INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 7

Due Thursday November 4 in class (no lates). Hand in seven of the following questions. You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument). Hand in six of the following problems. Make sure you know how to do the non-scheme problems you skip.

1. (a) *Hartshorne Ex. I.5.1.* Assume the characteristic is not 2. Locate the singular points. Which is which in Figure 1 below? (a) $x^2 = x^4 + y^4$; (b) $xy = x^6 + y^6$; (c) $x^3 = y^2 + x^4 + y^4$; (d) $x^2y + xy^2 = x^4 + y^4$.
   (b) *Hartshorne Ex. I.5.2.* Assume the characteristic is not 2. Locate the singular points in $\mathbb{A}^3$. Which is which in Figure 2 below?
   (c) Find the dimension of the Zariski-tangent space to the surfaces in (b) at the origin.
2. *Hartshorne Ex. I.5.3: Multiplicities.* Let $Y \subset \mathbb{A}^2$ be the curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Define the *multiplicity* of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y$ if and only if $\mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent directions* at P.
   (a) Show that $\mu_P(Y) = 1$ if and only if P is a nonsingular point of Y.
   (b) Find the multiplicity of each of the singular points in problem 1 (a) above.
3. *Intersection multiplicity.* If Y, $Z \subset \mathbb{A}^2$ are two distinct curves, given by equations $f = 0$, $g = 0$, and if $P \in Y \cap Z$, define the *intersection multiplicity* $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$-module $\mathcal{O}_P/(f, g)$, or equivalently its dimension as a $\overline{k}$-vector space. (It is also equivalent to replace $\mathcal{O}_{(a,b)}$ with the power series ring $\overline{k}[[x', y']]$, where $x' = x - a$ and $y' = y - b$; this makes calculations easier.)
   (a) Calculate the intersection number of $y^2 = x^5$ and $xy = 0$ at the origin. (Don't bother justifying all steps; just show how you computed the number.)
   (b) Show that $(Y \cdot Z)_P$ is finite, and $(Y \cdot Z) \geq \mu_P(Y) \cdot \mu_P(Z)$. *Possibly cut the first part of this, or give a hint.*
   (c) If $P \in Y$, show that for almost all lines L through P (i.e. all but a finite number), $(L \cdot Y)_P = \mu_P(Y)$.
   (d) If Y is a curve of degree d in $\mathbb{P}^2$, and if L is a line in $\mathbb{P}^2$, $L \neq Y$, show that $(L \cdot Y) = d$. Here we define $(L \cdot Y) = \sum_{P \in L \cap Y} (L \cdot Y)_P$, where $(L \cdot Y)_P$ is defined using a suitable affine cover of $\mathbb{P}^2$. (Hint: see problem 6 of the previous problem set.) *Possibly cut this.*

Date: October 28, 1999.

4. Suppose the characteristic of $\overline{k}$ is 0. Suppose a hypersurface $Y \subset \mathbb{P}^n$ is given by $f(x_0, \ldots, x_n) = 0$. Show that the locus of points $p \in \mathbb{P}^n$ where $\partial f/\partial x_i(p) = 0$ for all i are precisely the singular points of Y. (In particular, if $\partial f/\partial x_i(p) = 0$ for all i, then $f(p) = 0$, i.e. $p \in Y$! To see why, calculate $\sum_i \partial f/\partial x_i$; this algebraic trick is called Euler's Lemma.)
5. Consider the subvariety V of $\mathbb{A}^3$ cut out by the equation $z^2 = xy$. Explain why $\dim V = 2$. Let $R = \overline{k}[x, y, z]/(z^2 - xy)$ be its coordinate ring. Find (with proof) a subvariety W of codimension 1 whose coordinate ring is not of the form $R/(f)$ for any function $f \in R$. Show that W can be expressed as $V(f)$ for some $f \in R$. (f will cut out W with multiplicity 2.) Draw a picture of the situation.
6. Suppose $f : X \to Y$ is a morphism of varieties, with $f(p) = q$. Show that there are natural morphisms $T(f)^* : \mathfrak{m}_q/\mathfrak{m}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$ (the induced map on cotangent spaces) and $T(f) : (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \to (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$ (the induced map on tangent spaces). (If you imagine what is happening on the level of tangent spaces and cotangent spaces of smooth manifolds, this is quite reasonable.) If ϕ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map of tangent spaces at the origin is the zero map.
7. *Nonsingular schemes*
   (a) Show that both $\operatorname{Spec} \mathbb{Z}$ and $\operatorname{Spec} \mathbb{Z}[i]$ are nonsingular curves.
   (b) Let $\mathfrak{m} = (1+i)$ in $\mathbb{Z}[i]$. Then under the map $f : \operatorname{Spec} \mathbb{Z}[i] \to \mathbb{Z}$, $f(\mathfrak{m}) = (2)$. Check that the map on cotangent spaces (or equivalently, that the dual map on tangent spaces) is the zero-map. For all other primes of $\mathbb{Z}[i]$, calculate the map on cotangent spaces.
8. *Fibred products of affine schemes.* (This problem isn't as long as it looks!)
   (a) *Fibred products of sets.* Suppose X, Y, and Z are sets, and $f : X \to Z$ and $g : Y \to Z$ are (set) maps. Define a *fibred product of* X *and* Y *over* Z $X \times_Z Y$ as follows: it is a set W along with morphisms $p_1 : W \to X$ and $p_2 : W \to Y$, such that

$$\begin{array}{ccc} W & \to & Y \\ \downarrow & & \downarrow \\ X & \to & Z \end{array}$$

   is a commutative diagram, and for any maps $a : V \to X$, $b : V \to Y$, such that $f \circ a = g \circ b$ (i.e. the two maps $V \to Z$ are the same), there is a unique map $c : V \to W$ such that $a = p_1 \circ c$ and $b = p_2 \circ c$, and vice versa. (Translation: "maps $V \to X, Y$ that agree on Z correspond precisely to maps $V \to W$.)

   Prove that $W' = \{(x, y) \in X \times Y | f(x) = g(y) \in Z\}$ is a fibred product of X and Y over Z (so the above definition is an abstract nonsense way of saying something simple). Prove that if W is any fibred product, then there is a natural bijection $W = W'$. (Note that if Z is a one-element set, then $X \times_Z Y$ can be naturally identified with the product $X \times Y$.)
   (b) *Schemes.* Fibred products in arbitrary categories can be defined in the same way. Consider now the category of affine schemes (which is the same as the category of rings, with the arrows reversed). If $X = \operatorname{Spec} R$, and $Y = \operatorname{Spec} S$, $Z = \operatorname{Spec} T$, and f and g are induced by $f^* : T \to R$ and $g^* : S \to R$, explain why $W = \operatorname{Spec} R \otimes_T S$ is a fibred product of X and Y over Z. (You'll have to describe the morphisms p_1 and p_2.) Hence fibred products exist in the category of affine schemes. (One can then show that

fibred products exist in the category of schemes using the same argument as our proof that products exist in the category of prevarieties; see Hartshorne pp. 87–88.)

(c) *Varieties.* In the category of varieties, suppose the fibred product $W = X \times_Z Y$ exists. If $|Q|$ denotes the underlying set of a variety Q, show that $|W| = |X| \times_| Z||Y|$. (Translation: the points of the fibred product is/are the fibred product of the points.)

(d) *Fibred products of varieties don't always exist.* An affine scheme $\operatorname{Spec} R$ corresponds to an affine variety over $\overline{k}$ if R is a finitely generated $\overline{k}$ algebra that is a domain. Find an example of schemes X, Y, Z that correspond to varieties, and morphisms between them, such that the fibred product $X \times_Z Y$ in the category of affine schemes doesn't correspond to a variety. Hint: look in positive characteristic; you can even take X, Y, and Z to be the affine line. (This can easily be extended to prove that fibred products don't always exist in the category of affine varieties over $\overline{k}$, or even in the category of (pre)varieties over $\overline{k}$.)

(e) *A fun example, not for credit.* Calculate $\operatorname{Spec} \mathbb{C} \otimes_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$.