INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 8

Due Thursday November 11 in class (no lates). Hand in **how many?** of the following questions. You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument). *Ask someone else about the non-scheme problems you skip.*

1. **Essentially Harthshorne Ex. I.6.2** *An elliptic curve.* Let Y be the curve $y^2 = x^3 - x$ in $\mathbb{A}^2$, and assume that the characteristic of $\overline{k}$ is not 2. In this exercise we will show that Y is not a rational curve, or equivalently that $k(Y)$ is not a pure transcendental extension of $\overline{k}$.
(a) Show that Y is nonsingular, and deduce that $A = A(Y) \cong \overline{k}[x,y]/(y^2 - x^3 + x)$ is an integrally closed domain.
(b) Let $\overline{k}[x]$ be the subring of $k(Y)$ generated by the image of x in A. Show that $\overline{k}[x]$ is a polynomial ring, and that A is the integral closure of $\overline{k}[x]$ in $k(Y)$.
(c) Show that there is an automorphism $\sigma : A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the *norm* of a to be $N(a) = a\sigma(a)$. Show that $N(a) \in \overline{k}[x]$, $N(1) = 1$, and $N(ab) = N(a)N(b)$ for any $a, b \in A$.
(d) Using the norm, show that the units in A are precisely the nonzero elements of $\overline{k}$. Show that x and y are irreducible elements of A. Show that A is *not* a unique factorization domain.
(e) Show that if Q is a nonsingular rational curve which is not isomorphic to $\mathbb{P}^1$, then it is isomorphic to an open subset of $\mathbb{A}^1$, and hence affine. Show that $A(Q)$ is a unique factorization domain.
(f) Prove that Y is not a rational curve.

Date: November 4, 1999.