INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 9

Due Thursday November 18 in class. Hand in five of the following problems, including #3. You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument).

1. (a) (Assume the characteristic is 0.) Consider the point $(2,2)$ on the plane curve $y^2 = x^3 - 2x$. As this is a nonsingular point, it corresponds to a discrete valuation ring. Give an element of the function field with valuations 0, 1, and 2.
   (b) Let Y be the cone $xy = z^2$ in $\mathbb{A}^3$, and let C be the x-axis. Define the local ring at C (denoted $\mathcal{O}_{Y,C}$) as those rational functions (in $k(Y)$) defined at some point of C. (Recall that elements of $k(Y)$ have naturally defined "domains of definition".) Show that $\mathcal{O}_{Y,C}$ is a discrete valuation ring of $k(Y)/\overline{k}$, and give a uniformizer $u \in k(Y)$. (Scheme-theoretically, this corresponds to the fact that the generic point of C is a nonsingular codimension 1 point of the scheme (corresponding to) Y. Not for credit: can you think of a surface Y with a curve C such that the local ring is *not* a discrete valuation ring?)
2. *Shafarevich Ex. II.1.13.* Prove that if a hypersurface $X \subset \mathbb{P}^n$ contains a linear subspace of dimension $r \geq n/2$ then X is singular. (Hint: Choose the coordinate system so that L is given by $x_{r+1} = \cdots = x_n = 0$.) *Does characteristic have to be 0? Certainly X needs to be of degree greater than 1.*
3. Prove that the following three categories are equivalent:
   (i) nonsingular projective curves, and dominant morphisms;
   (ii) quasi-projective curves, and dominant rational maps;
   (iii) finitely-generated function fields of dimension 1 over $\overline{k}$, and $\overline{k}$-homomorphisms.
   You'll have to figure out precisely what the objects and morphisms are in these categories. *Explain this well!*
4. *Hartshorne I.6.1.* Let Y be a nonsingular rational curve which is separated and not isomorphic to $\mathbb{P}^1$. Show that Y is isomorphic to an open subset of $\mathbb{A}^1$, and hence that $A(Y)$ is a unique factorization domain. This will complete your proof of the elliptic curve problem from last week. *This is harder than it looks, given what I've told them.*
5. *Hartshorne I.6.6: Automorphisms of* $\mathbb{P}^1$. Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a *fractional linear transformation* of $\mathbb{P}^1$ by sending $x \mapsto (ax+b)/(cx+d)$, for $a, b, c, d \in \overline{k}$, $ad - bc \neq 0$.
   (a) Show that a fractional linear transformation induces an *automorphism* of $\mathbb{P}^1$. We denote the group of all these fractional linear transformations by $PGL(1)$.
   (b) Let $\operatorname{Aut} \mathbb{P}^1$ denote the group of all automorphisms of $\mathbb{P}^1$. Show that $\operatorname{Aut} \mathbb{P}^1 \cong \operatorname{Aut} \overline{k}(x)$, the group of $\overline{k}$-automorphisms of the field $\overline{k}(x)$.

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(c) Now show that every automorphism of $\overline{k}(x)$ is a fractional linear transformation, and deduce that $PGL(1) \to \operatorname{Aut} \mathbb{P}^1$ is an isomorphism.

6. *Essentially Hartshorne I.7.2.* Let Y be a variety of dimension r in $\mathbb{P}^n$, with Hilbert polynomial P_Y. We define the *arithmetic genus* of Y to be $p_a(Y) = (-1)^r(P_Y(0) - 1)$. This is an important invariant, which turns out to be independent of the projective embedding of Y. (In the case of a nonsingular curve, it is the *genus*, that we will discuss soon.)
(a) Show that $p_a(\mathbb{P}^n) = 0$.
(b) If H is a hypersurface of degree d in $\mathbb{P}^n$, then $p_a(H) = \binom{d-1}{n}$. (Hence if Y is a plane curve of degree d, $p_a(Y) = (d-1)(d-2)/2$.)
(c) If Y is a complete intersection of surfaces of degrees a, b in $\mathbb{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$. (See problem 6 on PS6 if you're unsure what this means.)