INTERSECTION THEORY CLASS 8

RAVI VAKIL

CONTENTS

1. PROOF OF KEY RESULT OF CHAPTER 2

Our goal now is to prove the key result of Chapter 2. It's not impressive in and of itself, but we used it to do a lot of other things.

Big Theorem 2.4. Let D and D' be Cartier divisors on an n-dimensional variety X. Then $D \cdot [D'] = D' \cdot [D]$ in $A_{n-1}(|D| \cap |D'|)$.

Last time, I discussed the case where D and D' have no common components, so $|D| \cap |D'|$ is codimension 2. I didn't prove it, but argued that it boils down to algebra. So the real problem is what to do if D and D' have a common component.

The proof involves an extremely clever use of blowing up. Given the background of the people in this class, I've had to make some decisions as to what arguments to include, and I think I'd most like to give you some feeling for blowing up, and then to outline the proof, rather than getting into the gory details.

1.1. **Crash course in blowing up.** Last time I began to talk about blowing up. Let X be a scheme, and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals on X. (Technical requirement automatically satisfied in our situation: $\mathcal{I}$ should be a coherent sheaf, i.e. finitely generated.) Here is the "universal property" definition of blowing-up. Then the blow-up of $\mathcal{O}_X$ along $\mathcal{I}$ is a morphism $\pi : \tilde{X} \to X$ satisfying the following universal property. $f^{-1}\mathcal{I}\mathcal{O}_{\tilde{X}}$ (the "inverse ideal sheaf") is an invertible sheaf of ideals, i.e. an effective Cartier divisor, called the *exceptional divisor*. (Alternatively: the scheme-theoretic pullback of the subscheme $\mathcal{O}/\mathcal{I}$ is a closed subscheme of $\tilde{X}$ which is (effective) Cartier, and this is called the exceptional (Cartier) divisor E.) If $f : Z \to X$ is any morphism such that $(f^{-1}\mathcal{I})\mathcal{O}_Z$ is an invertible sheaf

Date: Monday, October 18, 2004.

of ideals on Z (i.e. the pullback of $\mathcal{O}/\mathcal{I}$ is an effective Cartier divisor), then there exists a unique morphism $g : Z \to \tilde{X}$ factoring f.

$$\begin{array}{ccc} Z & \overset{g}{\dashrightarrow} & \tilde{X} \\ & \searrow^{f} & \downarrow \pi \\ & & X \end{array}$$

In other words, if you have a morphism to X, which, when you pull back the ideal I, you get an effective Cartier divisor, then this factors through $\tilde{X} \to X$.

As with all universal property statements, any two things satisfying the universal property are canonically isomorphic.

Theorem: Blow-ups exist. The proof is by construction: show that $\mathbf{Proj} \oplus_{d \geq 0} \mathcal{I}^d$ satisfies the universal property. (See Hartshorne II.7, although his presentation is opposite.)

This construction shows that in fact π is projective (hence proper).

Example 1. The "typical" first example is the blow-up of the plane at a point, $\mathrm{Bl}_0 \mathbb{A}^2$. Let $X = \{(p \in \mathbb{A}^2, \ell$ line in plane through p and $0\}$. Note that (i) X is smooth (it is an $\mathbb{A}^1$-bundle = total space of a line bundle over the $\mathbb{P}^1$ parametrizing the possible ℓ), (ii) it has a map π to $\mathbb{A}^2$, (iii) π is an isomorphism away from p, and $\pi^{-1} p \cong \mathbb{P}^1$. This $\mathbb{P}^1$ is codimension 1 on a smooth space, hence an effective Cartier divisor. Fact: This satisfies the universal property, hence is a blow-up. More generally, if you blow up a point on a smooth surface, the same story happens. More generally still, if you blow up a smooth variety X along a smooth subvariety V of codimension k, you get something that is isomorphic away from V, and the preimage of V is a $\mathbb{P}^{k-1}$-bundle over V; it is the projectivized normal bundle (i.e. points of the exceptional divisor E correspond to points of V along with a line in the normal bundle to V in X.)

Weirder things can happen.

Example 2. If you blow up X along an effective Cartier divisor D, then nothing changes. $(X, D) \to X$ already satisfies the universal property, tautologically.

Example 3. If you blow up X along itself, it disappears. For example, consider $X = \mathbb{A}^1$, and $\mathcal{I} = 0$. Then there is *no way* to pullback this ideal sheaf and get a Cartier divisor, which is codimension 1. Well, there *is* one way: via the morphism $\emptyset \to X$.

Example 3a. If you blow up X along one of its components, the component is blown away (disappears), and the rest will be affected too (blown up along their intersection with the old component).

Fun Example 4. Consider the cone, and blow it up along a line. The line is not a Cartier divisor, as we showed last day. Hence the blow-up does *something*. Moreover, it does nothing away from cone point. It turns out that this does indeed smooth out the cone! (It does the same thing as blowing up the cone point by itself.)

Remark. If X is a variety and $Y \neq X$, then $\tilde{X} \to X$ is birational.

1.2. **Back to the proof.** D and D' are two Cartier divisors, cut out locally by a single equation. Let $D \cap D'$ be the intersection scheme of D and D'. Let $\pi : \tilde{X} \to X$ be the blow-up of X along $D \cap D'$, and let $E = \pi^{-1}(D \cap D')$ be the exceptional divisor. The local equations for $\pi^* D$ and $\pi^* D'$ are divisible by the local equation for E. Translation: D and D' both lie in the ideal sheaf of $D \cap D'$, hence their pullback lies in the (Cartier) ideal sheaf of E. Hence we can write equalities of Cartier divisors:

$\pi^* D = E + C, \pi^* D' = E + C'$.

Let

$$\epsilon(D, D') := \max\{\operatorname{ord}_V(D) \operatorname{ord}_V(D') : \operatorname{codim}(V, X) = 1\}.$$

Note that we know the result when $\epsilon = 0$. We're going to work by induction on ϵ.

Omitted Lemma. (a) C and C' are disjoint. (This is a special case of Hartshorne Exercise II.7.12.) (b) If $\epsilon(D, D') > 0$, then $\epsilon(C, E), \epsilon(C', E) < \epsilon(D, D')$.

Proof is omitted. But caution: something very interesting is going on here. I'll give three examples to show you this. First, suppose L_1, L_2, and L_3 are three general lines in $\mathbb{P}^2$. If $D = L_1$ and $D' = L_1 + L_3$, then $D \cap D' = L_1$, and the blow-up does nothing. However, $E = L_1$, and then $C = \emptyset$ and $C' = L_3$.

Next, suppose $D = L_1 + L_2$ and $D' = L_1 + L_3$. Then the trouble occurs because $D \cap D'$ includes L_1. But the blow-up does something else; it blows up $L_2 \cap L_3$. Let E_{23} be the exceptional divisor of the blow-up of $L_2 \cap L_3$. Then the exceptional divisor of the blow-up that *we* care about is $L_1 + E_{23}$. Then we get C is the proper transform of L_2 and C' is the proper transform of L_3.

Finally, suppose $D = 2L_1 + L_2$ and $D' = L_1 + L_3$. Then the scheme-theoretic intersection $D \cap D'$ consists of the point $L_2 \cap L_3$, as well as L_1, *but also* some additional "fuzz" where L_1 meets L_3! When you blow this up, what happens? (Well, I can tell you what happens in this case — it's the same as blowing up the two points $L_1 \cap L_3$ and $L_2 \cap L_3$ — but in general this is quite complicated. I find it fascinating that we don't ever have to know precisely what happens to prove this lemma.)

Lemma. If D, D' are Cartier divisors on X, $\pi : \tilde{X} \to X$ is a proper birational morphism of varieties, $\pi^* D = B \pm C$, $\pi^* D = B' \pm C'$, for Cartier divisors B, C, B', C' on $\tilde{X}$ with $|B| \cup |C| \subset \pi^{-1}(|D|)$, $|B'| \cup |C'| \subset \pi^{-1}(|D'|)$, and the theorem holds for each pair (B, B'), (B, C'), (C, B'), (C, C') on $\tilde{X}$, then the theorem holds for (D, D') on X.

Proof.

$$\begin{aligned} D \cdot [D'] &= \pi_*((B \pm C) \cdot [B' \pm C']) \quad \text{(projection formula, note } \pi_*([B' \pm C']) = [D']\text{)} \\ &= \pi_*(B \cdot [B'] \pm B \cdot [C'] \pm C \cdot [B'] \pm C \cdot [C']) \quad \text{(linearity)} \\ &= \pi_*(B' \cdot [B] \pm C' \cdot B \pm [B'] \cdot [C] \pm C' \cdot [C]) \quad \text{(hypothesis)} \\ &= \pi_*((B' \pm C') \cdot [B \pm C]) \quad \text{(linearity)} \\ &= D' \cdot [D] \quad \text{(projection formula)} \end{aligned}$$

□

Now let's finish off the proof of the big theorem.

Case D *and* D' *effective.* We do this by induction on $\epsilon(D, D')$. The case $\epsilon = 0$ is already done (or more precisely, assumed!), as described earlier. If $\epsilon(D, D') > 0$, then blow up X along $D \cap D'$. Then the omitted lemma asserts that the theorem holds for (E, C') and (C, E). The theorem also holds for (E, E) stupidly (clearly $E \cdot [E] = E \cdot [E]$), and also for (C, C') for different stupid reasons ($C \cdot [C'] = 0 = C' \cdot [C]$). So the above lemma completes this proof.

Case D' *effective.* Let $\mathcal{J}$ be the ideal sheaf of denominators of D. (Translation: locally, on an open set $\operatorname{Spec} A$, is consists of those functions who, when multiplied by the generator of D in $R(X)$, turn it into a regular function.) Blow up X along $\mathcal{J}$. Then $\pi^* D = C - E$ where E is the exceptional divisor, and C is an effective Cartier divisor. Then the previous case covers $(C, \pi^* D')$ and $(E, \pi^* D')$ on $\tilde{X}$, so we're done by the Lemma.

General case. Blow up X along the ideal sheaf of denominators of D'. Then the pairs $(\pi^* D, C)$ and $(\pi^* D, E)$ are covered by the previous case, so we're done by the Lemma. □

2. Vector bundles, and Segre and Chern classes

In the next chapter, we're going to generalize the notion of the first Chern class of a line bundle to the notion of an arbitrary Chern class on an arbitrary vector bundle. These Chern classes will have similar properties to those you may have seen elsewhere, but we get at them in a strangely backwards way, by defining Segre classes first. The generating function for Segre classes will be inverse to that of Chern classes.

When you look through this chapter, you'll note that only a very small portion of it consists of propositions and theorems. The rest is full of useful examples.

2.1. **Segre classes of vector bundles.**

Let E be a vector bundle of rank $e+1$ on an algebraic scheme X. Let $P = \mathbb{P}E$ be the $\mathbb{P}^e$-bundle of lines on E, and let $p = p_E : P \to X$ be the projection. Note that it is both flat and proper (explain).

The line bundle $\mathcal{O}(1)$. On P there is a canonically defined line bundle, called the *tautological bundle*, denoted $\mathcal{O}(-1)$ or $\mathcal{O}_E(-1)$. For any point of P, I'll need to give you a one-dimensional vector space in some natural way. But each point of P corresponds to a line of E.

Define $\mathcal{O}(1)$ as the dual of $\mathcal{O}(-1)$, and let $\mathcal{O}(n)$ be $\mathcal{O}(1)^{\otimes n}$ (with the obvious convention if n is nonpositive).

Here's a second "definition" of $\mathcal{O}(1)$. This is somewhat informal; making it precise it a bit inefficient. Define the "projective completion" of E to be the projective bundle "compactifying" E. As sets, it is $E \coprod \mathbb{P}E$. It can also be described as $\mathbb{P}(E + \mathbf{1})$ where $\mathbf{1}$ is the trivial line bundle. ($\mathbf{1}$ is slightly unfortunate notation; but I'm following Fulton.) It is a $\mathbb{P}^{e+1}$-bundle. On it, $\mathbb{P}E$ is an effective Cartier divisor, and this divisor class is $\mathcal{O}_{\mathbb{P}(E+\mathbf{1})}(1)$. Restricting this divisor class to $\mathbb{P}E$ gives $\mathcal{O}_{\mathbb{P}E}(1)$. (Note that this is not automatically an effective Cartier divisor class on $\mathbb{P}E$.)

Remark. On $\mathbb{P}^e$, there is a line bundle / invertible sheaf $\mathcal{O}(1)$, and indeed $\mathcal{O}_E(1)$ restricts to each of the fibers to give $\mathcal{O}(1)$. But this doesn't determine the class $\mathcal{O}_E(1)$. Indeed, if I pull back any line bundle on X to P, I get a line bundle trivial on each of the fibers, so $\mathcal{O}_E(1) \otimes \mathcal{L}$ has this property for any invertible sheaf $\mathcal{L}$.

Definition. Define homomorphisms

$$s_i(E)\cap : A_k X \to A_{k-i} X$$

by $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha)$. Note that this indeed maps from $A_k X \to A_{k-i} X$.

Warm-up proposition. (First Segre class of a line bundle) If E is a line bundle on X, $\alpha \in A_* X$, then

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha.$$

Proof. In this case $\mathbb{P}E = X$, and $\mathcal{O}_E(-1) = E$ so $\mathcal{O}_E(1) = E^\vee$, hence $s_1(E) \cap \alpha = c_1(\mathcal{O}_E(1) \cap \alpha) = -c_1(E) \cap \alpha$. □

Segre class Theorem. (a) for all $\alpha \in A_k X$, (i) $s_i(E) \cap \alpha = 0$ for $i < 0$, and (ii) $s_0(E) \cap \alpha = \alpha$.

(b) (commutativity) If E and F are vector bundles on X, and $\alpha \in A_k X$, then for all i, j,

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha).$$

(c) (Segre classes behave well with respect to proper pushforward) If $f : X' \to X$ is proper, E a vector bundle on X, $\alpha \in A_* X'$, then for all i,

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha).$$

(d) (Segre classes behave well with respect to flat pullback) If $f : X' \to X$ is flat, E a vector bundle on X, $\alpha \in A_* X$

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha).$$

Corollary. The flat pullback $p^* : A_k X \to A_{k+e}(\mathbb{P}E)$ is a split monomorphism: by (a) (ii), an inverse is $\beta \mapsto p_*(c_1(\mathcal{O}_E(1))^e \cap \beta)$.

Corollary. It makes sense to multiply by various polynomials in Segre classes of various bundles, by the commutativity part (b).

Proof of theorem. I'll prove a smattering of these.

(c) Suppose $f : X' \to X$ is proper, E a vector bundle on X. There is a fibre square

$$\begin{array}{ccc} \mathbb{P}(f^*E) & \xrightarrow{f'} & \mathbb{P}E \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow \\ X' & \longrightarrow & X \end{array}$$

with $f'^*\mathcal{O}_E(1) = \mathcal{O}_{f^*E}(1)$. (All morphisms here are proper, the top one because proper morphisms are preserved by fibred squares.) Then

$$\begin{aligned} f_*(s_i(f^*E) \cap \alpha) &= f_* p'_*(c_1(\mathcal{O}_{\mathbb{P}f^*E}(1))^{e+i} \cap p'^*\alpha) \quad \text{(def'n of } s_i\cap) \\ &= p_* f'_*(c_1(f'^*\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap p'^*\alpha) \quad \text{(commutativity of proper pushforwards)} \\ &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap f'_* p'^*\alpha) \\ &\qquad \text{(proj. formula for } c_1 \text{, i.e. behaves well w.r.t. pr. push.)} \\ &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap p^* f_*\alpha) \quad \text{(pr. push. and flat pull. commute)} \\ &= s_i(E) \cap f_*\alpha \quad \text{(def'n of } s_i\cap) \end{aligned}$$

(d) **Exercise.**

(a) We may assume that $\alpha = [V]$. Then by (c), using the (proper) closed immersion $V \hookrightarrow X$, we may assme $X = V$. Then for $i < 0$, $s_i(E) \cap [V] \in A_{\dim V - i} X = 0$, so (i) is done. Similarly,

$$s_0(E) \cap [V] = p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^e \cap [P]) = m[V]$$

for some m. We will show that $m = 1$. We can check this on an open set of V, so restrict to an open set where E is a trivial bundle. Then $P = \mathbb{P}E = X \times \mathbb{P}^e$, and $\mathcal{O}(1)$ has sections whose zero scheme is $X \times \mathbb{P}^{e-1}$. Then $c_1(\mathcal{O}(1)) \cap [X \times \mathbb{P}^e] = [X \times \mathbb{P}^{e-1}]$ (from earlier theorem on c_1 of an effective Cartier divisor). Repeat this e times to get the desired result.

(b) next day...

E-mail address: `vakil@math.stanford.edu`