I’m going to start by telling you about this course, and about the field of algebraic geometry.

Goals:

- geometric insight
- concrete examples (geometric and arithmetic)
- hands on calculations (no fear of commutative algebra)
- no cohomology, flatness, differentials

Modern algebraic geometry lies somewhere between differential geometry, number theory, and topology. In a loose sense, it is polynomial equations, and sets defined by polynomial equations. This seems to be extremely narrow and low-tech, but it surprisingly ends up being extremely broad, powerful, and abstract.

Some of the philosophy — get at geometry via algebra, algebra via “pictures”. High school reference. Here’s a high-powered example of the link between geometry and arithmetic. \( x^n + y^n = z^n \). Finite number of solutions for each \( n > 1 \): the Mordell Conjecture, Faltings’ Theorem. Vojta’s conjecture. Well conjectures.

Give out handout of motivating problems.

This will be a tools course: examples and pictures, but with generality.

- fast-moving, but grounded by intuition
- exercises are important
- concepts really generalize, but become more abstract.

Date: September 9, 1999.
Objects:

smooth varieties over \( \mathbb{C} \) (over \( k \))

varieties over \( \mathbb{C} \) (over \( k \))

schemes

(stacks)

We won’t be shy about schemes in this course.


Theme: curves. Examples: \( \mathbb{Z} \) and \( \overline{k}[t] \).

1. Commutative algebra

Don’t worry about it. See books on handout.

Ideas you should know: ring (commutative, has 1), field, integral domain, has quotient field, prime ideal, maximal ideal.

Sample problem (to appear on problem set):

Let \( A \) be a (commutative) ring. An element \( a \in A \) is nilpotent (that is, \( a^n = 0 \) for some \( n > 0 \)) if and only if \( a \) belongs to every prime ideal of \( A \).

2. Algebraic sets

Throughout this course: \( k \) is a field. \( \overline{k} \) is an algebraically closed field.

For now we work over \( \overline{k} \). Feel free to think of this as \( \mathbb{C} \) for now.

\( \overline{k}^n \) will be rewritten \( \mathbb{A}^n(\overline{k}) \), affine \( n \)-space; we’ll often just write \( \mathbb{A}^n \) when there’s no confusion about the field. Coordinates \( x_1 \) to \( x_n \).

Algebraic geometry is about functions on the space, which form a ring. The only functions we will care about will be polynomials, i.e. \( \overline{k}[x_1, \ldots, x_n] \). We’ll eventually think of that ring as being the same thing as \( \mathbb{A}^n \).

We’ll next define subsets of \( \mathbb{A}^n \) that we’ll be interested in. Because we’re being very restrictive, we won’t take any subsets, or even analytic subsets; we’ll only think of subsets that are in some sense defined in terms of polynomials.
Let $S$ be a set of polynomials, and define $V(S)$ to be the locus where these polynomials are zero. ("Vanishing set"). Definition: Any subset of $\mathbb{A}^n(\mathbb{K})$ of the form $V(S)$ is an algebraic set.

Exercise (to appear on problem set): prove that the points of the form $(t, t^2, t^3)$ in $\mathbb{A}^3$ form an algebraic set. In other words, find a set of functions that vanish on these points, and no others.

Example/definition: hypersurface, defined by 1 polynomial.

Facts.

- If $I = (S)$, then $V(I) = V(S)$. So we usually will care only about ideals. Hence: subsets of $\mathbb{K}[x_1, \ldots, x_n]$ give us subsets of $\mathbb{A}^n$; specifically, ideals give us algebraic sets
- $V(\cup I_a) = \cap V(I_a)$ (Say it in english.)
- $I \subset J$, then $V(I) \supset V(J)$
- $V(FG) = V(F) \cup V(G)$

Note: Points are algebraic. Finite unions of points are algebraic.

Definition. A radical of an ideal $I \subset R$, denoted $\sqrt{I}$, is defined by

$$\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n\}.$$ 

Exercise. Show that $\sqrt{I}$ is an ideal.

Definition. An ideal $I$ is radical if $I = \sqrt{I}$.

Claim. $V(\sqrt{I}) = V(I)$. (Explain why.)

Conversely, subsets of $\mathbb{A}^n$ give us a subset of $\mathbb{K}[x_1, \ldots, x_n]$ For each subset $X$, let $I(X)$ be those polynomials vanishing on $X$.

Claim. $I(X)$ is a radical ideal. (Explain.)

Facts. If $X \subset Y$, then $I(X) \supset I(Y)$. $I(\emptyset) = \mathbb{K}[x_1, \ldots, x_n]$. $I(\mathbb{A}^n) = (0)$.

Question. What’s $I((a_1, \ldots, a_n))$?

(Discuss.)

Notice: ideal is maximal. Quotient is field. Quotient map can be interpreted as “value of function at that point”.

Exercise. (a) Let $V$ be an algebraic set in $\mathbb{A}^n$, $P$ a point not in $V$. Show that there is a polynomial $F$ in $\mathbb{K}[x_1, \ldots, x_n]$ such that $F(Q) = 0$ for all $Q$ in $V$, but $F(P) = 1$. Hint: $I(V) \neq I(V \cup P)$. 


(b) Let \( \{P_1, ..., P_2\} \) be a finite set of points in \( \mathbb{A}^n(\overline{k}) \). Show that there are polynomials \( F_1, ..., F_r \in \overline{k}[x_1, ..., x_n] \) such that \( F_i(P_j) = 0 \) if \( i \neq j \), and \( F_i(P_i) = 1 \).

**Exercise.** Show that for any ideal \( I \) in \( \overline{k}[x_1, ..., x_n] \), \( V(I) = V(\sqrt{I}) \), and \( \sqrt{I} \) is contained in \( I(V(I)) \).

### 3. Nullstellensatz (Theorem of Zeroes)

Earlier, we had: algebraic sets \( \rightarrow \) radical ideals and ideals \( \rightarrow \) algebraic sets.

This theorem makes an equivalence. In the literature, the word “nullstellensatz” is used to apply to a large number of results, not all of them equivalent.

**Nullstellensatz Version 1.** Suppose \( F_1, \ldots, F_m \in \overline{k}[x_1, \ldots, x_n] \). If the ideal \( (F_1, \ldots, F_m) \neq (1) = \overline{k}[x_1, \ldots, x_n] \) then the system of equations \( F_1 = \cdots = F_m = 0 \) has a solution in \( \overline{k} \).

Proof next day. (There is a better version for fields that are not necessarily algebraically closed, but we’re not worrying about that right now.)

**Nullstellensatz Version 2.** Suppose \( m \) is a maximal ideal of \( \overline{k}[x_1, \ldots, x_n] \). Then

\[
m = (x_1 - a_1, \ldots, x_n - a_n)
\]

for some \( a_1, \ldots, a_n \in \overline{k} \).

Show that this is equivalent to version 1, modulo fact that ideals are finitely generated.

**Nullstellensatz Version 3 (sometimes called the “Weak Nullstellensatz”).** If \( I \) is a proper ideal in \( \overline{k}[x_1, \ldots, x_n] \), then \( V(I) \) is nonempty. (From Version 2.)

**Nullstellensatz Version 4.** Let \( I \) be an ideal in \( \overline{k}[x_1, \ldots, x_n] \). Then \( I(V(I)) = \sqrt{I} \). Equivalently: Radical ideals are in 1-1 correspondence with algebraic sets: If \( I \) is a radical ideal in \( \overline{k}[x_1, \ldots, x_n] \) then \( I(V(I)) = I \). So there is a 1-1 correspondence between radical ideals and algebraic sets.

**Nullstellensatz Version 5.** A radical ideal of \( \overline{k}[x_1, \ldots, x_n] \) is the intersection of the maximal ideals containing it. This is the geometric rewording of 4. By version 4, a radical ideal is \( I(X) \) for some algebraic set \( X \). Functions vanishing on \( X \) are precisely those functions vanishing on all the points of \( X \).

**Nullstellensatz Version 6.** If \( F_1, \ldots, F_r, G \) are in \( \overline{k}[x_1, \ldots, x_n] \), and \( G \) vanishes wherever \( F_1, \ldots, F_r \) vanish, then there is an equation \( G^N = A_1 F_1 + \cdots + A_r F_r \) for some \( N > 0 \) and some \( A_i \) in \( \overline{k}[x_1, \ldots, x_n] \).

This has a cute proof, with a useful trick in it.
Proof. The case $G = 0$ is obvious, so assume $G \neq 0$. Introduce a new variable $U$, and consider the polynomials

$$F_1, \ldots, F_m, \text{ and } UG - 1 \in \overline{k}[x_1, \ldots, x_n, U].$$

They have no common solutions in $\overline{k}$, so by Version 1 they generate the unit ideal, so there are polynomials $P_1, \ldots, P_m, Q \in \overline{k}[x_1, \ldots, x_n, U]$ such that

$$P_1F_1 + \cdots + P_m F_m + Q(UG - 1) = 1.$$ 

Now set $U = 1/G$ in this formula, and multiply by some large power $G^N$ of $G$ to clear denominators. Then the right side is $G^N$, and the left side is in $(F_1, \ldots, F_m)$. 
