1. When is $F(k[V])$ a vector space?

Recall that if $V$ is a finite-dimensional $k$-vector space, we can define the Artin ring $k[V]$. I will now always assume $\epsilon^2 = 0$, so the dual numbers are $k[e]$.

Recall that I defined the categorical product of rings $A \times_C B$. Check that $k[V \oplus W] = k[V] \times_k k[W]$.

**Lemma.** Suppose $F$ is a functor (covariant, on $\mathcal{C}$) such that

$$F(k[V] \times_k k[W]) \sim F(k[W]) \times F(k[W])$$

for finite dimensional vector spaces $V$ and $W$ over $k$. Then $F(k[V])$ and in particular $t_F = F(k[e])$, has a canonical vector space structure, such that $F(k[V]) \cong t_F \otimes V$.

I already essentially gave the proof for $V = (e)$, and the general proof is essentially the same.

**Proof.** $k[V]$ is a “vector space object” in $\mathcal{C}$. In other words, for each $A \in \mathcal{C}$, $\text{Hom}(A, k[V])$ is a $k$-vector space. By:

$$\text{Hom}(A, k[V]) \cong \text{Der}_k(A, V).$$

The addition map is given by $k[V] \times_k k[V] \to k[V]$ $(x, 0), (0, x) \mapsto x$ ($x \in V$). Scalar multiplication by $a$ is given by the endomorphism $x \mapsto ax$ of $k[V]$.

So if $F$ commutes with the necessary products, $F(k[V])$ gets a vector space structure.
For the last statement, here’s a sketch. Note that $\text{Hom}(k[\epsilon]/\epsilon^2, k[V])$ is naturally identified with $V$. For any element of $\text{Hom}(k[\epsilon]/\epsilon^2, k[V])$ we get a map $t_F = F(k[\epsilon]/\epsilon^2) \to F(k[V])$, hence $t_F \times V \to F(k[V])$. In fact this is $\otimes$. The desired result is true if $V$ is one-dimensional; then use induction, as $V$ is finite-dimensional, and $k[V] = \otimes_{k}^{\dim V} k[\epsilon]/(\epsilon^2)$. (I said something wrong in class.)

Note that this isn’t so hard to check. For example, deformation functors of schemes of finite type over $k$ have this property (not even nonsingularity required).

2. Schlessinger’s Criterion for existence of universal deformations and hulls (miniversal deformations)

In $\mathcal{C}$, define a small extension to be a surjection $A'' \to A$, so $A = A''/I$, and $m_{A''}I = 0$, and $I$ is one-dimensional.

For the purposes of this course only, define a fairly small extension to be a surjection $A'' \to A$, so $A = A''/I$, and $m_{A''}I = 0$, without requiring that $I$ is one-dimensional.

Note: Then for any $A$ in $\mathcal{C}$, you can filter $A$ into a series of fairly small extensions (by powers of the maximal ideal).

Then you can filter $A$ into a series of small extensions (explain).

Fix our functor $F : \mathcal{C} \to \text{Sets}$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

\[ F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \]

(1)

Note that if $F$ is a prorepresentable functor, by $R \in \mathcal{C}$ say, then this map is

$\text{Hom}(R, A' \times_A A'') \to \text{Hom}(R, A') \times_{\text{Hom}(R, A)} \text{Hom}(R, A'')$

is always a bijection (explain). This is because $\times$ is a categorical product!

**Schlessinger’s Theorem.** [Put on one board permanently!]

It has two parts, and I’ll say it slowly, with translations and remarks.

(1) $F$ has a hull iff $F$ has properties H1–H3:

H1. (1) is a surjection whenever $A'' \to A$ is a small extension.

Translation: You can glue.

Remark: Hence equivalently whenever $A'' \to A$ is any surjection.
H2. (1) is a bijection when $A = k$, $A'' = k[e]/\epsilon^2$.

Translation: Uniqueness of gluing when adding $k[e]/\epsilon^2$.

Remark: Hence true when $A'' = k[V]$ by induction.

Remark: Hence the criterion of the lemma above are satisfied, so $t_F$ is a $k$-vector space.

H3. $\dim_k(t_F) < \infty$.

Translation: finite-dimensional tangent space.

(2) $F$ is pro-representable if and only if $F$ has the additional property

H4.

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A').$$

is a bijection for any small extension $A' \to A$.

Translation: bijection for gluing a small extension to itself.

That ends the statement. So we have four things to prove.

The first part is easy: if $F$ is prorepresentable, then H1–H4 are all satisfied. Before two of the remaining 3 are quite short.

2.1. Initial remarks. Before I get to them, I want to make some initial remarks.

Suppose $F$ satisfies H1–H3.

Consider any fairly small extension $p : A' \to A$, i.e. $0 \to I \to A' \to A \to 0$, so $m_{A'}I = 0$. We have an isomorphism

$$A' \times_{A'/I} A' \cong A' \times_k k[I]$$

induced by the map $(x, y) \mapsto (x, x_0 + y - x)$ (explain).

Now given a small extension $p : A' \to I$, By H2, we get

$$F(A' \times_A A') = F(A' \times_k k[I]) \cong F(A') \times_{F(k[I])} F(k[I]) = F(A') \times (t_F \otimes I).$$

Hence we get

$$F(A') \times (t_F \otimes I) \to F(A') \times_{F(A)} F(A').$$

For each $\eta \in F(A)$, this determines a group action of $t_F \otimes I$ on $F(p)^{-1}(\eta)$, i.e. those $F(A')$'s lifting $F(A)$, assuming the set is nonempty. The fact that this is a surjection (H1) means that the action is transitive. H4 is precisely the condition that this set is a principal homogeneous space under $t_F \otimes I$. (Say more here.)
So explicitly, what this is telling us is explicitly is that if $F$ already has a hull, then its obstruction to be representable is the existence of an automorphism of an object $y$ in some $F(A)$, that cannot be extended to an automorphism of some object $y' \in F(A')$ for some $A'$.

3. Proof of Schlessinger, Part 1

I'll show that hull and H4 imply prorepresentable. Then I'll show that hull implies H1–H3. Finally, next time I'll show that H1–H3 imply hull.

**Hull and H4 imply prorepresentable.**

Suppose we have hull + H4. Say $(R, r \in F(A))$ is a hull. Hence get $h_R(A) \to F(A)$. We want this to be an isomorphism.

We prove this by induction on the length of $A$. Trivially true for $A = k$.

Consider small $p : A' \to A$, ker $p = I$, one-dimensional.

Assume $h_R(A) \sim F(A)$. For each $a \in F(A)$, $h_R(p)^{-1}(a)$ $F(p)^{-1}(a)$ are both principal homogeneous spaces under $t_F \otimes I$ (or empty). Since $h_R(A')$ maps onto $F(A')$, we have $h_R(A') \sim F(A')$ (either both are empty, or both are principal homogeneous spaces).

Coming next day:

**Hull implies H1–H3, and vice versa.**