# INTERSECTION THEORY CLASS 6

RAVI VAKIL

CONTENTS



Where we are: proper pushforwards and flat pullbacks. We need a disproportionate amount of algebra to set ourselves up. This will decrease in later chapters.

Last day: Rob proved the *excision exact sequence*:

**Proposition.** Let $Y$ be a closed subscheme of $X$, and let $U = X - Y$. Let $i : Y \hookrightarrow X$ be the closed immersion (proper!) and $j : U \to X$ be the open immersion (flat!). Then

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \longrightarrow 0$$

is exact for all $k$.

(Aside: you certainly expect more on the left!)

*Proof.* We quickly check that

$$Z_k Y \xrightarrow{i_*} Z_k X \xrightarrow{j^*} Z_k U \longrightarrow 0$$

is exact. Hence we get exactness on the right in our desired sequence. We also get the composition of the two left arrows in our sequence is zero.

Next suppose $\alpha \in Z_k X$ and $j^* \alpha = 0$. That means $j^* \alpha = \sum_i \operatorname{div} r_i$ where each $r_i \in R(W_i)^*$, where $W_i$ are subvarieties of $U$. So $r_i$ is also a rational function on $R(\overline{W_i})$ where $\overline{W_i}$ is the closure in $X$. To be clearer, call this rational function $\overline{r}_i$ Hence $j^*(\alpha - \sum [\operatorname{div}(\overline{r}_i)]) = 0$ in $Z_k U$, and hence $j^*(\alpha - \sum [\operatorname{div}(\overline{r}_i)]) \in Z_k Y$, and we're done. $\square$

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**Definition.** $Y \to X$ is an *affine bundle of rank* $n$ over $X$ if there is an open covering $\cup U_\alpha$ of $X$ such that $f^{-1}(U_i) \cong U_i \times \mathbb{A}^n \to U_i$. This is a flat morphism.

**Proposition.** Let $p : E \to X$ be an affine bundle of rank $n$. Then the flat pullback $p^* : A_k X \to A_{k+n} E$ is surjective for all $k$.

Immediate corollary: $A_k \mathbb{A}^n = 0$ for $k \neq n$.

Homework: Example 1.9.3 (a). Show that $A_k(\mathbb{P}^n)$ is generated by the class of a $k$-dimensional linear space. (Hint: use the excision exact sequence.)

Example 1.9.4: Let $H$ be a reduced irreducible hypersurface of degree $d$ in $\mathbb{P}^n$. Then $[H] = d[L]$ for $L$ a hyperplane, and $A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d/Z$. Thus the codimension 1 Chow group is torsion. (Caution: where are you using reduced and irreducible?)

$Z_k X \otimes Z_l Y \to Z_{k+l}(X \times Y)$ by $[V] \times [W] = [V \times W]$.

**Proposition.**

(a) if $\alpha \sim 0$ then $\alpha \times \beta \sim 0$. There are exterior products $A_k X \otimes A_l Y \to A_{k+l}(X \times Y)$.
(b) If $f$ and $g$ are proper, then so is $f \times g$, and $(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$. Hence exterior product respects proper pushforward.
(c) If $f$ and $g$ are flat of relative dimensions $m$ and $n$, (so $f \times g$ is flat of relative dimension $m+n$), then

$$(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta.$$

Hence exterior product respects flat pullback.

## 1. Divisors

There are three related concepts of divisors: Weil divisors, Cartier divisors, and (a concept local to intersection theory) pseudodivisors.

A *Weil divisor* on a variety $X$ is a formal sum of codimension 1 subvarieties.

The notion of Cartier divisor looks more unusual when you first see it. A *Cartier divisor* is defined by data $(U_\alpha, f_\alpha)$ where the $U_\alpha$ form an open covering of $X$ and $f_\alpha$ are non-zero functions in $R(U_\alpha) = R(X)$, subject to the condition that $f_\alpha / f_\beta$ is a unit (regular, nowhere vanishing function) on the intersection $U_\alpha \cap U_\beta$. There is an equivalence classes of Cartier divisors.

The rational functions are called *local equations*. Local equations are defined up to multiplication by a unit.

*Baby Example:* $X = U_\alpha = \mathbb{A}^1 - \{1\}$, local equation $1/t^2$. Another local equation for the same Cartier divisor: $(t-1)/t^2$.

1.1. **Crash course in Cartier divisors and invertible sheaves (aka line bundles).** (See Appendix B.4 for an even faster introduction!)

$$\begin{tikzcd}
\operatorname{Pic} X = \{\text{ Car. div.}\} \ / \text{ lin. equiv.} \arrow[r, leftrightarrow] & \{\text{invertible sheaves}\} \\
= \{\text{Car. div. / princ. Car. div. }\} \arrow[d] & = \{\text{line bundles}\} \arrow[d] \\
\{\text{ Cartier divisors }\} \arrow[r, leftrightarrow] & \{\text{ inv. sheaves w. nonzero rat'l sec. }\}/\text{inv. funcs.}\Gamma(X, \mathcal{O}_X^*)
\end{tikzcd}$$

Given a Cartier divisor $(U_\alpha, f_\alpha)$, here's how you produce an invertible sheaf $\mathcal{F}$. I need to tell you $\mathcal{F}(U)$. $\mathcal{F}(U) = \{(g_\alpha \in \mathcal{O}_X^*(U_\alpha \cap U))_\alpha = R(X) : g_\alpha f_\alpha \in \mathcal{O}_X(U_\alpha \cap U)), g_\alpha f_\alpha = g_\beta f_\beta \in \mathcal{O}_X(U \cap U_\alpha \cap U_\beta)\}$. You can check that this is indeed a sheaf, and it is locally trivial: check for $U = U_\alpha$ that $\mathcal{F}(U_\alpha)$ consists of rational functions $g_\alpha$ on $U_\alpha$ such that $g_\alpha f_\alpha$ is a regular function. Thus the $g_\alpha$ are all of the form $\mathcal{O}(U_\alpha)/f_\alpha$ (regular functions divided by $f_\alpha$), and thus as a $\mathcal{O}(U_\alpha)$-module, it is isomorphic to $\mathcal{O}(U_\alpha)$ itself.

*Baby Example:* $X = U_\alpha = \mathbb{A}^1 - \{1\}$, $f_\alpha = 1/t^2$. The rational functions on X are $K[t, 1/(t-1)]$. The module corresponding to this Cartier divisor is $K[t, 1/(t-1)]t^2$, which is clearly isomorphic to $K[t, 1/(t-1)]$.

A Cartier divisor is *effective* if the $f_\alpha$ are all *regular functions* ("have no poles"). Thus we can add to that square above: { effective Cartier divisors } correspond to invertible sheaves with nonzero sections.

A Cartier divisor is *principal* if it is the divisor of a rational function i.e. $\operatorname{div}(r)$ where $r \in R(X)^*$. Two Cartier divisors differing by a principal Cartier divisor give rise to the same invertible sheaf.

Rob told you that the Cartier divisor form an abelian group $\operatorname{Div}(X)$. When you mod out by the subgroup of principal Cartier divisors, you get the group of invertible sheaves $\operatorname{Pic} X$.

The *support* of a Cartier divisor D, denoted $|D|$, is the union of all subvarieties Z of X such that the local equation for D in the ring $\mathcal{O}_{Z,X}$ is not a unit. This is a closed algebraic subset of X of pure codimension one.

Notice: invertible sheaves pull back, but Cartier divisors don't necessarily. (Give an example.)

We have a map from Cartier divisors to Weil divisors. Linear equivalence of Cartier divisors rational equivalence of Weil divisors, hence this map descends to $\operatorname{Pic} X \to A_{\dim X-1} X$.

1.2. **Pseudo-divisors.** A *pseudo-divisor* on a scheme X is a triple where L is an invertible sheaf on X, Z is a closed subset, and s is a nowhere vanishing section of L on $X - Z$.

As of last day, you know: Pseudo-divisors pull back. And if $X$ is a *variety*, any pseudo-divisor on $X$ is represented by some Cartier divisor on $X$. (A Cartier divisor $D$ represents a pseudo-divisor $(L, Z, s)$ if $|D| \subset Z$, and there is an isomorphism $\mathcal{O}_X(D) \to L$ which away form $Z$ takes $s_D$ (the "canonical section") to $s$.) Furthermore, if $Z \neq X$, $D$ is uniquely determined. If $Z = X$, then $D$ is determined up to linear equivalence.

Hence given any pseudo-divisor $D$, we get a Weil divisor class in $A_{n-1}X$. But we can do better. Given a pseudo-divisor $D$, we get a Weil divisor class $[D] \in A_{n-1}(|D|)$.

## 2. Intersecting with divisors

We will now define our first intersections, that with Cartier divisors, or more generally pseudo-divisors. Let $D$ be a pseudo-divisor on a scheme $X$. We define $D \cdot [V]$ where $V$ is a $k$-dimensional subvariety. $D \cdot [V] := [j^*D]$ where $j$ is the inclusion $V \hookrightarrow X$. This lies in $A_{k-1}V \cap |D|$. Hence we can do this with any finite combination of varieties. Note that we get a map $Z_kX \to A_{k-1}X$, but we're asserting more: we're getting classes not just on $X$, but on subsets smaller than $X$.

**Proposition 2.3.**

(a) (linearity in $\alpha$) If $D$ is a pseudo-divisor on $X$, and $\alpha$ and $\alpha'$ are $k$-cycles on $X$, then
$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha' \quad \text{in } A_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|)).$$

(b) (linearity in $D$) If $D$ and $D'$ are pseudo-divisors on $X$, and $\alpha$ is a $k$-cycle on $X$, then
$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha \quad \text{in } A_{k-1}((|D| \cup |D'|) \cap |\alpha|).$$

(c) (projection formula) Let $D$ be a pseudo-divisor on $X$, $f : X' \to X$ a proper morphism, $\alpha$ a $k$-cycle on $X'$, and $g$ the morphism from $f^{-1}(|D|) \cap |\alpha|$ to $|D| \cap f(|\alpha|)$ induced by $f$. Then
$$g_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha) \quad \text{in } A_{k-1}(|D| \cap f(|\alpha|)).$$

(d) (commutes with flat base change) Let $D$ be a pseudo-divisor on $X$, $f : X' \to X$ a flat morphism of relative dimension $n$, $\alpha$ a $k$-cycle on $X$, and $g$ the induced morphism from $f^{-1}(|D| \cap |\alpha|)$ to $|D| \cap |\alpha|$. Then
$$f^*D \cdot f^*\alpha = g^*(D \cdot \alpha) \quad \text{in } A_{k+n-1}(f^{-1}(|D| \cap |\alpha|).$$

(e) If $D$ is a pseudodivisor on $X$ whose line bundle $\mathcal{O}_X(D)$ is trivial, and $\alpha$ is a $k$-cycle on $X$, then
$$D \cdot \alpha = 0 \quad \text{in } A_{k-1}(|\alpha|).$$

Proof next day.

Example: Intersection of two curves in $\mathbb{P}^2$, $C_1$ and $C_2$. We get a number. Old-fashioned intersection theory (Hartshorne V): $\mathcal{O}(C_1)|_{C_2}$ gives you a number.

This tells you a bit more: the class has "local contributions" from each connected component of the intersection.

Excess intersection can happen! Example: A line meeting itself.

**Remark:** This proves some of the things Fulton said about Bezout in the first chapter.

Here's a natural question: if you intersect two effective Cartier divisors, then if you reverse the order of intersection, you had better get the same thing!

$D \cdot [D'] = D' \cdot [D]$?

**Big Theorem 2.4** Let $D$ and $D'$ be Cartier divisors on an $n$-dimensional variety $X$. Then $D \cdot [D'] = D' \cdot [D]$ in $A_{n-2}(|D| \cap |D'|)$.

We'll prove this next day, or the day after.

**Corollary.** Let $D$ be a pseudo-divisor on a scheme $X$, and $\alpha$ a $k$-cycle on $X$ which is rationally equivalent to zero. Then $D \cdot \alpha = 0$ in $A_{k-1}|D|$.

**Corollary.** Let $D$ and $D'$ be pseudo-divisors on a scheme $X$. Then for any $k$-cycle $\alpha$ on $X$,

$$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$$

in $A_{k-2}(|D| \cap |D'| \cap |\alpha|)$.

Hence we can make sense of phrases such as $D_1 \cdot D_2 \cdot \dots \cdot D_n \cdot \alpha$.

2.1. **The first Chern class of a line bundle.** If $L$ is a line bundle on $X$, we define "$c_1(L)\cap$". If $V$ is a subvariety, then write the restriction of $L$ to $C$ as $\mathcal{O}_V(C)$ for some Cartier divisor $C$. Then define $c_1(L) \cap [V] = [C]$. ($C$ is well-defined up to linear equivalence, so this makes sense in $A_{\dim V-1}V \hookrightarrow A_{\dim V-1}X$.) Extend this by linearity to define $c_1(L)\cap : Z_kX \to A_{k-1}X$.

**Proposition 2.5.**

(a) If $\alpha$ is rationally equivalent to $0$ on $X$, then $c_1(L) \cap \alpha = 0$. There is therefore an induced homomorphism $c_1(L)\cap : A_kX \to A_{k-1}X$. (That's what we'll usually mean by $c_1(L) \cap \cdot$.)

(b) (commutativity) If $L$, $L'$ are line bundles on $X$, $\alpha$ a $k$-cycle on $X$, then
$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha) \quad \text{in } A_{k-2}X.$$

(c) (projection formula) If $f : X' \to X$ is a proper morphism, $L$ a line bundle on $X$, $\alpha$ a $k$-cycle on $X'$, then
$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha) \quad \text{in } A_{k-1}X.$$

(d) (flat pullback) If $f : X' \to X$ is flat of relative dimension $n$, $l$ a line bundle on $X$, $\alpha$ a $k$-cycle on $X$, then
$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha) \quad \text{in } A_{k+n-1}X'.$$

(e) (additivity) If $L$ and $L'$ are line bundles on $X$, $\alpha$ a $k$-cycle on $X$, then
$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L' \cap \alpha) \quad \text{and}$$
$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha \quad \text{in } A_{k-1}X.$$

We'll prove this next day.

2.2. **Gysin pullback.** Define the *Gysin pullback* as follows. Suppose $i : D \to X$ is an inclusion of an effective Cartier divisor. Define $i^* : Z_k X \to A_{k-1} D$ by

$$i^* \alpha = D \cdot \alpha.$$

**Proposition.**

(a) If $\alpha$ is rationally equivalent to zero on $X$ then $i^* \alpha = 0$. (Hence we get induced homomorphisms $i^* : A_k X \to A_{k-1} D$.)
(b) If $\alpha$ is a $k$-cycle on $X$, then $i_* i^* \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ in $A_{k-1} X$.
(c) If $\alpha$ is a $k$-cycle on $D$, then $i^* i_* \alpha = c_1(N) \cap \alpha$ in $A_{k-1} D$, where $N = i^* \mathcal{O}_X(D)$. $N$ is the normal (line) bundle. (Caution to differential geometers: $D$ could be singular, and then you'll be confused as to why this should be called the normal bundle.)
(d) If $X$ is purely $n$-dimensional, then $i^*[X] = [D]$ in $A_{n-1} D$.
(e) (Gysin pullback commutes with $c_1(L) \cap$) If $L$ is a line bundle on $X$, then

$$i^*(c_1(L) \cap \alpha) = c_1(i^* L) \cap i^* \alpha$$

in $A_{k-2} D$ for any $k$-cycle $\alpha$ on $X$.

Proof next day; although in fact you may be able to see how all but (d) comes from what we've said earlier today. (Part (d) comes from something we discussed earlier, but I'll leave that for next time.)

*E-mail address*: `vakil@math.stanford.edu`