INTERSECTION THEORY CLASS 19

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CONTENTS

Today I'm going to try to finish the proof of Grothendieck-Riemann-Roch in the case of projective morphisms from smooth varieties to smooth varieties. We'll see that we're essentially going to prove it more generally for projective lci morphisms.

1. Recap of Last day

Recall the definition of the Chern character and Todd class. Suppose $\mathcal{F}$ is a coherent sheaf. Let $\alpha_1, \dots, \alpha_n$ be the Chern roots of the vector bundle, so $\alpha_1 + \cdots + \alpha_n = c_1(\mathcal{F})$, etc. Define $ch(\mathcal{F}) = \sum_{i=1}^r \exp(\alpha_i)$ This is *additive* on exact sequences. For vector bundles, we have $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E')$.

The *Todd class* $\mathrm{td}(E)$ of a vector bundle is defined by $\mathrm{td}(E) = \prod_{i=1}^r Q(\alpha_i)$ where

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}.$$

It is multiplicative in exact sequences.

We defined the Grothendieck groups K^0X and K_0X. They are vector bundles, respectively coherent sheaves, modulo the relation $[E] = [E'] + [E'']$. We have a pullback on K^0: $f^* : K^0X \to K^0Y$. K^0X is a *ring*: $[E] \cdot [F] = [E \otimes F]$. We have a pushforward on K_0: $f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}]$.

We obviously have a homomorphism $K^0X \to K_0X$. K_0X is a K^0X-module: $K^0X \otimes K_0X \to X$ is given by $[E] \cdot [\mathcal{F}] = [E \otimes \mathcal{F}]$. Unproved fact: If X is nonsingular and projective, the map $K^0X \to K_0X$ is an isomorphism. (Reason: If X is nonsingular, then $\mathcal{F}$ has a finite resolution by locally free sheaves.)

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The Chern character map descends to $K(X)$: $ch : K(X) \to A(X)_{\mathbb{Q}}$. This does not commute with proper pushforward; Grothendieck-Riemann-Roch explains how to fix this.

1.1. **New facts.** Here are some useful facts, that I didn't mention last time. We have an excision exact sequence for K_0: If $Z \hookrightarrow X$ is a closed immersion, and U is the open complement, we have an excision exact sequence

$$K_0(Z) \to K_0(X) \to K_0(U) \to 0.$$

The proof is similar to our proof for Chow; this is Hartshorne Exercise II.6.10(c).

Similarly, we have $K_0(\mathbb{A}^1 \times Y) \cong K(Y)$.

Last time I showed: **Lemma.** The group $K_0(\mathbb{P}^m)$ is generated by the classes $[\mathcal{O}_{\mathbb{P}^m}(n)]$, with $0 \le n \le m$.

(Incidentally, I mentioned an interesting algebraic problem coming out of my previous proof. Joe gave a nice proof of it. If I have time, I'll type it up and put it in the posted notes.)

I'd like to do it differently today. Instead, I'll show it is generated by the classes $[\mathcal{O}_{\mathbb{P}^m}(-n)]$, with $0 \le n \le m$.

Using the excision exact sequence for K-theory, and $\mathbb{P}^m = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^m$, we get inductively: $K_0(\mathbb{P}^m)$ is generated by $n+1$ things: $[\mathcal{O}_{\mathbb{P}^0}]$, $[\mathcal{O}_{\mathbb{P}^1}]$, …, $[\mathcal{O}_{\mathbb{P}^m}]$.

I'll now express these in terms of $\mathcal{O}_{\mathbb{P}^m}(n)$'s. From

$$0 \to \mathcal{O}_{\mathbb{P}^m}(-1) \to \mathcal{O}_{\mathbb{P}^m} \to \mathcal{O}_{\mathbb{P}^{m-1}} \to 0$$

shows $[\mathcal{O}_{\mathbb{P}^{m-1}}] = [\mathcal{O}_{\mathbb{P}^m}] - [\mathcal{O}_{\mathbb{P}^m}(-1)]$. Similarly,

$$\begin{aligned}
[\mathcal{O}_{\mathbb{P}^{m-2}}] &= [\mathcal{O}_{\mathbb{P}^{m-1}}] - [\mathcal{O}_{\mathbb{P}^{m-1}}(-1)] \\
&= ([\mathcal{O}_{\mathbb{P}^m}] - [\mathcal{O}_{\mathbb{P}^m}(-1)]) - ([\mathcal{O}_{\mathbb{P}^m}(-1)] - [\mathcal{O}_{\mathbb{P}^m}(-2)]) \\
&= [\mathcal{O}_{\mathbb{P}^m}] - 2[\mathcal{O}_{\mathbb{P}^m}(-1)] - [\mathcal{O}_{\mathbb{P}^m}(-2)]
\end{aligned}$$

and you see the pattern (established by the obvious induction). $\square$

Important philosophy behind Riemann-Roch: $K(\mathbb{P}^m)$ and $A_*(\mathbb{P}^m)$ are both m-dimensional vector spaces; Chern character provides an isomorphism between them. Multiplying by the Todd class provides a "better" isomorphism between them.

More generally, the identical proof shows that for any Y, $K(Y) \otimes K(\mathbb{P}^m) \to K(\mathbb{P}^m \times Y)$ is surjective: cut up $\mathbb{P}^m \times Y$ into $Y \coprod \mathbb{A}^1 \times Y \coprod \cdots \coprod \mathbb{A}^m \times Y$, and proceed as before.

2. Statement of the theorem

Grothendieck-Riemann-Roch Theorem. Suppose $f : X \to Y$ is a proper morphism of smooth varieties. Then for any $\alpha \in K(X)$,

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X)).$$

Interesting exercise: how do you make sense of this when X and Y are singular? For example, what if $X \to Y$ is a smooth morphism, we get $\mathrm{ch}(f_*\alpha) \cdot = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(T_{X/Y}))$ where X/Y is the *relative tangent bundle*. As another example, what if $X \to Y$ is a complete intersection? Then T_X and T_Y don't make sense, but $N_{X/Y}$ *is* a vector bundle, and then $\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(N_{X/Y}) = f_*(\mathrm{ch}(\alpha))$. Combining these two, you can now make sense of GRR in the case when f is an lci morphism (i.e. closed immersion followed by a smooth morphism).

The theorem may be interpreted to say that the homomorphism

$$\tau_X : K(X) \to A(X)_{\mathbb{Q}}$$

given by $\tau_X(\alpha) = \mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X)$ commutes with proper pushforward: $f_* \circ \tau_X = \tau_Y \circ f_*$. Last time we showed that this implies **Lemma.** Given $X \xrightarrow{f} Z \xrightarrow{g} Y$. Suppose GRR holds for f and g. Then it holds for $g \circ f$.

Hence the strategy is now to show GRR for $Y \times \mathbb{P}^m \to Y$, and for closed immersions.

We'll use this interpretation of the theorem to show

Theorem. GRR is true for $\mathbb{P}^m \times Y \to Y$.

Proof. We showed last time that this is true in the case where Y is a point. Consider the following diagram.

$$\begin{tikzcd}
K(Y) \otimes K(\mathbb{P}^m) \arrow[r, "\tau_Y \otimes \tau_{\mathbb{P}^m}"] \arrow[d, "\times"] & A(Y)_{\mathbb{Q}} \otimes A(\mathbb{P}^m)_{\mathbb{Q}} \arrow[d, "\times"] \\
K(Y \times \mathbb{P}^m) \arrow[r, "\tau_Y \times \tau_{\mathbb{P}^m}"] \arrow[d, "f_*"] & A(Y \times \mathbb{P}^m)_{\mathbb{Q}} \arrow[d, "f_*"] \\
K(Y) \arrow[r, "\tau_Y"] & A(Y)_{\mathbb{Q}}
\end{tikzcd}$$

(I won't be using anything special about $\mathbb{P}^m$ now.) We want to show that the bottom square commutes.

Note that the top square commutes. Reason: $T_{Y \times \mathbb{P}^m} = p_1^* T_Y \oplus p_2^* T_{\mathbb{P}^m}$ (where p_1 and p_2 are the projections) from which $\mathrm{td}(T_{Y \times \mathbb{P}^m}) = \mathrm{td}(p_1^* T_Y) \times \mathrm{td}(p_2^* T_{\mathbb{P}^m})$.

Moreover the upper left vertical arrow is surjective.

So it suffices to show that the big rectangle commutes. But it does because we've already shown that GRR holds for $\mathbb{P}^m \to \mathrm{pt}$. $\square$

2.1. **GRR for a special case of closed immersions** $f : X \to Y = \mathbb{P}(N \oplus \mathbf{1})$**.** Suppose f is a closed immersion into a projective completion of a normal bundle. Let $d = \operatorname{rank} N$. We want to prove GRR for a vector bundle E. As the vector bundles generate $K(X)$, this will suffice.

This example comes the closest to telling me why the Todd class wants to be what it is. Let $p : Y = \mathbb{P}(N \oplus \mathbf{1}) \to X$ be the projection. Let $\mathcal{O}_Y(-1)$ be the tautological line bundle on $Y = \mathbb{P}^{(}N \oplus \mathbf{1})$. Then as in previous lectures we have a tautological exact sequence of vector bundles on Y:

$$0 \to \mathcal{O}_Y(-1) \to p^*(N \oplus \mathbf{1}) \to Q \to 0$$

where Q is the universal quotient bundle. (Recall that $f^*Q = N_{X/Y}$.) Here is something you have to think through, although we've implicitly used it before. We have a natural section of $p^*(Q \oplus \mathbf{1})$, the $\mathbf{1}$. This gives a section s of Q. This section vanishes precisely (scheme-theoretically) along X. In particular, for any $\alpha \in A(Y)$, $\boxed{f_*(f^*\alpha) = c_d(Q) \cdot \alpha}$. (This was one of our results about the top Chern class. f_*f^* will knock the degree down by d, and we found that this operator was the same as capping with the top Chern class.)

Lemma. We can resolve the sheaf $f_*\mathcal{O}_X$ on Y by

$$0 \longrightarrow \wedge^d Q^\vee \longrightarrow \cdots \longrightarrow \wedge^2 Q^\vee \longrightarrow Q^\vee \xrightarrow{s^\vee} \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \longrightarrow 0. \tag{1}$$

Note that everything except $f_*\mathcal{O}_X$ is a vector bundle on Y.

Proof. Rather than proving this precisely, I'll do a special case, to get across the main idea. This in fact becomes a proof, once the "naturality" of my argument is established. Suppose $Y = \operatorname{Spec} k[x_1, \dots, x_n]$, so $\mathcal{O}_Y = k[x_1, \dots, x_n]$ (a bit sloppily) and $X = \vec{0} \subset Y$. Then let's build a resolution of $\mathcal{O}_X$. We start with

$$\mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

We have a big kernel obviously: the ideal sheaf of $\mathcal{O}_X$. So our next step is:

$$\mathcal{O}_Y x_1 \oplus \mathcal{O}_Y x_2 \cdots \oplus \mathcal{O}_Y x_n \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

We still have a kernel; $(-x_2x_1, x_1x_2, 0, \cdots, 0)$ is in the kernel, for example. So our next step is:

$$\mathcal{O}_Y x_1x_2 \oplus \cdots \oplus \mathcal{O}_Y x_{n-1}x_n \to$$

(We need to check that we've surjected onto the kernel! But that's not hard; you can try to prove that yourself.) And the pattern continues. We get:

$$0 \to \mathcal{O}_Y x_1 \cdots x_n \to \oplus_{i=1}^n \mathcal{O}_Y x_1 \cdots \hat{x_i} \cdots x_n \to \cdots \to \oplus_{i=1}^n \mathcal{O}_Y x_i \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

And this is what we wanted (in this special case).

(All that is missing for this to be a proof is to realize that $\oplus_{i=1}^n \mathcal{O}_Y x_i \to \mathcal{O}_Y$ is canonically $Q^\vee$.) $\square$

If E is a vector bundle on X, then we have an explicit resolution of f_*E, by tensoring (1) with p^*E:

$$0 \longrightarrow \wedge^d Q^\vee \otimes p^*E \longrightarrow \cdots \longrightarrow Q^\vee \otimes p^*E \xrightarrow{s^\vee} p^*E \longrightarrow f_*E \longrightarrow 0.$$

(Tensoring with a vector bundle is exact, and $(p^*E) \otimes \mathcal{O}_X \cong f_*E$.)

Therefore

$$\boxed{\operatorname{ch} f_*[E] = \sum_{p=0}^{d} (-1)^p \operatorname{ch}(\wedge^p Q^\vee) \cdot \operatorname{ch}(p^*E).}$$

Lemma.

$$\sum_{p=0}^{d} (-1)^p \operatorname{ch}(\wedge^p Q^\vee) = c_d(Q) \cdot \operatorname{td}(Q)^{-1}.$$

This tells you why the Todd class is what it is!

Proof. This is remarkably easy. Let $\alpha_1, \ldots, \alpha_d$ be the Chern roots of Q. Then the Chern roots of $\wedge^p Q^\vee$ are $-\sum \alpha_{i_1} \cdots \alpha_{i_p}$. Hence $\operatorname{ch}(\wedge^p Q^\vee) = \sum e^{-\sum \alpha_{i_1} \cdots \alpha_{i_p}}$ from which

$$\begin{aligned}
\sum_{p=0}^{d} (-1)^p \operatorname{ch}(\wedge^p Q^\vee) &= \sum_{p=0}^{d} (-1)^p \sum e^{-\alpha_{i_1}} \cdots e^{-\alpha_{i_p}} \\
&= \prod_{i=1}^{d} (1 - e^{-\alpha_i}) \\
&= (\alpha_1 \cdots \alpha_d) \prod_{i=1}^{d} \frac{1 - e^{-\alpha_i}}{\alpha_i} \\
&= c_d(Q) \cdots \operatorname{td}(Q)^{-1}.
\end{aligned}$$

□

Hence

$$\begin{aligned}
\operatorname{ch} f_*[E] &= c_d(Q)\operatorname{td}(Q)^{-1} \cdot \operatorname{ch}(p^*E) \\
&= f_*(f^* \operatorname{td}(Q)^{-1} \cdot f^* \operatorname{ch}(p^*E)) \quad \text{(using } c_d(Q) \cap \beta = f_*(f^*\beta)\text{, see 1st par of Section 2)} \\
&= f_*(\operatorname{td}(N_{X/Y})^{-1} \operatorname{ch}(E)) \quad \text{(using } f^*Q = N_{X/Y},\ f^*p^*E = E) \\
&= f_*(\operatorname{td}(T_X) f^* \operatorname{td}(T_Y)^{-1} \operatorname{ch}(E)) \\
&= \operatorname{td}(T_Y) f_*(\operatorname{td}(T_X) \operatorname{ch}(E)) \quad \text{(projection formula)}
\end{aligned}$$

as desired!

This ends the proof of GRR for a closed immersion of X into the projective completion of a normal bundle. □

2.2. **GRR for closed immersions in general.** Suppose $f : X \to Y$ is a closed immersion. We'll prove GRR in this case; again, we need only to consider a generator of $K(X)$, a vector bundle E on X.

We'll show GRR by deformation to the normal cone.

Let $M = \mathrm{Bl}_{X \times \{\infty\}} Y \times \mathbb{P}^1$. (Draw picture.) Recall that the fiber over ∞ is $M_\infty = \mathrm{Bl}_X Y \coprod \mathbb{P}(N \oplus \mathbf{1})$.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X \times \mathbb{P}^1 \arrow[d, "F"] & X \arrow[l] \arrow[d] \\
Y = M_0 \arrow[r] \arrow[d] & M = \mathrm{Bl}_{X \times 0} Y \times \mathbb{P}^1 \arrow[d] & M_\infty = \mathrm{Bl}_X Y \coprod \mathbb{P}(N \oplus \mathbf{1}) \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{P}^1 & \{\infty\} \arrow[l]
\end{tikzcd}$$

Define F (above), $p : X \times \mathbb{P}^1 \to X$. Resolve p^*E on M:

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to F_*(p^*E) \to 0.$$

Both $X \times \mathbb{P}^1$ and M are flat over $\mathbb{P}^1$ (recall that dominant morphisms from irreducible varieties to a smooth curve are always flat), so restriction of these exact sequences to the fibers M_0 and M_∞ (also known as tensoring with the structure sheaves of the fibers) preserves exactness.

Let $j_0 : Y \cong M_0 \hookrightarrow M$, $j_\infty : \mathrm{Bl}_X Y \cup \mathbb{P}(N \oplus \mathbf{1}) = M_\infty \hookrightarrow M$, $k : \mathbb{P}(N \oplus \mathbf{1}) \hookrightarrow M$, $l : \mathrm{Bl}_X Y \hookrightarrow M$.

Now $j_0^* G.$ resolves f_* on $Y = M_0$. So

$$\begin{aligned}
j_0 * (\mathrm{ch}(f_* E)) &= j_{0*} \mathrm{ch}(j_0^* G\cdot) \\
&= \mathrm{ch}(G.) \cap j_{0*}[Y] \quad \text{(proj. formula)} \\
&= \mathrm{ch}(G.) \cap j_{\infty *}[M_\infty] \quad \text{(pulling back rat'l equivalence } 0 \sim \infty \in \mathbb{P}^1) \\
&= \mathrm{ch}(G.) \cap (k_*[\mathbb{P}(N \oplus \mathbf{1})] + l_*[\mathrm{Bl}_X Y])
\end{aligned}$$

Now $G.$ is exact away from $X \times \mathbb{P}^1$, so it is exact on $\mathrm{Bl}_X Y$, so the Chern character of the complex (the alternating sums of the Chern characters of the terms) is 0. Hence:

$$= \mathrm{ch}(G.) \cap (k_*[\mathbb{P}(N \oplus \mathbf{1})])$$

Using the projection formula again:

$$= k_*(\mathrm{ch}(\overline{f}_* E) \cap [\mathbb{P}(N \oplus \mathbf{1})])$$

(where $\overline{f}$ is the map $X \hookrightarrow \mathbb{P}(N \oplus \mathbf{1})$). (We're writing this as $k_*(\mathrm{ch}(\overline{f}_* E)$.) So now we're dealing with the case $X \hookrightarrow \mathbb{P}(N \oplus \mathbf{1})$! We've already calculated that this is $f_*(\mathrm{td}(N)^{-1} \cdot \mathrm{ch}(E))$. As $[N] = [f^* T_Y] - [T_X]$:

$$\mathrm{ch}(f_* E)\, \mathrm{td}(T_Y) = f_*(\mathrm{ch}(E)\, \mathrm{td}(T_X))$$

and we're done! □

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