1. Extending rational maps of nonsingular curves

1.1. More on integral closure in a field extension. I neglected to say a few small facts about integral closure in a field extension. Recall:

Let $A$ be an integral domain, which is a finitely generated algebra over $k$. Let $K$ be the quotient field of $A$, and let $L$ be a finite algebraic extensions of $K$. Then the integral closure of $A$ in $L$ consists of those elements of $L$ satisfying $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in A$.

For example, the integral closure of $A$ in its quotient field is the integral closure of $A$.

It isn’t hard to show that integral closure $A'$ of $A$ in $L$ must be integrally closed, i.e. the only solutions to such equations in $L$, where now the $a_i$ are supposed to lie in $A'$, lie in $A'$.

Remark 1. $R \subset S$. Indeed, if $r \in R$, then it is a solution to the equation $x - r = 0$.

Remark 2. The quotient field of $A'$ is $L$. This follows from:

Remark 3. If $l \in L$, then some multiple of it $al$ ($a \in A$) is in $A'$. Proof. $l$ satisfies some $l^n + a_{n-1}l^{n-1} + \cdots + a_0 = 0$ for some $a_i \in K$. Clear denominators,
to write \( b_n l^n + b_{n-1} l^{n-1} + \cdots + b_0 = 0 \) for some \( b_i \in A \). Then rewrite as \( (b_n l)^n + b_{n-1} b_n l^{n-1} + \cdots + b_0 b_n^n = 0 \). Thus \( b_n l \in A' \).

Remark 4. A reminder: A theorem I proved (using a couple of theorems I didn’t prove), that we’ll use today:

**Theorem.** Take any Dedekind domain \( R \) that is a finitely generated algebra over \( \overline{\mathbb{k}} \), and let \( K \) be its field of fractions. Let \( L \) be a finite extension of \( K \), and let \( S \) be the integral closure of \( R \) in \( L \). Then \( S \) is also a Dedekind domain, and is also a finitely generated algebra over \( \overline{\mathbb{k}} \).

1.2. **Last time.** Goal: Rational maps of nonsingular curves to projective varieties can be extended to morphisms.

We discussed reasons why you can’t extend \( \mathbb{P}^1 \rightarrow \mathbb{A}^1, \mathbb{A}^2 \rightarrow \mathbb{P}^1 \).

**Lemma.** Let \( Y \) be a prevariety, and suppose \( P \) and \( Q \) are two points contained in a single affine open \( U \), and \( \mathcal{O}_{Y, P} \subseteq \mathcal{O}_{Y, Q} \) (as subrings of \( k(Y) \)). Then \( P = Q \).

**Key Technical Theorem (tricky).** Let \( K \) be a finitely generated function field of dimension one over \( k \), and let \( x \in K \). Then the set of discrete valuations of \( K/\overline{k} \) where \( v(x) < 0 \) is finite.

Warning: We’re going to get a lot of mileage out of the proof this, so pay attention to it, even though it will get heavy!

Here’s the geometric idea. Suppose \( K = \overline{k}(t) \), the function field of \( \mathbb{P}^1 \). \( x \in K \), so \( x \) is a rational function of \( \mathbb{P}^1 \). Then the discrete valuations of \( K/\overline{k} \) correspond to points of \( \mathbb{P}^1 \) (exercise). The points/valuations \( v_p \) where \( v_p(x) < 0 \) are precisely the points where \( x \) has poles; there are only a finite number of such points/valuations.

1.3. **New material starts here.** The proof will use the following:

**Commutative algebra lemma.** Suppose \( (S, \mathfrak{n}) \), \( (R, \mathfrak{m}) \) are two discrete valuation rings of \( K/\overline{k} \), with \( S \subseteq R \) and \( \mathfrak{n} = S \cap \mathfrak{m} \). Then \( S = R \) and \( \mathfrak{n} = \mathfrak{m} \): they are the same valuation ring.

(Compare to lemma just above.)

**Proof.** The two valuation rings give two discrete valuations \( v_n \) and \( v_m \) on \( K \) (so \( S \) are the elements \( s \) of \( K \) with \( v_n(s) \geq 0 \), and \( R \) are the elements \( r \) of \( K \) with \( v_m(r) \geq 0 \); we can recover the DVRs from the valuations.) Let \( u \) be a uniformizer of \( S \) (i.e. a generator of \( \mathfrak{n} \)), so \( v_n(u) = 1 \) (and \( v_m(u) > 0 \)). Suppose \( r \in K \), \( r \notin S \); we’ll see that \( r \notin R \). As \( v \notin S \), \( v_n(r) < 0 \), so \( r = u^{-n} w \) with \( n > 0, v_n(w) = 0 \), so \( w \in S \setminus \mathfrak{n} \). Then \( w \in R \setminus \mathfrak{m} \), hence \( v_m(w) = 0 \). But then \( v_m(r) = -nv_n(u) + v_m(w) < 0 \), so \( r \notin R \) as desired. \( \square \)
Proof of “Key Technical Theorem”. i) The set-up. Note that if \((R, \mathfrak{m}_R)\) is a valuation ring of \(K/\kbar\), then if \(x\) is not in \(R\) then it has negative valuation, so \(y := 1/x\), having positive valuation, is in \(\mathfrak{m}_R\). So we have to show that if \(y \in K\), \(y \neq 0\), then the set of discrete valuations where \(y \in \mathfrak{m}_R\) is a finite set. (Geometric picture: show that a non-zero rational function on \(\mathbb{P}^1\) has only a finite number of zeroes.) If \(y \in \kbar\), then \(v(y) = 0\) for all discrete valuations of \(K/\kbar\), so we can assume \(y\) is not in \(\kbar\). (Geometric picture: can ignore constant functions.)

\[\text{ii) Part of proof involving } y. \text{ Consider } \kbar[y], \text{ the subring of } K \text{ generated by } y. \text{ Since } \kbar \text{ is algebraically closed, } y \text{ is transcendental over } \kbar, \text{ so } \kbar[y] \text{ is a polynomial} \]

\[\text{ring, i.e. } y \text{ doesn’t satisfy any relations. Hence } K \text{ is a finite field extension of } \kbar(y). \text{ Let } B \text{ be the integral closure of } \kbar[y] \text{ in } K. \text{ Then by an earlier theorem proved last day (and repeated as Remark 4 above), } B \text{ is a Dedekind domain, and is also a finitely generated algebra over } \kbar.\]

Hence \(B\) corresponds to an affine variety \(Y\), which maps to \(\mathbb{A}^1\) with coordinate \(y\); draw it! It is dimension 1 and nonsingular. Also, \(y\) is a function on \(Y\).

\[\text{iii) Now bring in the valuation. If } y \text{ is contained in a discrete valuation ring } (R, \mathfrak{m}_R) \text{ of } K/\kbar, \text{ then } \kbar[y] \subset R, \text{ and since } R \text{ is integrally closed in } K, \text{ we have } B \subset R \text{ as well. Let } n = \mathfrak{m}_R \cap B. \text{ Then } n \text{ is a maximal ideal of } B. \text{ (Hence it corresponds to a point of } Y \text{ mapping to zero.) Also, } B_n \subset R. \text{ And also } nB_n \subset \mathfrak{m}_R. \text{ Now } B_n \text{ is also a } DVR \text{ of } K/\kbar, \text{ so } B_n = R \text{ by the Commutative Algebra Lemma.}\]

Hence if \(y\) is in \(\mathfrak{m}_R\), then \(y\) is in \(n\). To say that \(y\) is in \(n\), means that \(y\), as a regular function on \(Y\), vanishes at the corresponding point. As non-zero functions vanish at only finitely many points (closed zero-dimensional subsets are only finite numbers of points), we’re done. \(\square\)

From the proof, we can also extract another result, which we can use later.

**Corollary.** Let \(v\) be a discrete valuation of the field \(K/\kbar\), a finitely-generated function field of transcendence degree 1. Then there is a nonsingular curve \(C\), with function field \(K\), and a point \(p \in C\), such that \(v\) is the valuation induced by \(p\).

**Proof.** Take any \(y \in K\) of positive valuation. Then construct \(Y\) as in the previous proof. \(\square\)

**Corollary.** Given \(y \in K \setminus \kbar\), all discrete valuations of \(K/\kbar\) such that \(v(y) \geq 0\) (i.e. that \(y\) is in the corresponding DVR) are accounted for by points on the curve \(Y\) above.

*(Remark: just proof above.)*

The only “missing” valuations are those for which \(v(y) < 0\), so by the Key Technical Theorem, we’re only missing a finite number.
Proof. Let \((R, m)\) be the corresponding DVR. As \(v_m(y) \geq 0, y \in R\). Then let \(a \in \overline{k}\) be the residue of \(y\) modulo \(m\), i.e. \(y \equiv a \pmod{m}\). If \(a = 0\), then the proof above shows that \((R, m)\) is the local ring of one of the points of \(Y\) mapping to 0, so we’re done.

If \(a \neq 0\), then replace \(y\) with \(y - a\), and we’re done again. \(\Box\)

This confused them.

1.4. Extension of morphisms to projective varieties, over nonsingular points of curves. Now we’re ready to prove the main result of this section.

Theorem. Let \(X\) be a nonsingular curve, \(p\) a point of \(X\), \(Y\) a projective variety, and \(\phi : X - p \to Y\) a morphism. Then there exists a unique morphism \(\tilde{\phi} : X \to Y\) extending \(\phi\).

By separatedness, as \(Y\) is a variety, we have uniqueness; all that is necessary is existence.

Last day, we discussed an example of a map from \(\mathbb{P}^1\) to \(\mathbb{P}^n\): \([x; y] \mapsto [x^3; x^2y; xy^2]\) is defined away from \([0; 1]\); it’s clear how to extend. You divide by \(x\). We’ll carry this example along as we do the proof.

Hence this is l’Hopital’s rule in a vague sense.

Proof. It is sufficient to show that \(f\) extends to a morphism of \(X\) into \(\mathbb{P}^n\) (with coordinates \(x_0, \ldots, x_n\)). Let \(U\) be the open set where all \(x_i\) are non-zero. By changing coordinates if necessary, assume \(f(X - p)\) meets \(U\).

For each \(i, j\), \(x_i/x_j\) is a regular function on \(U\); pulling back by \(f\), we have regular function \(f_{ij}\) on an open subset of \(X\), which we view as a rational function on \(X\), so \(f_{ij}\) is in \(k(X)\).

Let \(v\) be the valuation associated with \(p\). Let \(r_i = v(f_{i0})\) for \(i = 0, 1, \ldots, n\). Then \(v(f_{ij}) = r_i - r_j\). Choose \(r_k\) minimal, so \(v(f_{ik}) \geq 0\).

Consider the map of sets \(\tilde{\phi} : X \to \mathbb{P}^n\), given by \(\tilde{\phi}\) is the same as \(\phi\) on \(X - p\), and \(\tilde{\phi}(p) = (f_{0k}(p); \ldots; f_{nk}(p))\) (observe that not all coordinates are 0). I claim this is a morphism. To show this, I need only show an affine neighbourhood \(X_{\text{aff}}\) of \(X\) that maps pointwise to an affine \(U\) of \(\mathbb{P}^n\), such that the pullback of every regular function on \(U\) is a regular function on \(X_{\text{aff}}\).

Let \(U = U_k\) be the open set where \(x_k \neq 0\) (so \(\tilde{\phi}(p) \in U_k\), since \(f_{kk}(p) = 1\)). The coordinate ring of the affine \(U_k\) is \(k[x_0/x_k, \ldots, x_i/x_k, \ldots, x_n/x_k]\). These functions pull back to \(f_{0k}, \ldots, f_{nk}\) which are regular at \(p\) by construction. Let \(X_{\text{aff}}\) be any affine neighbourhood of \(p\) where the rational functions \(f_{ik}\) are defined.
So we’re done.

As consequences, we will get (next day):

**Theorem.** Every nonsingular separated curve $C$ is quasiprojective.

**Corollary.** Hence every nonsingular separated curve is birational to a projective curve.

**Proposition.** Let $C_1$ and $C_2$ be two separated nonsingular curves, and let $\alpha : C_1 \to C_2$ be a birational map between them. Then they can be glued together via $\alpha$, i.e. there is another nonsingular curve $C = U_1 \cup U_2$, with $C_i$ isomorphic to $U_i$, and the birational map $U_1 \to U_2$ induced by $\alpha$.

**Proposition.** If two nonsingular projective curves $C_1$ and $C_2$ are birational, then they are isomorphic.

Coming up: We’ll discuss these, and begin talking about why finitely-generated fields of transcendence dimension 1 correspond to nonsingular projective curves (over $\bar{k}$). started just before the introduction