COMMUTATIVE ALGEBRA USED IN INTRO TO A.G. COURSE (UP TO CLASS 16)

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Topics: Rings, localization, primes, completion, dimension theory.

Class 3: If k is a field, and B is an integral domain which is a finitely-generated k-algebra. Then $\dim B = tr.deg_k K(B)$ (where $K(B)$ is the quotient field of B), and for any prime ideal $\mathfrak{p}$ in B,

$$ht\mathfrak{p} + \dim B/\mathfrak{p} = \dim B.$$

(I can't remember if this was central to anything I said.

Class 11: The tensor product of two domains is also a domain. (In case of finitely-generated domain over $\overline{k}$, and the tensor is over $\overline{k}$.)

Class 13: Transcendence degree. Krull's Hauptidealsatz.

Class 15: If A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\overline{k}$, then $\dim_{\overline{k}} \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.

Class 16: Let $(R, \mathfrak{m})$ be a noetherian local domain of dimension one. Then the following are equivalent.

(i) R is a discrete valuation ring;
(ii) R is integrally closed (I'll speak about integral closures next day);
(iii) R is a regular local ring;
(iv) $\mathfrak{m}$ is a principal ideal.

Class 17: **Theorem (Integral closure is a local property).** If R is integrally closed if and only if the localization of R at each of its maximal ideals is integrally closed.

Theorem on finiteness of integral closure (in finite field extension).

Take any Dedekind domain R, and let K be its field of fractions. Let L be a finite extension of K, and let S be the integral closure of R in L. Then S is also a Dedekind domain. (Not used in this generality, but stated for the benefit of the subset of students who are thinking about schemes.) From Hartshorne I.6.3.

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Class 18: I may use the following fact (but this weekend I might try to come up with a proof of the special case I'll need). **Maximality of valuation rings.** If $(A, \mathfrak{m}_A)$, $(B, \mathfrak{m}_B)$ are local rings contained in a field K, we say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$) Let K be a field. A local ring R contained in K is a valuation ring of K iff it is a maximal element of the set of local rings contained in K, with respect to domination.