Gamma-ag-ef+ae+gf} \pm X_{\Gamma-ag-ef+af+eg} = 0,$$

and similarly for the X' variables. By applying the argument of §5.14 with e and a swapped, we have $X'_{\Gamma-ag-ef+af+eg} = X_{\Gamma-ag-ef+af+eg}$, and by applying the argument of §5.14 with f and a swapped, we have $X'_{\Gamma-ag-ef+ae+gf} = X_{\Gamma-ag-ef+ae+gf}$, from which $X'_\Gamma = X_\Gamma$, concluding the $n = 10$ case.

5.16. Suppose finally that $n = 12$. If $\phi(c) \neq \phi(d)$, we are done (by the same argument as §5.14, with ab replaced by cd), and similarly if $\phi(e) \neq \phi(f)$. Hence the only case left is if

$\phi(a) = \phi(b)$, $\phi(c) = \phi(d)$, and $\phi(e) = \phi(f)$, and (by stability of the 6 points $\phi(a), \dots, \phi(f)$) these are three distinct points of $\mathbb{P}^1$. Consider the Plücker relation for Γ with respect to $bhci$. One of the other two terms is $\Gamma - bh - ci + bi + ch$, and $X'_{\Gamma-bh-ci+bi+ch} = X_{\Gamma-bh-ci+bi+ch}$ (by the same argument as in §5.14, as $\phi(a) \neq \phi(c)$). We thus have to prove that "$X' = X$" for the third term in the Plücker relation: $X_{\Gamma'} = X'_{\Gamma'}$ for

$$\Gamma' = ag \cdot bc \cdot hi \cdot dj \cdot ek \cdot fl.$$

For this, apply the argument of §5.15 applies, with $abghef$ replaced by $felkbc$ respectively. □

5.17. Proof of Main Theorem 3.13 in general. (If $n = 3$, $M_{\mathbf{w}}$ is a point, and the result is trivially true, so we assume $n \geq 4$.) The general case follows in the same way as it did for Kempe's Theorem 3.3. If $|\mathbf{w}|$ is odd, it suffices to consider the case $2\mathbf{w}$, so by replacing $\mathbf{w}$ by $2\mathbf{w}$, we may assume $\epsilon_{\mathbf{w}} = 1$. (The number $\epsilon_{\mathbf{w}}$ was defined just after the statement of Kempe's Theorem 3.3.)

If $n = |\mathbf{w}|$, then our space is a linear section of M_n, and the Plücker relations and simple binomial relations of M_n restrict to the same relations for our moduli space (or restrict to 0 relations). This is slightly subtle (it is not a proper intersection of M_n with a linear space, as is shown by the fact that $\deg M_6 = 3$ but $\deg M_{2,1,1,1,1} = 1$, see Figure 14), so we discuss this at greater length.

This is best understood via the moduli map $\pi : (\mathbb{P}^1)^n - U \to [X_\Gamma]$, where U is the unstable locus. We group the n vertices into $\mathbf{w}$ "clumps" (of size w_1, etc.). Let $\Delta \cong (\mathbb{P}^1)^{\#\mathbf{w}} \subset (\mathbb{P}^1)^n$ be the locus where the points in each clump are equal, so we get a composition of morphisms $\Delta \hookrightarrow (\mathbb{P}^1)^n \dashrightarrow [X_\Gamma]$, and the pullback of U is precisely the unstable locus for $\mathbf{w}$. The pullback of X_Γ to $\Delta \cong (\mathbb{P}^1)^{\#\mathbf{w}}$ is as follows. If Γ has an edge contained in a clump, it pulls back to 0. Otherwise, choose any Γ' of degree $\mathbf{w}$ "lifting" Γ. Then X_Γ lifts to $X_{\Gamma'}$. This is independent of choice of Γ', as for any two choices Γ' and Γ'', the Plücker relations may be used to show that $X_{\Gamma'} \pm X_{\Gamma''}$ a combination of variables with edges contained in clumps. (This argument is essentially that given in §3.14.) Note that the Plücker relations for the X_Γ "lift" to Plücker relations for $X_{\Gamma'}$, or lift to the 0 relation, and similarly for the simple binomial relations (and indeed all the binomial relations, although that is irrelevant). This construction may be interpreted as describing $M_{\mathbf{w}}$ as a linear section of M_n, where all $X_\Gamma = 0$ for those Γ with an edge in a clump. □

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