INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 14

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Problem sets back at end.

If pages 26–27 of Eisenbud-Harris are missing in your copy, let me know; Alex Ghitza has kindly made a bunch of copies.

Come by and ask me questions!

Last time I said that we would call a dominant rational map $Y \rightarrow X$ finite if the induced morphism of function fields $k(X) \rightarrow k(Y)$ is a finite extension. That’s not really in keeping with standard usage, so let me redefine it as generically finite.

1. Dimension

I’ll start by reviewing what I mentioned last time on dimension.

But first, let me make a few algebraic remarks, on things such as transcendence degrees of finitely-generated field extensions. You should think about the statements I make; I think you’ll find them all believable. Of course, one needs to properly prove everything. If you’ve already seen these remarks, great; if you’re taking Commutative Algebra, that’s great too; and otherwise, you should convince yourself that the statements are reasonable, and treat them as black boxes.

Date: Tuesday, October 26, 1999.
1.1. **Last time. Definition.** If $X$ is a prevariety defined over $\overline{k}$, define $\dim X = tr.d_{\overline{k}}k(X)$. If $Z$ is a closed subset of $X$, then $Z$ has pure dimension $r$ if each of its components has dimension $r$. A variety of dimension 1 is a curve, a variety of dimension 2 is a surface, a variety of dimension $n$ is an $n$-fold.

**Observation.** If $U \subset X$ is a nonempty open subset, then $k(U) = k(X)$, so $\dim U = \dim X$.

I mentioned an algebraic fact, which I proved (albeit imperfectly).

**Lemma.** Let $R$ be an integral domain over $k$, $p \subset R$ a prime ideal. Then $tr.d_{\overline{k}}R \geq tr.d_{\overline{k}}R/p$, with equality iff $P = \{0\}$ or both sides are infinite.

Geometrically, this translates into:

**Proposition.** If $Y$ is a closed subprevariety of $X$, then $\dim Y < \dim X$.

Summary of proof: can reduce to an affine open meeting $Y$ by earlier observation, then use lemma.

**Definition.** Call the difference the codimension of $Y$ in $X$.

So for example, if the codimension is 1, there are no other subvarieties in between $Y$ and $X$.

I then quoted a result from commutative algebra:

**Theorem (Krull’s Hauptidealsatz, or Principal Ideal Theorem).** Suppose $R$ is a finitely generated integral domain over $\overline{k}$, $f \in R$, $p$ a minimal prime of $(f)$ (i.e. minimal among the prime ideals containing it). Then if $f \neq 0$, $tr.d._R R = tr.d._R R - 1$. (Proof omitted.)

Let me repeat why, in geometrical situations, this is very reasonable. For example, let $R = \overline{k}[x, y, z]$, and let $f$ be some polynomial, say $xy(x - y^3 - z^4)$. Notice that the vanishing set $V(f) = \{f = 0\}$ consists of two planes and this weird surface. The minimal primes containing $(f)$ correspond to maximum subvarieties of $\mathbb{A}^3$ contained in the vanishing set of $f$, so there are 3 of them. Intuitively, it is reasonable that all 3 have dimension 2.

The immediate geometric consequence of this is:

**Theorem.** Let $X$ be a variety, $U \subset X$ open, $g \in \mathcal{O}_X(U)$ a regular function on $U$, $Z$ an irreducible component of $V(g) \cap U$. Then if $g \neq 0$, $\dim Z = \dim X - 1$.

($U$ is a red herring, and doesn’t add any complexity to the proof. Essentially: $V(g)$ has pure codimension 1 for any non-zero $g \in \mathcal{O}_X(X)$.)

**Proof.** Take $U_0 \subset U$ to be any open affine meeting $Z$. Let $R = \mathcal{O}_X(U_0)$ be its coordinate ring, and $f = \text{res}_{U \cap U_0} g \in R$ the restriction of our function $g$. 

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Then $Z \cap U_0$ (being irreducible) corresponds to some prime ideal $\mathfrak{p} \subset R$. $Z$ is a maximal irreducible subset of $V(g) \subset U$, so $Z \cap U_0$ is a maximal irreducible subset of $V(f) \subset U_0$, so $\mathfrak{p}$ is a minimal prime containing $f$, and we’re in the situation of the Hauptidealsatz.

Conversely, if $Z$ is an irreducible closed subset of $X$ of codimension 1, then for any open $U$ meeting $Z$ and for all non-zero functions $f$ on $U$ vanishing on $Z$, $Z \cap U$ is a component of $f = 0$.

**Corollary.** If $X$ is a variety, with subvariety $Z$ of codimension at least 2. Then there is a subvariety of $W$ of codimension 1 containing $Z$.

**Proof.** We can restrict to an affine open meeting $Z$, so without loss of generality assume $X$ is affine. Then $Z$ corresponds to a prime ideal $\mathfrak{p} \neq (0)$. Let $f$ be any nonzero element of $\mathfrak{p}$. Then all the components of $V(f)$ have codimension 1 (by the Principal Ideal Theorem). Hence $Z$ isn’t a component, so it must be contained in one.

**Corollary.** If $X$ is a variety, and $Z$ is a maximal closed irreducible subset, smaller than $X$. Then $\dim Z = \dim X - 1$.

**Corollary.** Suppose $\emptyset \neq Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_r = X$ (where no $\subset$ is an inclusion) is a maximal chain of closed irreducible subsets of $X$. Then $\dim X = r$.

**Proof.** By induction on $X$.

(Discuss a bit.)

**Remark.** This is a good initial definition to make for schemes; our original variety definition involves the field $\bar{k}$. Exercise (possibly on the next problem set). Check that Spec $\mathbb{Z}$ is a curve.

**Corollary.** Let $X$ be a variety and let $Z$ be a component of $V((f_1, \ldots, f_r))$, where $f_i \in \mathcal{O}_X(X)$. Then $\text{codim} Z \leq r$.

(Explain.)

**Corollary.** Let $U$ be an affine variety, $Z$ a closed irreducible subset. Let $r = \text{codim } Z$. Then there are $f_1, \ldots, f_r$ in $A(U) = \mathcal{O}_U(U)$ such that $Z$ is a component of $V((f_1, \ldots, f_r))$.

**Proof.** We did the case $r = 1$ earlier, and the proof is the same.

### 1.2. An algebraic definition of dimension.
Given a Noetherian local ring $\mathcal{O}$, you can attach an integer called the *Krull dimension*. (This will come up towards the end of this semester’s commutative algebra class.)
It’s defined as the length \( r \) of the longest chain of prime ideals \( P_0 \subset P_1 \subset \cdots \subset P_n = m \subset \mathcal{O} \). (Here \( m \) is the maximal ideal.)

**Corollary.** The dimension of a variety \( X \) is the Krull dimension of any of the stalks of the structure sheaf.

**Proof.** Fix a point \( p \). Translating the Krull definition into geometry, we’re asking about the longest chain of subvarieties of \( X \) containing \( p \). But we already know that this is \( \dim X \). \( \square \)

1.3. **Other facts that are not hard to prove.** (Proofs are omitted for the first 2.)

**Proposition.** If \( X \) is an affine variety with coordinate ring \( R \), where \( R \) is a unique factorization domain. Then every closed codimension 1 subset equals \( V((f)) \) for some \( f \in R \).

**Proposition.** \( \dim X \times Y = \dim X + \dim Y \).

**Proposition.** The Zariski topology on a dimension 1 prevariety is the cofinite topology.

(Explain.) We’ll use this later when we study curves.

2. **Non-singularity: a beginning**

For a reference, see Hartshorne I.5 or Shafarevich, the start of Ch. II.

Some intuition, in the classical topology. Consider the plane curve \( y = x + x^2 \) in \( \mathbb{C}^2 \). Why is it smooth? What is the tangent line? (Discuss.)

What about \( y = x + z + y^2 \), \( x = y + z + x^3 \)? How about \( y = x + z + y^2 \), \( y = x + z + y^4 \)?

There are no constant terms. All we care about are the linear terms. In the ring \( k[x, y, z] \) with maximal ideal \( m \), we care about \( m/m^2 \).

Classically, something is smooth of dimension \( n \) if there is a local isomorphism with \( \mathbb{C}^n \). I’ll let you check that this is the same as the following definition.

**Definition.** Let \( Y \) be a dimension \( d \) affine variety in \( \mathbb{A}^n \) (with coordinates \( x_1, \ldots, x_n \)). Suppose \( Y \) is defined by equations \( f_1, \ldots, f_t \) (i.e. \( I(Y) \) is generated by the \( f_i \); recall that any ideal in \( k[x_1, \ldots, x_n] \) is finitely generated!). Warning: We know that \( t \) is at least the codimension \( n - d \), but they two aren’t necessarily equal! Then \( Y \subset \mathbb{A}^n \) is **nonsingular at a point** \( p \in Y \) if the rank of the Jacobian matrix \( (\partial f_i/\partial x_j(p))_{i,j} \) is \( n - d \).
I still must convince you that this is a reasonable definition, and in particular, that this agrees with the old definition. I’ll let you check that in the classical topology, if \( Y \) is nonsingular at \( a \) in this sense, then there is a (classical) neighbourhood of \( a \) isomorphic to an open set in \( \mathbb{C}^n \). I’ll do an example first.

Consider our earlier example, \( x - y + z + y^2 = 0, -x + y + z + x^3 = 0 \). So our point is the origin, \( f_1 = x - y + z + y^2, f_2 = -x + y + z + x^3 \). The two intersect in a curve, of dimension 1. The Jacobian matrix is

\[
J = \begin{pmatrix}
1 & -1 & 1 \\
-1 & 1 & 1 \\
\end{pmatrix}
\]

Indeed the rank of the matrix \( J \) is 3-1=2.

I argued (with too much hand-waving) that the implicit function theorem shows (in the case \( k = \mathbb{C} \)) that the Jacobian condition implies that \( X \) is a manifold at \( p \); in the case above, consider the inverse of the projection \((x, y, z) \mapsto t = x + y + z\).

2.1. A more algebraic definition of nonsingularity; hence nonsingularity is intrinsic. You’d think that the nonsingularity of a point of \( Y \) wouldn’t depend on how you stuck it in an affine space, and you’d be right; but the above definition does depend on that, so it isn’t clear that nonsingularity really is intrinsic. We’ll show this now.

**Algebraic Definition.** Let \( A \) be a noetherian local ring with maximal ideal \( m \) and algebraically residue field \( K \). Then \( A \) is a regular local ring if \( \dim_k m/m^2 = \dim A \).

The reason this will be relevant is:

**Theorem.** Let \( Y \subset \mathbb{A}^n \) be an affine variety. Let \( p \in Y \) be a point. Then \( Y \) is nonsingular at \( p \) if and only if the local ring \( \mathcal{O}_{Y,p} \) is a regular local ring.

We’ll prove this soon.

Thus the concept of nonsingularity is intrinsic, so we can make the following definitions:

**Definition.** Let \( Y \) be any prevariety. Then \( Y \) is nonsingular at a point \( p \in Y \) if the local ring \( \mathcal{O}_{Y,p} \) is a regular local ring; otherwise it is singular at \( p \). \( Y \) is nonsingular if it is nonsingular at any point. Otherwise it is singular.

**Remark.** To check that something is singular, it is still easier to use the Jacobian definition. We make this more general definition because it is, well, more general.

**Theorem.** Let \( A \) be the localization of \( \mathbb{K}[x_1, \ldots, x_n] \) at the origin, so \( A \) has dimension \( n \). Then \( m/m^2 \) are naturally isomorphic to the vector space \( (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n \) (call it \( V \)), where points of the vector space can be associated with linear forms \( \alpha_1 x_1 + \ldots + \alpha_n x_n \). Hence \( A \) is a regular local ring.
The proof will be enlightening (hopefully) for several reasons.

Proof. The morphism \( V \to m/m^2 \) is just given by
\[
(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 x_1 + \ldots \alpha_n x_n.
\]
The morphism \( m/m^2 \) is given by
\[
f \in A \mapsto (\partial f/\partial x_1(0, \ldots, 0), \ldots, \partial f/\partial x_n(0, \ldots, 0))
\]
(where \( f \) vanishes at the origin). To show that this is well-defined, we need to check that if \( f \in m^2 \), then \( f \mapsto (0, \ldots, 0) \). But if \( f \in m^2 \), then \( f = \sum_i g_i h_i \), where \( g_i, h_i \) are in \( m \). Then by the chain rule, \( \partial(g_i h_i)/\partial x_1 = g_i \partial h_i/\partial x_1 + h_i \partial g_i/\partial x_1 \), so indeed if \( f \in m^2 \) then \( f \mapsto (0, \ldots, 0) \).

Finally, we need to show that they compositions are the identity. The map \( V \to m/m^2 \to V \) is the identity (show it), so what’s left is to show that \( m/m^2 \to V \to m/m^2 \) is also the identity; this comes down to the fact that if \( f \mapsto 0 \), then \( f \in m^2 \).

Important observation. Notice that the elements of \( m/m^2 \) are naturally identified with linear functions on \( A^n \). Now \( A^n \) can canonically be identified with the tangent space of \( A^n \) at the origin. So we’ve made an identification of \( m/m^2 \) with the cotangent space of \( A^n \) at the origin.

Based on this observation:

Definition. Let \((A, m)\) be the local ring of a point \( p \in Y \). Call \( m/m^2 \) the Zariski co-tangent space to \( Y \) at \( p \), and \((m/m^2)^*\) the Zariski tangent space.

Exercise (that I will give on Thursday). Suppose \( f : X \to Y \) is a morphism of varieties, with \( f(p) = q \). Show that there are natural morphisms \( f^* : m_q/m_q^2 \to m_p/m_p^2 \) (the induced map on cotangent spaces) and \( f^* : (m_q/m_q^2)^* \to (m_p/m_p^2)^* \) (the induced map on tangent spaces). (If you imagine what is happening on the level of tangent spaces and cotangent spaces of smooth manifolds, this is quite reasonable.) If \( \phi \) is the vertical projection of the parabola \( x = y^2 \) onto the \( x \)-axis, show that the induced map of tangent spaces at the origin is the zero map.

Coming next: Examples. Checking for nonsingularity in projective space. The singular points form a closed subset.