INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 1

RAVI VAKIL

1. Commutative algebra

Don't worry about it. See books on handout.

Ideas you should know: ring (commutative, has 1), field, integral domain, has quotient field, prime ideal, maximal ideal.

Sample problem (to appear on problem set):

Let A be a (commutative) ring. An element $a \in A$ is nilpotent (that is, $a^n = 0$ for some $n > 0$) if and only if a belongs to every prime ideal of A.

2. Algebraic sets

Throughout this course: k is a field. $\overline{k}$ is an algebraically closed field.

For now we work over $\overline{k}$. Feel free to think of this as $\mathbb{C}$ for now.

$\overline{k}^n$ will be rewritten $\mathbb{A}^n(\overline{k})$, affine n-space; we'll often just write $\mathbb{A}^n$ when there's no confusion about the field. Coordinates x_1 to x_n.

Algebraic geometry is about functions on the space, which form a ring. The only functions we will care about will be polynomials, i.e. $\overline{k}[x_1, \dots, x_n]$. We'll eventually think of that ring as being the same thing as $\mathbb{A}^n$.

We'll next define subsets of $\mathbb{A}^n$ that we'll be interested in. Because we're being very restrictive, we won't take any subsets, or even analytic subsets; we'll only think of subsets that are in some sense defined in terms of polynomials.

Let S be a set of polynomials, and define $V(S)$ to be the locus where these polynomials are zero. ("Vanishing set".) Definition: Any subset of $\mathbb{A}^n(\overline{k})$ of the form $V(S)$ is an *algebraic set*.

Exercise (to appear on problem set): prove that the points of the form (t, t^2, t^3) in $\mathbb{A}^3$ form an algebraic set. In other words, find a set of functions that vanish on these points, and no others.

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Example/definition: *hypersurface*, defined by 1 polynomial.

Facts.

- If $I = (S)$, then $V(I) = V(S)$. So we usually will care only about ideals. Hence: subsets of $\overline{k}[x_1, ..., x_n]$ give us subsets of $\mathbb{A}^n$; specifically, *ideals* give us *algebraic sets*
- $V(\cup I_a) = \cap V(I_a)$ (Say it in english.)
- $I \subset J$, then $V(I) \supset V(J)$
- $V(FG) = V(F) \cup V(G)$

Note: Points are algebraic. Finite unions of points are algebraic.

Definition. A radical of an ideal $I \subset R$, denoted $\sqrt{I}$, is defined by

$$\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n\}.$$

Exercise. Show that $\sqrt{I}$ is an ideal.

Definition. An ideal I is *radical* if $I = \sqrt{I}$.

Claim. $V(\sqrt{I}) = V(I)$. (Explain why.)

Conversely, subsets of $\mathbb{A}^n$ give us a subset of $\overline{k}[x_1, ..., x_n]$ For each subset X, let $I(X)$ be those polynomials vanishing on X.

Claim. $I(X)$ is a radical ideal. (Explain.)

Facts. If $X \subset Y$, then $I(X) \supset I(Y)$. $I(\emptyset) = \overline{k}[x_1, ..., x_n]$. $I(\mathbb{A}^n) = (0)$.

Question. What's $I((a_1, ..., a_n))$?

(Discuss.)

Notice: ideal is maximal. Quotient is field. Quotient map can be interpreted as "value of function at that point".

Exercise. (a) Let V be an algebraic set in $\mathbb{A}^n$, P a point not in V. Show that there is a polynomial F in $\overline{k}[x_1, ..., x_n]$ such that $F(Q) = 0$ for all Q in V, but $F(P) = 1$. Hint: $I(V) \neq I(V \cup P)$.

(b) Let $\{P_1, ..., P_2\}$ be a finite set of points in $\mathbb{A}^n(\overline{k})$. Show that there are polynomials $F_1, ..., F_r \in \overline{k}[x_1, ..., x_n]$ such that $F_i(P_j) = 0$ if $i \neq j$, and $F_i(P_i) = 1$.

Exercise. Show that for any ideal I in $\overline{k}[x_1, ..., x_n]$, $V(I) = V(\sqrt{I})$, and $\sqrt{I}$ is contained in $I(V(I))$.

3. Nullstellensatz (theorem of zeroes)

Earlier, we had: algebraic sets → radical ideals and ideals → algebraic sets.

This theorem makes an equivalence. In the literature, the word "nullstellensatz" is used to apply to a large number of results, not all of them equivalent.

Nullstellensatz Version 1. Suppose $F_1, \dots, F_m \in \overline{k}[x_1, \dots, x_n]$. If the ideal $(F_1, \dots, F_m) \neq (1) = \overline{k}[x_1, \dots, x_n]$ then the system of equations $F_1 = \cdots = F_m = 0$ has a solution in $\overline{k}$.

Proof next day. (There is a better version for fields that are not necessarily algebraically closed, but we're not worrying about that right now.)

Nullstellensatz Version 2. Supopse $\mathfrak{m}$ is a maximal ideal of $\overline{k}[x_1, \dots, x_n]$. Then

$$\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$$

for some $a_1, \dots, a_n \in \overline{k}$.

Show that this is equivalent to version 1, modulo fact that ideals are finitely generated.

Nullstellensatz Version 3 (sometimes called the "Weak Nullstellensatz"). If I is a proper ideal in $\overline{k}[x_1, ..., x_n]$, then $V(I)$ is nonempty. (From Version 2.)

Nullstellensatz Version 4. Let I be an ideal in $\overline{k}[x_1, ..., x_n]$. Then $I(V(I)) = \sqrt{I}$. Equivalently: Radical ideals are in 1-1 correspondence with algebraic sets: If I is a radical ideal in $\overline{k}[x_1, ..., x_n]$ then $I(V(I)) = I$. So there is a 1-1 correspondence between radical ideals and algebraic sets.

Nullstellensatz Version 5. A radical ideal of $\overline{k}[x_1, \dots, x_n]$ is the intersection of the maximal ideals containing it. This is the geometric rewording of 4. By version 4, a radical ideal is $I(X)$ for some algebraic set X. Functions vanishing on X are precisely those functions vanishing on all the points of X.

Nullstellensatz Version 6. If F_1, ..., F_r, G are in $\overline{k}[x_1, ..., x_n]$, and G vanishes wherever F_1, ..., F_r vanish, then there is an equation $G^N = A_1F_1 + ... + A_rF_r$ for some $N > 0$ and some A_i in $\overline{k}[x_1, ..., x_n]$.

This has a cute proof, with a useful trick in it.

Proof. The case $G = 0$ is obvious, so assume $G \neq 0$. Introduce a new variable U, and consider the polynomials

$$F_1, \dots, F_m, \text{ and } UG - 1 \in \overline{k}[x_1, \dots, x_n, U].$$

They have no common solutions in $\overline{k}$, so by Version 1 they generate the unit ideal, so there are polynomials $P_1, \dots, P_m$, $Q \in \overline{k}[x_1, \dots, x_n, U]$ such that

$$P_1F_1 + \cdots + P_mF_m + Q(UG - 1) = 1.$$

Now set $U = 1/G$ in this formula, and multiply by some large power G^N of G to clear denominators. Then the right side is G^N, and the left side is in $(F_1, \dots, F_m)$.

Next day. Recap what we know: we're studying $\mathbb{A}^n(\overline{k})$, via the ring $\overline{k}[x_1, \dots, x_n]$ (the "ring of functions" we're interested in). Algebraic sets correspond to radical ideals. Maximal ideals correspond to points. We'll do some examples of how this dictionary works. We'll prove the Nullstellensatz (a proof you needn't know, but one you should see).

In the next major section, we'll address the question: why is each algebraic sets cut out by a finite number of equations? In answering this, we'll talk about Noetherian rings, the Hilbert Basis Theorem, and the game of Chomp. I'll give out the first problem set.