INTERSECTION THEORY CLASS 18

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CONTENTS

Where we're going, by popular demand: Grothendieck Riemann-Roch (chapter 15); bivariant intersection theory and A^* (chapter 17).

1. LAST DAY

We defined the Gysin pullback $i^!$ in a rather general circumstance. I have only a few additional comments to make. Recall that a morphism $f : X \to Y$ is a *local complete intersection morphism* if f can be factored as a local complete intersection followed by a smooth morphism.

I don't know why one wouldn't more generally think of factorizations into a local complete intersection followed by a *flat* morphism.

I gave you a few examples as to why you might care about such morphisms. Here is another. If X and Y are smooth then *any* morphism between them is an lci morphism. Reason: factor it into

$$X \hookrightarrow X \times Y \to Y.$$

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2. TOWARDS GROTHENDIECK-RIEMANN-ROCH

I'm going to first explain the terminology behind the statement, then give the statement. I will then give some examples to show you that the statement is in fact very powerful. And finally, I hope to sketch a proof in an important special case,

2.1. **The Chern character and the Todd class.**

Suppose E is a rank n vector bundle. Let $\alpha_1, \dots, \alpha_n$ be the Chern roots of the vector bundle, so $\alpha_1 + \cdots + \alpha_n = c_1(E)$, etc. Define

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i)$$

When you expand this out, you get:

$$\begin{aligned} \mathrm{ch}(E) &= \mathrm{rk}(E) + c_1 + \frac{1}{2}(c_1^2 - c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) \\ &\quad + \frac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4\right) + \cdots \end{aligned}$$

So this makes sense for any coherent sheaf, not just a vector bundle. In that case, rank refers to the rank at the generic point.

Exercise. For any exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$. (This is true for coherent sheaves in general.)

For comparison, the Chern polynomial is *multiplicative* in exact sequences; the Chern character is *additive*.

Exercise. For tensor products of vector bundles, $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E')$. I don't think this is true for coherent sheaves in general, but haven't checked. (I would expect $\sum_{i \geq 0} \mathrm{ch}(\underline{\mathrm{Tor}}^i(E, E')) = \mathrm{ch}(E) \cdot \mathrm{ch}(E')$.)

The *Todd class* $\mathrm{td}(E)$ of a vector bundle is defined by

$$\mathrm{td}(E) = \prod_{i=1}^{r} Q(\alpha_i)$$

where

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}.$$

The first few terms are

$$\begin{aligned} \mathrm{td}(E) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &\quad + \frac{1}{720}\left(-c_1^4 + 4c_1 2c_2 + 3c_2^2 + c_1c_3 - c_4\right) + \cdots \end{aligned}$$

If $0 \to E' \to E \to E'' \to 0$ is exact, then

$$\mathrm{td}(E) = \mathrm{td}(E')\,\mathrm{td}(E'').$$

Like the Chern polynomial, it is multiplicative in exact sequences.

2.2. **The Grothendieck groups** K^0X **and** K_0X**.** If you went to Dan Ramras' K-theory talks, you will have seen these.

The *Grothendieck group of vector bundles* K^0X on X is the group generated by vector bundles, modulo the relations on exact sequences $[E] = [E'] + [E'']$. Vector bundles pull back to vector bundles, and exact sequences of vector bundles pull back to exact sequences of vector bundles, so if a morphism $f : X \to Y$ induces a homomorphism $f^* : K^0X \to K^0Y$. However, vector bundles seldom pushforward to vector bundles.

K^0X is a *ring*: $[E] \cdot [F] = [E \otimes F]$.

The *Grothendieck group of coherent sheaves* K_0X on X is the group generated by coherent sheaves, modulo the same relations on exact sequences. Bad news: coherent sheaves pull back to coherent sheaves, but exact sequences of coherent sheaves don't pull back to exact sequences of coherent sheaves. So we don't have a pullback map $f^* : K_0X \to K_0Y$. Good news: we can make sense of pushforwards; if $f : X \to Y$ is a proper morphism, then coherent sheaves pushforward to coherent sheaves (see Hartshorne). Bad news: exact sequences don't push forward to exact sequences: If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence on X, then we only get left exactness of pushforwards:

$$0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}''.$$

Good news: we can extend this to a long exact sequence:

$$\begin{array}{l} 0 \longrightarrow R^0f_*\mathcal{F}' \longrightarrow R^0f_*\mathcal{F} \longrightarrow R^0f_*\mathcal{F}'' \longrightarrow \\ \\ \qquad R^1f_*\mathcal{F}' \longrightarrow R^1f_*\mathcal{F} \longrightarrow R^1f_*\mathcal{F}'' \longrightarrow \\ \\ \qquad R^2f_*\mathcal{F}' \longrightarrow R^2f_*\mathcal{F} \longrightarrow R^2f_*\mathcal{F}'' \longrightarrow \cdots \end{array}$$

So this tell us how to define $f_* : K_0X \to K_0Y$, by

$$f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}].$$

(See Hartshorne for more on these "higher direct image sheaves. They can be defined as follows: $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $U \to H^i(f^{-1}(U), \mathcal{F})$.

We obviously have a homomorphism $K^0X \to K_0X$.

K_0X is a K^0X-module: $K^0X \otimes K_0X \to X$ is given by $[E] \cdot [\mathcal{F}] = [E \otimes \mathcal{F}]$. (Exercise: this is well-defined. Key fact: if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves, and E is a vector bundle, then $0 \to E \otimes \mathcal{F}' \to E \otimes \mathcal{F} \to E \otimes cF'' \to 0$ is exact. (Explain. Tensoring with locally frees is exact.)

Lemma. If $\alpha \in K^0Y$ and $\beta \in K_0X$, and $f : X \to Y$, then $f_*(f^*\alpha \cdot \beta) = \alpha f_* \beta$.

Proof. The projection formula $R^i f_*(f^*E \otimes \mathcal{F}) = E \otimes R^i f_* \mathcal{F}$, shown in Hartshorne. $\square$

Fact. If X is nonsingular, the map $K^0 X \to K_0 X$ is an isomorphism.

Reason: If X is nonsingular, then $\mathcal{F}$ has a finite resolution by locally free sheaves:

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to \mathcal{F} \to 0,$$

where the $n \leq \dim X$. Hence the inverse map is $[\mathcal{F}] = \sum_{i=0}^{n} (-1)^i [E_i]$. A sketch of the reason: show that there is a vector bundle surjecting onto $\mathcal{F}$. ("There are enough locally free's.") Build the sequence from right to left. By the time you reach E_n, you will run out of steam — the kernel at some point will already be locally free. How do you show this? You have a cohomological measure of the "non local freeness" of a coherent sheaf. If the measure is 0, the sheaf is the 0 sheaf. If $0 \to \mathcal{F}' \to E \to \mathcal{F} \to 0$, then you show that the cohomological measure of $\mathcal{F}'$ is one less than that of $\mathcal{F}$. (Hence if the cohomological measure is 1, then the sheaf is locally free.)

From now on, X will be smooth, so $K^0 X = K_0 X$, so I'll just call this group $K(X)$.

The Chern character map descends to $K(X)$:

$$ch : K(X) \to A(X)_{\mathbb{Q}}.$$

3. Statement of the theorem

Grothendieck-Riemann-Roch Theorem. For any $\alpha \in K(X)$,

$$\mathrm{ch}(f_* \alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot td(T_X)).$$

Here $f : X \to Y$ is a proper morphism of smooth varieties.

(I should point out where all the intersections take place, and where the pushforwards take place!)

This can be generalized further to singular schemes, but this is enough generality for now.

3.1. **Why you should care.**

Before we get into proving it, let me first try to convince you how powerful it is. I'll first show that it gives you old-fashioned Riemann-Roch. (I won't try to convince you why you should care about Riemann-Roch for curves — that is a whole lecture in itself, or more!)

Let's apply this to Y a point, X a smooth curve, and α a line bundle L. Then we get

$$h^0(X, L) - h^1(X, L) = \cdots$$

On the right side, we have

$$f_*((1 + c_1(L))(1 + \frac{1}{2} c_1(T)) = f_*(1 + c_1(L) - \frac{1}{2} c_1(K)) = \deg(c_1(L) - \frac{1}{2} c_1(K)).$$

Recall that $c_1(K) = -c_1(T) = 2g - 2$. Thus the right side is $d - g + 1$.

That's the baby-est case. Let's make things more interesting. We'll keep Y a point and X a nonsingular curve, and now α is the class of a vector bundle V of rank r. Then we get

$$h^0(X,V) - h^1(X,V) = f_*((r + c_1(V))(1 + \frac{1}{2}c_1(T)) = f_*(c_1(V) + \frac{r}{2}c_1(T)) = d + r(1-g).$$

Let's generalize further; now V is a coherent sheaf, of "rank" r (rank at the generic point). The same formula holds!

Next let's go to the case where X is now a smooth surface, Y a point, and to keep things calm, let's make α the class of a line bundle L. Then the left side is

$$h^0(X,L) - h^1(X,L) + h^2(X,L).$$

The right side is

$$\begin{aligned} & f_*((1 + c_1(L) + \frac{1}{2}(c_1^2(L) - c_2(L))) \left(1 + \frac{c_1(T)}{2} + \frac{c_1^2(T) + c_2(T)}{12}\right) \\ = \quad & \deg\left(\frac{c_1^2(L)}{2} - c_1(L) \cdot K/2 + \frac{K^2 + c_2(T)}{12}\right) \end{aligned}$$

which is Riemann-Roch for surfaces, which you can read about in Hartshorne chapter V.

More generally still, if X is a smooth surface, and E is a vector bundle, and Y is still a point, we get

$$\chi(X,E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(T_X).$$

We have reproved the *Hirzebruch-Riemann-Roch* theorem. And this also works for coherent sheaves.

What about if Y is *not* a point? I'll describe why you care somewhat philosophically. Suppose you have a nice morphism $X \to Y$, interpreted as "nice family" (say of smooth surfaces). Say you have a vector bundle on the family. On each of the elements of the family (the fibers of the morphism), you have a vector bundle; let's say to make things nice that for every element of the family, this vector bundle has vanishing higher cohomology. Then $h^0(V)$ is constant, as $h^0(V) = \chi(V)$, and $\chi(V)$ is constant on connected families. Thus for each point of the base Y, you have a vector space of some rank $h^0(V)$. You should expect this to glue together into a vector bundle, and indeed it does: f_*V. (Again, to make this precise requires Hartshorne chapter III or its equivalent.) *Which vector bundle do you get?* For example, what are its Chern classes? Grothendieck-Riemann-Roch will answer this for you!

So let me emphasize: you're going to see a proof of GRR that will not be too bad; and as a special case you'll get old-fashioned Riemann-Roch for curves. I think the difficulty of this proof is comparable to the difficulty of building up the machinery behind the "usual" proof of Riemann-Roch in the algebraic category; so you may as well get a much more powerful result for the same price.

4. TOWARD A PROOF

I'll prove this in the case where $X \to Y$ factors through $X \hookrightarrow \mathbb{P}^n \times Y \to Y$ where the first is a closed immersion. (This is a projective morphism in the sense of Hartshorne, and a special case of a projective morphism according to other people, such as EGA. I don't want to get into this.) This isn't such an outrageous assumption; for example, if X is projective, then $X \hookrightarrow \mathbb{P}^n$, and then $X \hookrightarrow \mathbb{P}^n \times Y$.

Lemma. Given $X \xrightarrow{f} Z \xrightarrow{g} Y$. Suppose GRR holds for f and g. Then it holds for $g \circ f$.

Proof. This has been cooked up to be easy! (That Grothendieck is quite a tricky guy!) The pushforward of $\mathrm{ch}(\alpha)\,\mathrm{td}(T_X)$ by f is $\mathrm{ch}(f_*\alpha)\,\mathrm{td}(T_Z)$, by GRR for f. The pushforward of this in turn is $\mathrm{ch}(g_*f_*\alpha)\,\mathrm{td}(T_Y)$, by GRR for g. But then we have GRR for $g \circ f$: $(g \circ f)_*(\mathrm{ch}(\alpha)\,\mathrm{td}(T_X)) = \mathrm{ch}(g_*f_*\alpha)\,\mathrm{td}(T_Y)$. □

So our strategy is clear. We're going to prove GRR for closed immersions $X \hookrightarrow Y$, and we'll prove it for $\mathbb{P}^n \times Y \to Y$.

5. GROTHENDIECK-RIEMANN-ROCH FOR $\mathbb{P}^n \to \mathrm{pt}$

Let me first work out $K(\mathbb{P}^n)$.

Theorem. The group $K_0(\mathbb{P}^m)$ is generated by the classes $[\mathcal{O}(n)]$, with $0 \le n \le m$.

First we show:

Lemma. $K_0(\mathbb{P}^m)$ is generated by the classes of line bundles $[\mathcal{O}(n)]$, without any restriction on n.

Proof. I will need some machinery we have not developed. How much extra you will need to consider as a "black box" will depend on how much you already know. Let $\mathcal{F}$ be any coherent sheaf. Our goal is to get a resolution of $\mathcal{F}$ by direct sums of line bundles:

$$0 \to \oplus\mathcal{O}(?) \to \oplus\mathcal{O}(?) \to \cdots \to \oplus\mathcal{O}(?) \to \mathcal{F} \to 0.$$

By an earlier statement, we need only show that for any coherent sheaf $\mathcal{F}$, we can find a surjection $\oplus_{i=1}^{j}\mathcal{O}(n) \to \mathcal{F}$, because then we can iterate this, and at some point we will get a 0.

By a property of ample line bundles, for $N \gg 0$, $\mathcal{F} \otimes \mathcal{O}(N)$ is generated by global sections. (It is then generated by a finite number of global sections, by a Noetherian argument.) That means that there is a surjection $\oplus_{i=1}^{j}\mathcal{O} \to \mathcal{F}(N)$. Twisting by $\mathcal{O}(-N)$, we get our desired surjection $\oplus_{i=1}^{j}\mathcal{O}(-N) \to \mathcal{F}$. □

The theorem is then proved once we know the next lemma:

Lemma. There is an exact sequence on $\mathbb{P}^m$

$$0 \to \mathcal{O} \to \oplus^{m+1}\mathcal{O}(1) \to \oplus^{\binom{m+1}{2}}\mathcal{O}(2) \to \cdots \oplus^{\binom{m+1}{j}}\mathcal{O}(j) \to \cdots \oplus^{m+1}\mathcal{O}(m) \to \mathcal{O}(m+1) \to 0.$$

Here's how this implies the theorem. Twisting this by $\mathcal{O}(N)$ we get:

$$0 \to \mathcal{O}(N) \to \oplus^{m+1}\mathcal{O}(N+1) \to \oplus^{\binom{m+1}{2}}\mathcal{O}(N+2) \to \cdots \oplus^{m+1}\mathcal{O}(N+m) \to \mathcal{O}(N+m+1) \to 0.$$

This expresses $[\mathcal{O}(N+m)]$ in terms of the classes of the $m+1$ smaller line bundles. Similarly, it expresses $[\mathcal{O}(N)]$ in terms of the classes of the $m+1$ larger line bundles. Thus by using this repeatedly, any line bundle can be expressed in terms of $\mathcal{O}$, $\mathcal{O}(1), \dots, \mathcal{O}(m)$.

Aside: you also get some interesting algebra out of this. Apply the Chern polynomial to this exact sequence. You get

$$\prod_{i=0}^{m+1}(1+iH)^{(-1)^i\binom{m+1}{i}} \equiv 1 \pmod{H^{m+1}}$$

Example $m=1$: $(1+H)^{-2}(1+2H)^1 \equiv 1 \pmod{H^2}$. Joe Rabinoff gave me a nice explanation of why this is true; I'll give it next day.

Sketch of proof of Lemma. We'll prove instead an exact sequence

$$0 \to \mathcal{O}(-m-1) \to \oplus^{m+1}\mathcal{O}(-m) \to \cdots \oplus^{m+1}\mathcal{O}(-1) \to \mathcal{O} \to 0$$

which is the dual (or alternatively, a twist) of the one we want. Let $V = \oplus^{m+1}\mathcal{O}(-1)$. Then this sequence is

$$0 \to \wedge^{m+1}V \to \wedge^m V \to \cdots \to \wedge^1 V \to \wedge^0 V \to 0.$$

You can check this on the level of graded modules. Let $S = k[x_0, \dots, x_m]$, with the usual grading. Let $\oplus^{m+1}S[-1]$. ($S[-1]$ is the same as S, except the grading is shifted by 1, so $S[-1]_1$ has dimension 1.) Define the map $V \to S$ by multiplication by $(x_0, \dots, x_m)$. This induces maps $\wedge^{j+1}V \to \wedge^j V$. Then you can check by hand that this is exact everywhere.; $\square$

Theorem. GRR is true for $\mathbb{P}^m \to \mathrm{pt}$ for the line bundles $\mathcal{O}(n)$ ($0 \le n \le m$). Hence GRR is true for $\mathbb{P}^m \to \mathrm{pt}$.

Proof. Now $p_*[\mathcal{O}(n)] = \chi(\mathbb{P}^m, \mathcal{O}(n))$. Now we can compute the cohomology groups of $\mathcal{O}(n)$ by hand, and we find that $h^i(\mathcal{O}(n)) = 0$ for $n \ge 0$. Thus $\chi(\mathbb{P}^m, \mathcal{O}(n)) = h^0(\mathbb{P}^m, \mathcal{O}(n))$. And this corresponds to the vector space of degree n polynomials with $m+1$ variables. This turns out to be $\binom{n+m}{m}$.

Hence we wish to prove that

$$\int_{\mathbb{P}^m} \mathrm{ch}(\mathcal{O}(n))\,\mathrm{td}(T_{\mathbb{P}^m}) = \binom{n+m}{m}.$$

Let's do this.

Let's first calculate $\mathrm{td}(T_{\mathbb{P}^m})$. The "Euler exact sequence" for the tangent bundle of projective space is

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus m+1} \to T_{\mathbb{P}^m} \to 0.$$

(Aside: notice that this is the beginning of that big exact sequence of direct sums of line bundles in the proof of the previous lemma (that was omitted in class)! This shouldn't be a coincidence, but I'm not precisely sure why not.) The Todd class is *multiplicative* for exact sequences, so we get

$$\mathrm{td}(T_{\mathbb{P}^m}) = \left(\frac{x}{1-e^{-x}}\right)^{m+1}$$

where $x = c_1(\mathcal{O}(1))$. We also have $\mathrm{ch}(\mathcal{O}(n)) = e^{nx}$. Thus we want to show that

$$\int_{\mathbb{P}^m} \frac{e^{nx}x^{m+1}}{(1-e^{-x})^{m+1}} = \binom{n+m}{m}.$$

The thing on the left says: "extract the x^m term the power series". So we want to prove

$$[x^m]\frac{e^{nx}x^{m+1}}{(1-e^{-x})^{m+1}} = \binom{n+m}{m}.$$

Now the left side

$$= [x^{-1}]\frac{e^{nx}}{(1-e^{-x})^{m+1}}$$

so we've turned this into a residue calculation, which is a reasonable quals problem. □

Next, we'll show that knowing the result for $\mathbb{P}^m \to pt$ will imply the result for $\mathbb{P}^m \times Y \to Y$.

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