INTERSECTION THEORY CLASS 17

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Where we're going, by popular demand: Grothendieck Riemann-Roch (15); comparison to Borel-Moore homology (chapter 19).

1. WHERE WE ARE

We defined the Gysin pullback $i^!$ and a rather general intersection product. Let $i : X \hookrightarrow Y$ be a local complete intersection of codimension d. Y is arbitrarily horrible. Suppose V is a scheme of pure dimension k, with a map $f : V \to Y$. Here I am *not* assuming V is a closed subscheme of Y. Then define W to be the closed subscheme of V given by pulling back the equations of X in Y:

$$\begin{array}{ccc} W & \xrightarrow{\text{cl. imm.}} & V \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\text{cl. imm.}} & Y \end{array}$$

(notice definition of g).

The cone of X in Y is in fact a vector bundle (as $X \hookrightarrow Y$ is a local complete intersection); call it $N_X Y$. The cone $C_W Y$ to W in Y may be quite nasty; but we saw that $C_W Y \hookrightarrow g^* N_X Y$. Then we define

$$X \cdot V = s^*[C_W V]$$

where $s : W \to g^* N_X Y$ is the zero-section. (Recall that the Gysin pullback lets us map classes in a vector bundle to classes in the base, dropping the dimension by the rank. Algebraic black box from appendix: as V is purely k-dimensional scheme, $C_W V$ is also.)

Last time I proved:

Date: Wednesday, November 17, 2004.

Proposition. If ξ is the universal quotient bundle of rank d on $\mathbb{P}(g^*N_{X/Y} \oplus 1)$, and $q : \mathbb{P}(g^*N_{X/Y} \oplus 1) \to W$ is the projection, then

$$X \cdot V = q_*(c_d(\xi) \cap [\mathbb{P}(C_{W/V} \oplus 1)]).$$

and

Proposition. $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$. (Here $\{\cdot\}_{k-d}$ means "take the dimension $k - d$ piece of $\cdot$.)

and stated (without proof):

Proposition. If $d = 1$ (X is a Cartier divisor on Y), V is a variety, and f is a closed immersion, then $X \cdot V$ is the intersection class we defined earlier ("cutting with a pseudo-divisor g^*X").

1.1. **Refined Gysin homomorphisms $i^!$.** Let $i : X \to Y$ be a local complete intersection of codimension d as before, and let $f : Y' \to Y$ be any morphism.

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

As before, $C' = C_{X'}Y' \hookrightarrow g^*N_XY$. Define the *refined Gysin homomorphism* $i^!$ as the composition:

$$A_kY' \xrightarrow{\sigma} A_kC' \longrightarrow A_kN \xrightarrow{s^*} A_{k-d}X' .$$

Note what we can now do: we used to be able to intersect with a local complete intersection of codimension d. Now we can intersect in a more general setting.

We'll next show that these homomorphisms behave well with respect to everything we've done before. These are all important, but similar to what we've done before, so I'll state the various results. I'll just sporadically give proofs.

Handy fact: Say we want to prove something about $i^![V]$. Consider

$$\begin{array}{ccc} X' \cap V & \longrightarrow & V \\ \downarrow{\scriptstyle h} & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow{\scriptstyle g} & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Then

$$i^![V] = c(g^*N_{X/Y}) \cap h_*s(X' \cap V, V).$$

Reason we like this: we already know Chern and Segre classes behave well. So we can reduce calculations about $i^!$ to things we've already proved. Reason for fact: Calculate

$X \cdot V$ using

$$\begin{array}{ccc} X' \cap V & \longrightarrow & V \\ \downarrow{\scriptstyle g\circ h} & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

We get $c(h^*g^*N) \cap s(X' \cap V, V)$. Push this forward to X':

$$h_*(c(h^*g^*N) \cap s(X' \cap V, V)) = c(g^*N) \cap h_*s(X' \cap V, V))$$

using the projection formula. We now have to show that this really gives $i^![V]$. (Fulton uses this second version as the original definition.) Omitted.

Refined Gysin commutes with proper pushforward and proper pullback. Consider the fiber diagram

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

where i is a locally closed intersection of codimension d.

(a) If p is proper and $\alpha \in A_kY''$, then $i^!p_*(\alpha) = q_*(i^!\alpha)$ in $A_{k-d}X'$. (Caution: $i^!$ means two different things here!)
(b) If p is flat of relative dimension n, and $\alpha \in A_kY'$, then $i^!p^*(\alpha) = q^*(i^!\alpha)$ in $A_{k+n-d}X''$.

Proof. (a) We may assume $\alpha = [V']$ (on Y''). Let $V = p(V')$ (on Y').

$$\begin{aligned} i^!p_*\alpha &= \deg(V'/V)\{c(g^*N_{X/Y}) \cap s(X' \cap V, V)\}_{k-d} \quad \text{previous proposition} \\ &= \{c(g^*N_{X/Y}) \cap q_*s(X'' \cap V', V')\}_{k-d} \quad \text{Segre classes push forward well} \\ &= q_*\{c(q^*g^*N_{X/Y}) \cap s(X'' \cap V', V')\}_{k-d} \quad \text{projection formula} \\ &= q_*i^![V'] \end{aligned}$$

Compatibility. If i' is also a local complete intersection of codimension d, and $\alpha \in A_kY''$, then $i^!\alpha = (i')^!\alpha$ in $A_{k-d}X''$.

It suffices to verify that $g^*N_XY \cong N_{X'}Y'$. Reason: If $\mathcal{I}$ and $\mathcal{I}'$ are the respective ideal sheaves, the canonical epimorphism $g^*(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}'/(\mathcal{I}')^2$ must be an isomorphism. (Details omitted. X is locally cut out in Y by d equations. X' is cut out in Y' by (the pullbacks of) the same d equations.)

1.2. **Excess intersection formula.** Consider the same fiber diagram as before

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i'} & Y \end{array}$$

where now i is still a locally closed intersection of codimension d, and i' is *also* a locally closed intersection, of possibly different dimension d'. Let N and N' be the two normal bundles; as before we have a canonical closed immersion $N' \hookrightarrow g^*N$. Let $E = g^*N/N'$ be the quotient vector bundle, of rank $e = d - d'$.

For any $\alpha \in A_k Y''$, note that $i^!(\alpha)$ and $(i')^!(\alpha)$ differ in dimension by e. What is their relationship? Answer:

Excess intersection formula. For any $\alpha \in A_k Y''$, $i^!(\alpha) = c_e(q^*E) \cap (i')^!(\alpha)$ in $A_{k-d}X''$.

(Proof short but omitted.)

Immediate corollary. Specialize to the case where the top row is the same as the middle row, and i' is an isomorphism:

$$\begin{array}{ccc} X' & \xrightarrow[i']{\sim} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

Then $i^!\alpha = c_d(g^*N) \cap \alpha$. Specialize again to $X' = Y' = X$ to get the self-intersection formula: $i^*i_*\alpha = c_d(N) \cap \alpha$.

Intersection products commute with Chern classes. Let $i : X \to Y$ be a locally closed intersection of codimension d,

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow[i]{} & Y \end{array}$$

a fiber square, and F a vector bundle on Y'. Then for all $\alpha \in A_k Y'$ and all $m \geq 0$,

$$i^!(c_m(F) \cap \alpha) = c_m(i'^*F) \cap i^!\alpha$$

in $A_{k-d-m}(X')$

Proof omitted.

Refined Gysin homomorphisms commute with each other. Let $i : X \to Y$ be a locally closed intersection of codimension d, $j : S \to T$ a locally closed intersection of codimension e. Let Y' be a scheme, $f : Y' \to Y$, $g : Y' \to T$ two morphisms. Form the fiber

diagram

$$\begin{array}{ccccc} X'' & \longrightarrow & Y'' & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle j'} & & \downarrow \\ X' & \xrightarrow[i']{} & Y' & \xrightarrow[g]{} & T \\ \downarrow & & \downarrow{\scriptstyle f} & & \\ X & \xrightarrow[i]{} & Y & & \end{array}.$$

Then for all $\alpha \in A_k Y'$, $j^! i^! \alpha = i^! j^! \alpha$ in $A_{k-d-e} X''$.

Proof (long!) omitted. Idea: by blowing up to reduce to the case of divisors, as we did when we showed that the intersection of two divisors was independent of the order of intersection, long ago.

Functoriality.

The refined Gysin homomorphisms for a composite of locally closed intersections is the composite of the refined Gysin homomorphisms of the factors.

Consider a fiber diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{j'} & Z' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y & \xrightarrow[j]{} & Z. \end{array}$$

If i (resp. j) is a locally closed intersection of codimension d (resp. e), then ji is a locally closed intersection of codimension $d+e$, and for all $\alpha \in A_k Z'$, $(ji)^! \alpha = i^! j^! \alpha$ in $A_{k-d-e} X'$.

Proof omitted. Similarly:

Second functoriality proposition. Consider a fiber diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y & \xrightarrow[p]{} & Z. \end{array}$$

(a) Assume that i is a locally closed intersection of codimension d, and that p and pi are flat of relative dimensions n and $n-d$. Then i' is a locally closed intersection of codimension d, p' and $p'i'$ are flat, and for $\alpha \in A_k Z'$,

$$(p'i')^* \alpha = i^! p'^* \alpha$$

in $A_{k+n-d} X'$.

(b) Assume that i is a locally closed intersection of codimension d, p is smooth of relative dimension n, and pi is locally closed intersection of codimension $d-n$. Then for all $\alpha \in A_k Z'$,

$$(pi)^! \alpha = i^! (p'^* \alpha)$$

in $A_{k+n-d} X'$.

Short proof, omitted.

2. Local complete intersection morphisms

A morphism $f : X \to Y$ is called a lci morphism of codimension d if it factors into a locally closed intersection $X \to P$ followed by a smooth morphism $p : X \to Y$. Examples: families of nodal curves over an arbitrary base; families of surfaces with mild singularities. Reason we care: often we want to consider families of things degenerating. We won't need this in the next two weeks, but it's worth at least giving the definition.

For any lci morphism $f : X \to Y$ of codimension d, and any morphism $h : Y' \to Y$, we have the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & Y \end{array}$$

We want to define a refined Gysin homomorphism

$$f^! : A_k Y' \to A_{k-d} X'.$$

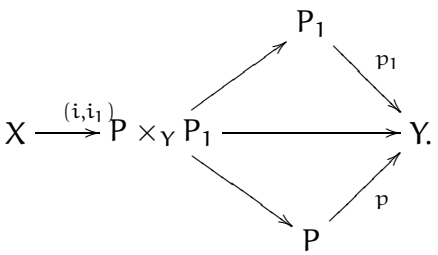
Here's how. Factor f into $p \circ i$ where $p : P \to Y$ is a smooth morphism of relative dimension $d + e$ and $i : X \hookrightarrow P$ is a local complete intersection of codimension e. Then form the fiber diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & P' & \xrightarrow{p'} & Y' \\ \downarrow{\scriptstyle h'} & & \downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[i]{} & P & \xrightarrow[p]{} & Y. \end{array}$$

Then p' is smooth (smooth morphisms behave well under base change), and we define $f^! \alpha = i^!((p')^* \alpha)$ (smooth morphisms are flat; this is part of the definition).

Proposition (a) The definition of $f^!$ is independent of the factorization of f. (!!!) (b) If f is both lci and flat, then $f^! = f'^*$. (c) The assertions earlier (pushforward and pullback compatibility; commutativity; functoriality) for locally closed intersections are valid for arbitrary lci morphisms. There is also an excess intersection formula, that I won't bother telling you precisely.

Because (a) seems surprising, and the roof is short, I'll give it to you. If $X \xrightarrow{i_1} P_1 \xrightarrow{p_1} Y$ is another factorization of f, compare them both with the diagonal:



Use the second functoriality proposition (b).

Then (b) follows from (a). (c) is omitted.

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