SOME AMUSING PROBLEMS RELATED TO ALGEBRAIC GEOMETRY

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Here are some problems to pique your interest; we will solve (or mostly solve) them during the semester. Some are deep, and others are just fun. You won't actually need algebraic geometry to solve a fair number of these problems, but if you solve them (or make progress), you'll secretly pick up a lot of algebro-geometrical insight.

1. *Infinite Chomp.* The game of Chomp is played as follows. Fix non-negative integers m and n. Cookies are placed in a rectangular array at the points (x, y) where $0 \leq x \leq m$ and $0 \leq y \leq n$ (x, y are integers). The cookie at (0,0) is poisoned. Two players alternate moving; a move involves picking a cookie, and eating it and every cookie above and to the right of it. The player who dies loses. (There's a neat proof that if m and n aren't both zero, the first player has a winning strategy, that doesn't reveal what that strategy is.) Clearly the game ends in a finite number of terms.

Infinite Chomp is the same, except cookies are placed at (x, y) where x and y run through all non-negative integers. Prove that the game is guaranteed to end in a finite number of terms. Generalize this to where cookies are placed on $(\mathbb{Z}^+)^n$ for any n.

(This is secretly related to the Hilbert Basis Theorem.)

2. Suppose $f_1(x_1, \ldots, x_n) = 0, \ldots, f_r(x_1, \ldots, x_n) = 0$ is a system of r polynomial equations in n unknowns, with integral coefficients, and suppose this system has a finite number of complex solutions. Show that each solution is algebraic, i.e. if $(x_1, \ldots, x_n)$ is a solution, then $x_i \in \overline{Q}$ for all i.

3. *Pascal's theorem.* If a hexagon is inscribed in an irreducible conic, then the opposite sides meet in collinear points.

4. *Pappas' theorem.* Let L_1 and L_2 be two lines. Let p_1, p_2, and p_3 be distinct points of L_1, and let q_1, q_2, and q_3 be distinct points of L_2 (none lying on $L_1 \cap L_2$). Let L_{ij} be the line between P_i and Q_j. For each i, j, k with $\{i, j, k\} = \{1, 2, 3\}$, let $R_k = L_{ij} \cap L_{ji}$. Then R_1, R_2, and R_3 are collinear.

5. We'll see that the integers $\mathbb{Z}$ should really be thought of as some weird curve.

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6. *Group law on an elliptic curve.* We'll define the group law on an elliptic curve, and see that it is associative. (We'll first define elliptic curves, which secretly come up in the next three problems.)

7. Find all rational solutions to $y^2 = ax^3 + bx^2$ where $a, b \in \mathbb{Q}$.

8. Prove that if $p(t)$ and $q(t)$ are polynomials (with complex coefficients) such that $p(t)^2 = q(t)^3 + 1$, then $p(t)$ and $q(t)$ are both constant.

9. *Poncelet's theorem.* Let C and D be two ellipses, with C contained in D. Pick a point p_0 on C, and draw a tangent to D from p_0, which meets C again at some other point p_1. Repeat this process (picking the "other" tangent from p_1). Suppose after n repetitions, you return to your starting point, i.e. $p_n = p_0$. Prove that this is true no matter where you start.

10. If $f(x)$ is a polynomial with integer coefficients, of degree $d = 2g + 1$ or $2g + 2$, square-free mod p, then

$$\left| \sum_{i=0}^{p-1} \left(\frac{f(i)}{p} \right) \right| \leq 2g\sqrt{p} + 1.$$

(The "fraction" denotes the symbol for quadratic residues.) If $g > 1$, the proof uses the Weil conjectures for curves. (In specific cases, an explicit formula can be given for the left-hand side.)

11. The degree of the discriminant $b^2 - 4ac$ of the quadratic $ax^2 + bx + c$ is 2. Show that the degree of the discriminant of a degree d polynomial is $2(d - 1)$.

12. Fix a positive integer d, and let $\sigma_1, \ldots, \sigma_n$ be n transpositions in S_d (i.e. they swap two elements of $\{1, \ldots, d\}$). Suppose they generate S_d, and

$$\sigma_1 \ldots \sigma_n = e,$$

where e is the identity. Show that $n \geq 2(d - 1)$.