1. Products

We know what we mean when we discuss products of sets. We'll now define products more generally.

_Categorical products_. Suppose you have two sets $X$ and $Y$. Then the product set $Z$ has two natural “projection” maps $p_X$ and $p_Y$ to $X$ and $Y$ respectively. Moreover, if you have any other set $W$ with maps to both $X$ and $Y$, then there...
is a unique map $W \rightarrow Z$ such that the maps to $X$ and $Y$ can be obtained by composition with the projections.

**Definition.** Suppose $X$ and $Y$ are objects in a category. Then a *product of $X$ and $Y$* is the data of another object $Z$, along with projection maps $p_X : Z \rightarrow X$ and $p_Y : Z \rightarrow Y$, such that for any other object $W$, maps from $W$ to $Z$ correspond to the data of maps from $W$ to $X$, and maps from $W$ to $Y$. Equivalently,

$$\text{Mor}(Z, W) \rightarrow \text{Mor}(Z, X) \times \text{Mor}(Z, Y)$$

is a bijection.

You should check that this works with sets, i.e. in the category of sets.

**Remark.** Then the product is defined up to unique isomorphism.

**Products in the category of topological spaces.** Define product topology. Explain why this is called the product topology: this is a product in the category of topological spaces.

**Products in the category of prevarieties?**

If there is a product of $X$ and $Y$ in the category of prevarieties, then the points of the product are the products of the points. Reason: take $Z$ to be a point.

To save you the suspense: products indeed exist, and we’ll construct them.

Topologies will be weird. It will turn out that $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, but the Zariski topology on $\mathbb{A}^2$ isn’t the same as the product of the Zariski topologies on the factor (Exercise).

1.1. **Products in the category of affine varieties, and in the category of varieties.** Recall that the category of affine varieties (over $\overline{k}$) is just the category of “nice” rings, with the arrows reversed, where nice means “finitely generated algebra over $\overline{k}$ that is an integral domain”.

Let $X$ and $Y$ be affine varieties (over $\overline{k}$). Let’s diagram-chase.

If we have some “nice” ring $A(W)$, the question is: given maps (ring morphisms over $\overline{k}$) $A(X) \rightarrow A(W)$ and $A(Y) \rightarrow A(W)$, is there some ring $A(X \times Y)$ along with morphisms $A(X), A(Y) \rightarrow A(X \times Y)$ through which this has to factor? Answer: Yes, $A(X) \otimes_{\overline{k}} A(Y)$, with the morphism $A(X) \rightarrow A(X) \otimes_{\overline{k}} A(Y)$ given by $x \mapsto x \otimes 1$, and similarly for $A(Y)$. (Remind them what tensor product is; show that tensor product has this property.)

All that’s left to show is that the tensor product is also “nice”.
Finitely-generated is easiest: if \( f_1, \ldots, f_m \) are generators for \( A(X) \) and \( g_1, \ldots, g_n \) are generators for \( A(Y) \), then \( f_i \otimes 1 \) and \( 1 \otimes g_j \) generate the tensor product.

Then we invoke a fact from commutative algebra: Let \( R \) and \( S \) be integral domains over \( k \). Then \( R \otimes_k S \) is also an integral domain.

**Remarks.** Hence \( \otimes \) is the *coproduct* in the category of rings. And we’ve also shown that the product in the category of affine schemes is given by tensor product.

**Theorem.** Let \( X \) and \( Y \) be affine varieties. Then there is product previety \( Z := X \times Y \); it is affine with coordinate ring \( A(X) \otimes_k A(Y) \).

**Proof.** We’ve shown that this is the product in the category of affine varieties; what’s different is that \( W \) may now be *any* previety. Suppose we have morphism \( W \to X \) and \( W \to Y \). How do we get the morphism \( W \to Z \)?

(This is where patching arguments make life really easy.) Cover \( W \) with affines \( U_i \). We have maps \( U_i \to X, Y \) so (as \( Z \) is a product in the category of affine varieties) we get map \( U_i \to Z \).

We now need to show that these maps “glue together” to give us a map from \( W \) to \( Z \), i.e. that if you consider the overlap \( U_{ij} := U_i \cap U_j \), then the induced morphism \( U_{ij} \to U_i \to Z \) is the same as the induced morphism \( U_{ij} \to U_j \to Z \).

Cover \( U_{ij} \) with affines \( V_k \); then again because \( Z \) is a product in the category of affine varieties, and we have a morphism \( V_k \to X, Y \), there is only one morphism \( V_k \to Z \) compatible with them.

**Examples.** It isn’t hard working out what products of affine varieties actually are. For example, \( \mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2 \). If \( X \) is the affine variety in \( \mathbb{A}^2 \) cut out by \( v^3 + w^3 = 1 \), and \( Y \) is the affine variety in \( \mathbb{A}^3 \) cut out by \( xyz = 3 \), then \( X \times Y \) is the affine variety in \( \mathbb{A}^5 \) cut out by \( v^3 + w^3 = 1 \) and \( xyz = 3 \). Keep these examples in mind.

**Looking at the product of affines more closely.**

i) **Topology.** Note that the distinguished opens on \( X \times Y \) are of the form \( D(\sum f_i \otimes g_i) \) where \( f_i \in A(X), g_i \in A(Y) \); this gives the beginning of an insight as to why the topology on the product is not the product of the topologies.

ii) **Function field.** Note also that the function field of the product \( k(X \times Y) \) is the *quotient field* of the tensor product of the function fields: \( k(X) \otimes_k k(Y) \).

iii) **Stalks of the structure sheaf.** Let’s interpret the local ring \( \mathcal{O}_{X \times Y, (x, y)} \) in terms of the local rings \( \mathcal{O}_{X, x} \) and \( \mathcal{O}_{Y, y} \). Let the maximal ideals of these local rings be \( \mathfrak{m}_x \) and \( \mathfrak{m}_y \) respectively. (This is concrete! The local rings are quotients of polynomials where the denominator *does not* vanish at the point, and the maximal ideal is where the numerator *does* vanish at the point.)
Lemma. Then $\mathcal{O}_{X \times Y,(x,y)}$ is the localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y,y}$ at the maximal ideal $m_x \mathcal{O}_{Y,y} + m_y \mathcal{O}_{X,x}$.

We’ll need this technical fact (which isn’t too hard) only once, in a few minutes, and then you can forget about it.

If you try to parse what this means, you’ll realize that it’s reasonable.

Proof. Here’s a real explanation. $\mathcal{O}_{X \times Y,(x,y)}$ is the localization of $A(X) \otimes_k A(Y)$ at the maximal ideal of all functions vanishing at $(x,y)$. Now $A(X) \times A(Y) \subset \mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y,y} \subset \mathcal{O}_{X \times Y, x \times y}$, so we can describe the last term by localizing at the maximal ideal of all functions vanishing at $(x,y)$, so we need to check that this really is $m_x \mathcal{O}_{Y,y} + m_y \mathcal{O}_{X,x}$. One inclusion is clear. Conversely, if
\[ h = \sum f_i \otimes g_i \in \mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y,y} \]
vanishes at $(x,y)$, with $f_i(x) = a_i$ and $g_i(y) = b_i$, then
\[ h = h - \sum a_i b_i = \sum (f_i - a_i) \otimes g_i + \sum a_i \otimes (g_i - b_i) \]
which lies in $m_x \mathcal{O}_{Y,y} + m_y \mathcal{O}_{X,x}$ as desired.

Theorem. Let $X$ and $Y$ be prevarieties over $\mathcal{K}$. Then they have a product.

Proof. Let’s first build a reasonable candidate from affines, and then later check that it really is a product. We’ll start with the set, add the topology, and finally the structure sheaf.

For the underlying set, we just take the product set.

For the topology, we give a base: For all open affines $U \subset X$, $V \subset Y$, and all finite sets of elements $f_i \in A(X)$, $g_i \in A(Y)$, consider $D(\sum f_i \otimes g_i) \subset U \times V$. (Small check required to make sure that this really gives the topology you want on $U \times V$.)

Now for the structure sheaf. Let $K$ be the quotient field of $k(X) \otimes_k k(Y)$, which is our candidate for the function field of the product. For $x \in X$, $y \in Y$, let $\mathcal{O}_{X \times Y,(x,y)} \subset K$ be the localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y,y}$ at the ideal $m_x \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_y$, and define
\[ \mathcal{O}_{X \times Y}(U) = \bigcap_{(x,y) \in U} \mathcal{O}_{X \times Y,(x,y)}. \]

This is a sheaf of functions, which coincides on each $U \times V$ ($U$, $V$ affine) with the structure sheaf of $\mathcal{O}_{U \times V}$ (by our analysis of the affine case).

Then this is a prevariety! (Check: covered by finitely many affines, and connected.)
Next to check: that this prevariety is a product. We have our projection maps $X \times Y$ to $X$ and $Y$. Suppose we’re given some morphisms $f_X : W \to X$, $f_Y : W \to Y$. Then we automatically get a map of sets $W \to X \times Y$, as the underlying set of $X \times Y$ is just the product of the underlying sets of $X$ and $Y$. We just need to check that this is a morphism. To do that, we can cover $X$ and $Y$ by affines $U_i$ and $V_j$ respectively, and cover $W$ with $f_X^{-1}U_i \cap f_Y^{-1}V_j$; we need only show that $f_X^{-1}U_i \cap f_Y^{-1}V_j \to U_i \times V_j \subset U \times V$ is a morphism; but it is because we’ve already shown that $U_i \times V_j$ is a product in the category of prevarieties when $U_i$ and $V_j$ are affines.

It isn’t hard to check:

**Corollary.** If $U$ is an open subprevariety of $X$, then $U \times Y$ is an open subprevariety of $X \times Y$. If $Z$ is a closed subprevariety of $X$, then $Z \times Y$ is a closed subprevariety of $X \times Y$.

**Remark.** If $\mathbb{K} = \mathbb{C}$, then you might reasonably have the classical topology in mind. It is true that if $X$ and $Y$ are complex varieties, then the classical topology on $X \times Y$ is the same as the product of the classical topologies.

2. **Coming soon**

1. products of projective prevarieties are projective prevarieties; the Segre map
2. rational maps; open sets of definition; birational maps