INTERSECTION THEORY CLASS 16

RAVI VAKIL

CONTENTS

1. WHERE WE ARE

We've covered a lot of ground so far. I want to remind you that we've essentially defined a very few things, and spent all our energy on showing that they behave well with respect to each other. In particular: proper pushforward, flat pullback, $c_\cdot$, $s_\cdot$, $s_\cdot(X, Y)$. Gysin pullback for divisors; intersecting with pseudo-divisors. Gysin pullback for 0-sections of vector bundles.

We know how to calculate the Segre class of a cone.

$$s(C) := q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}(C \oplus \mathbf{1})] \right)$$

where q is the morphism $\underline{\mathrm{Proj}}(C \oplus \mathbf{1}) \to X$.

Last day, Andy talked about linear systems.

1.1. **Deformation to the normal cone.** This is the central construction. Suppose $X \to Y$ is a closed immersion of schemes.

Goal: We will define a *specialization homomorphism* $\sigma : A_k Y \to A_k C$ where C is the normal cone $\sum_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1}$.

If $W \hookrightarrow Z$ is a closed immersion, recall that $\mathrm{Bl}_W Z$ is the blow-up of Z along W. For the purposes of the next few lectures, let $E_W Z$ be the exceptional divisor, and let $\mathcal{I}_W Z$ be the ideal sheaf. Then recall:

Date: Monday, November 15, 2004.

- $\mathrm{Bl}_W Z = \underline{\mathrm{Proj}} \oplus (\mathcal{I}_W Z)^n$
- $E_W Z = \underline{\mathrm{Proj}} \oplus (\mathcal{I}_W Z)^n / (\mathcal{I}_W Z)^{n+1}$
- $E_W Z \hookrightarrow \mathrm{Bl}_W Z$ is a closed immersion, and describes $E_W Z$ as an effective Cartier divisor, in fact in class $\mathcal{O}_{\underline{\mathrm{Proj}} \oplus (\mathcal{I}_W Z)^n}(1)$. The closed immersion is visible at the level of graded algebras.

Blow up $Y \times \mathbb{P}^1$ along $X \times 0$. The central fiber turns into $\mathrm{Bl}_X Y$, union (the exceptional divisor of the blow-up) $\underline{\mathrm{Proj}}(C_X Y \oplus \mathbf{1}) = C_X Y \coprod \mathbb{P} C_X Y \cong E_X Y$. We glue these two pieces together along $E_X Y$.

We throw out $\mathrm{Bl}_X Y$: let $M^\circ = \mathrm{Bl}_{X \times 0}(Y \times \mathbb{P}^1) - \mathrm{Bl}_X Y$. (A picture is helpful here.) Away from 0, M° is still $Y \times \mathbb{A}^1$. Over 0, we see the normal cone $C_X Y$. So we have really deformed Y to the normal cone. Let $i : C \hookrightarrow M^\circ$ be the closed immersion of the normal cone, and let $j : Y \times (\mathbb{P}^1 - 0) \hookrightarrow M^\circ$ be the open immersion of the complement.

The argument from last week was slick enough that I'm going to repeat it (quickly). Consider the following diagram:

$$\begin{array}{ccccccc}
A_{k+1}C & \xrightarrow{i_*} & A_{k+1}M^\circ & \xrightarrow{j^*} & A_{k+1}(Y \times \mathbb{A}^1) & \longrightarrow & 0 \\
 & \text{Gysin map for divisors} & \downarrow i^* & & \uparrow \sim & & \\
 & & A_k C & & A_k Y. & &
\end{array}$$

The top row is the excision exact sequence. The right column is flat pullback and is an isomorphism, as flat pullback to the total space of a line bundle is always an isomorphism. The left column is the Gysin pullback map to divisors.

We have shown $i^* i_* : A_{k+1} C \to A_k C$ is the same as capping with c_1 of the normal (line) bundle to the divisor C in M°. (Reminder for future use: if $i : W \hookrightarrow Z$ is the closed immersion of W into a vector bundle over W, as the zero section, then the map $i_* i^* : A_* W \to A_* W$ is capping with the top Chern class of the vector bundle.) In this case the normal line bundle is trivial: it is the pullback of the normal bundle to $t = 0$ in $\mathbb{P}^1$. Thus $i^* i_* = 0$. Hence $A_{k+1} M^\circ \to A_k C$ descends to a map $A_{k+1}(Y \times \mathbb{A}^1) \to A_k C$, and hence we get a map $\sigma : A_k Y \to A_k C$, which is what we wanted! The final diagram:

$$\begin{array}{ccccccc}
A_{k+1}C & \xrightarrow{i_*} & A_{k+1}M^\circ & \xrightarrow{j^*} & A_{k+1}(Y \times \mathbb{A}^1) & \longrightarrow & 0 \\
 & \searrow^{i^* i_* = 0} & \downarrow i^* & \swarrow^{\therefore} & \uparrow \sim & & \\
 & & A_k C & \xleftarrow[\therefore \sigma]{} & A_k Y. & &
\end{array}$$

1.2. **Gysin pullback for local complete intersections.** We already had defined the Gysin pullback or Gysin homomorphism in the case where Y is a vector bundle over X: $A_k Y \to A_{k-d} X$. We now extend it to when "Y looks like a vector bundle over X": when X is a local complete intersection inside Y. Define the *Gysin pullback* $i^* : A_k Y \to A_{k-d}$ as the composition

$$A_k Y \xrightarrow{\sigma} A_k N \xrightarrow{s_N^*} A_{k-d} X$$

where s_N^* is the old Gysin morphism for vector bundles. We're going to generalize this further soon!

I showed that the two definitions agree, by observing that the normal cone to a the zero section of a vector bundle is the vector bundle itself (which is true). Also, we showed earlier that the Gysin pullback for vector bundles satisfied all sorts of nice properties; if we show that σ satisfies these nice properties too, then we'll know it for Gysin pullbacks to local complete intersections.

Note: $i^* i_*(\alpha) = c_d(N) \cap \alpha$. Reason: we know this for vector bundles.

Note also: If Y is purely n-dimensional, notice that $i^*[Y] = [X]$. Because $\sigma[Y] = [C]$, and $s_N^*[C] = [X]$.

I concluded with:

1.3. **Intersection products on smooth varieties.** If X is an n-dimensional variety which is smooth over the ground field, then the diagonal morphism $\Delta : X \to X \times X$ is a local complete intersection of codimension n. Then we get an intersection product on A_*X!

$$A_p X \otimes A_q X \xrightarrow{\times} A_{p+q}(X \times X) \xrightarrow{\Delta^*} A_{p+q-n} X.$$

(Notice that we don't need X to be proper!)

I should probably be a bit clearer about that first map, which might reasonably be called $\boxtimes$. (You can see a discussion in Chapter 1 if you want.) Here's what we need: consider the map

$$Z_p X \otimes Z_q Y \xrightarrow{\times} Z_{p+q}(X \times Y)$$

defined on varieties by $[V] \times [W] = [V \times W]$, and defined generally by linearity. (We'll take $X = Y$, but we might as well do this in some generality.) We want this to descend to the level of Chow classes:

Lemma. If $\alpha \sim 0$ (or, symmetrically, $\beta \sim 0$) then $\alpha \times \beta \sim 0$.

(This is Prop. 1.10 (a) in the book.) Likely exercise: finish this proof.

2. Intersection Products

We're now ready to discuss the last chapter in the core of the book, on intersection products. We'll define the intersection product, and then we'll verify that it has a host of properties. This verification will involve lots of diagram-chasing and symbol-pushing, so I'm going to try to concentrate on helping you keep your eye on the big picture.

What we know so far: proper pushforward, flat pullback, $s.$ of cones, e.g. $s.(X, Y)$, $c.$. Gysin pullbacks have gotten more and more complicated: i) $X \hookrightarrow Y$ as a divisor. More generally

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ X & \xrightarrow{\text{eff. Car. div.}} & Y \end{array}$$

Then $X \xrightarrow{\text{loc. com. int.}} Y$. Now we go to the logical extreme:

$$\begin{array}{ccc} & & V\ . \\ & & \downarrow \\ X & \xrightarrow{\text{loc. comp. int.}} & Y \end{array}$$

Here's the context in which we'll work. $i : X \hookrightarrow Y$ will be a local complete intersection of codimension d. Y is arbitrarily horrible. Suppose V is a scheme of pure dimension k, with a map $f : V \to Y$. Here I am *not* assuming V is a closed subscheme of Y. Then define W to be the closed subscheme of V given by pulling back the equations of X in Y:

$$\begin{array}{ccc} W & \xrightarrow{\text{cl. imm.}} & V \\ \downarrow g & & \downarrow f \\ X & \xrightarrow{\text{cl. imm.}} & Y \end{array}$$

(notice definiton of g). We'll define the intersection product $X \cdot V \in A_{k-d}W$. (We'll most obviously care about the case where $V \hookrightarrow Y$, but you'll see that this more general case will be handy too!)

The cone of X in Y is in fact a vector bundle (as $X \hookrightarrow Y$ is a local complete intersection); call it $N_X Y$. The cone $C_W Y$ to W in Y may be quite nasty; but we'll see (in just a moment) that it lives in the pullback of the normal bundle: $C_W Y \hookrightarrow g^* N_X Y$. Then we can define

$$X \cdot V = s^*[C_W V]$$

where $s : W \to g^* N_X Y$ is the zero-section. (Recall that the Gysin pullback lets us map classes in a vector bundle to classes in the base, dropping the dimension by the rank.)

Let's check that $C_W V \hookrightarrow g^* N_X Y$: The ideal sheaf $\mathcal{I}$ of X in Y generates the ideal sheaf $\mathcal{J}$ of W in V, hence there is a surjection

$$\oplus_n f^*(\mathcal{I}^n / \mathcal{I}^{n+1}) \to \oplus_n \mathcal{J}^n / \mathcal{J}^{n+1}.$$

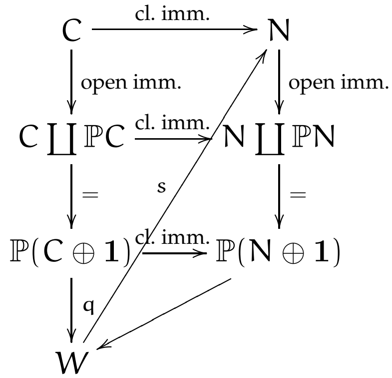
This determines a closed imbedding of the normal cone $C_W V$ into the vector bundle N.

Algebraic fact (black box from appendix): as V is purely k-dimensional scheme, $C_W V$ is also. Then we may define $X \cdot V$ as I said we would: $X \cdot V = s^* C_{W/V}$.

Proposition. If ξ is the universal quotient bundle of rank d on $\mathbb{P}(g^* N_{X/Y} \oplus 1)$, and $q : \mathbb{P}(g^* N_{X/Y} \oplus 1) \to W$ is the projection, then

$$X \cdot V = q_*(c_d(\xi) \cap [\mathbb{P}(C_{W/V} \oplus 1)]).$$

Proof. Let $C = C_{W/V}$.



We want to take the cone C and intersect it with the zero section s of the vector bundle N (the top row of this diagram). We we can do this on the second row of the diagram. Recall that we proved: if $\beta \in A_k N$ and $\overline{\beta} \in A_k(\mathbb{P}(N \oplus \mathbf{1}))$ which restricts to β. Then $s^*\beta = q_*(c_r(\xi) \cap \overline{\beta})$ where ξ is the universal (rank r) quotient bundle of $q^*(N \oplus \mathbf{1})$. Then we're done. □

Proposition. $X \cdot V = \{c(g^* N_{X/Y}) \cap s(W, V)\}_{k-d}$. (Here $\{\cdot\}_{k-d}$ means "take the dimension $k - d$ piece of $\cdot$.)

Proof. Consider the universal (or tautological) exact sequence

$$0 \to \mathcal{O}(-1) \to q^* N \oplus \mathbf{1} \to \xi \to 0$$

on $\mathbb{P}(N \oplus \mathbf{1})$. By the Whitney sum formula, $c(\xi) c(\mathcal{O}(-1)) = c(q^* N)$. Hence

$$q_*(c_d(\xi) \cap [\mathbb{P}(C \oplus \mathbf{1})]) = \{q_*(c(\xi) \cap [\mathbb{P}(C \oplus \mathbf{1})])\}_{k-d}$$

(essentially the previous proposition, but note that we've replaced $c_d(\xi)$ with $c(\xi)$)

$$= \{q_*(c(q^* N) s(\mathcal{O}(-1)) \cap [\mathbb{P}(C \oplus \mathbf{1})])\}_{k-d}$$

(using Whitney sum formula)

$$= \{c(N) \cap q_*(s(\mathcal{O}(-1)) \cap [\mathbb{P}(C \oplus \mathbf{1})])\}_{k-d}$$

(projection formula)

$$= \{c(N) \cap s(C)\}_{k-d}$$

(definition of Segre class of a cone). □

Proposition. If $d = 1$ (X is a Cartier divisor on Y), V is a variety, and f is a closed immersion, then $X \cdot V$ is the intersection class we defined earlier ("cutting with a pseudo-divisor $g^* X$").

Proof omitted.

3. Refined Gysin homomorphisms

We now come to the last fundamental construction of the subject.

Let $i : X \to Y$ be a local complete intersection of codimension d as before, and let $f : Y' \to Y$ be any morphism.

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

As before, the normal cone $C' = C_{X'}Y'$ is a closed subcone of g^*N_XY. Define the *refined Gysin homomorphism* $i^!$ (pronounced i shriek, which is what people sometimes do when they first hear about this) as the composition:

$$A_kY' \xrightarrow{\sigma} A_kC' \longrightarrow A_kN \xrightarrow{s^*} A_{k-d}X' .$$

Note what we can now do: we used to be able to intersect with a local complete intersection of codimension d. Now we can intersect in a more general setting.

We'll next show that these homomorphisms behave well with respect to everything we've done before.

E-mail address: vakil@math.stanford.edu