INTERSECTION THEORY CLASS 10

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CONTENTS

1. LAST TIME

Let E be a vector bundle of rank $e+1$ on an algebraic scheme X. Let $P = \mathbb{P}E$ be the $\mathbb{P}^e$-bundle of lines on E, and let $p = p_E : P \to X$ be the projection. The Segre classes are defined by:

$$s_i(E) \cap : A_k X \to A_{k-i} X$$

by $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^* \alpha)$.

Corollary to Segre class theorem. The flat pullback $p^* : A_k X \to A_{k+e}(\mathbb{P}E)$ is a split monomorphism: by (a) (ii), an inverse is $\beta \mapsto p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^e \cap \beta)$.

2. CHERN CLASSES

We then defined Chern classes. Define the Segre power series $s_t(E)$ to be the generating function of the s_i. Define the *Chern power series* (soon to be Chern polynomial!) as the inverse of $s_t(E)$.

We're in the process of proving parts of the Chern class theorem. Left to do:

Chern class Theorem. The Chern classes satisfy the following properties.

(a) (vanishing) For all bundles E on X, and all $i > \operatorname{rank} E$, $c_i(E) = 0$.

(e) (Whitney sum) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

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of vector bundles on X, then $c_t(E) = c_t(E') \cdot c_t(E'')$, i.e. $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$.

Notation. The Chern classes and Segre classes of all vector bundles determine a ring of operators on Chow groups. I won't give this ring a name (or I may tentatively call it the Segre-Chern ring); later we will define a ring A^*X of operators, in which these Chern and Segre classes will lie.

Splitting principle. I introduced the splitting principle, which tells that we can pretend that every vector bundle splits, not into a direct sum, but into a nice filtration.

Given a vector bundle E on a scheme X, there is a flat morphism $f : X' \to X$ such that

(1) $f^* : A_*X \to A_*X'$ is injective, and
(2) f^*E has a filtration by subbundles

$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0.$$

Injectivity shows that if we can show some equality involving Chern classes on the pull-back to X', then it will imply the same equality downstairs on X.

The construction was pretty simple: we took a tower of projective bundles.

I should have said explicitly: we've shown how to split a single vector bundle. But clearly we can split any finite number of vector bundles in this way as well.

Lemma. Assume that E is filtered with line bundle quotients $L_1, \dots, L_r$. Let s be a section of E, and let Z be the closed subset of X where s vanishes. Then for any k-cycle α on X, there is a $(k-r)$-cycle class β on Z (i.e. an element of $A_{k-r}Z$) with

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = \beta$$

in $A_{k-r}X$. (Even better, we will see that we will get equality in $A_{k-r}(Z)$: we have pinned down (or "localized") this class even further.) In particular, if s is nowhere zero, then $\prod_{i=1}^{r} c_1(L_i) = 0$. (Recall $r = \operatorname{rank} E$.)

Proof. For simplicity of exposition, let me show you how this works for $r = 2$. We have $0 \to L_1 \to E \to L_2 = 0$. The section s of E induces a section $\overline{s}$ of L_2. If Y is the zero scheme of $\overline{s}$, then $(L_2, Y, \overline{s})$ is a pseudodivisor D_2 on X. Let $j : Y \hookrightarrow X$ be the closed immersion. Intersecting with D_2 gives a class $D_2 \cdot \alpha$ in $A_{k-1}Y$ such that $c_1(L_2) \cap \alpha = j_*(D_2 \cdot \alpha)$. By the projection formula ("proper pushforward behaves with respect to c_1"):

$$c_1(L_1) \cap c_1(L_2) \cap \alpha = j_*(c_1(j^*L_1) \cap (D_2 \cdot \alpha)).$$

The bundle $L_1Y = j^*E$ has a section, induced by s, whose zero set is Z. So $c_1(j^*L_1) \cap (D_2 \cdot \alpha) \in A_{k-2}Z$ as desired.

The general argument is just the same (an induction). $\square$

Lemma. Suppose E has a filtration by subbundles $E = E_r \supset E_{r-1} \supset \cdots \supset E_0 = 0$ with quotients $L_r, \dots, L_1$. Then

$$c_t(E) = \prod_{i=1}^{r}(1 + c_1(L_i)t).$$

Proof. Let $p : \mathbb{P}E \to X$ be the associated projective bundle. We have a tautological subbundle $\mathcal{O}_{\mathbb{P}E}(-1) \to p^*E$ on $\mathbb{P}E$. Twisting (tensoring) this inclusion by the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$, we get

$$\mathcal{O}_{\mathbb{P}E} \to (p^*E) \otimes \mathcal{O}_{\mathbb{P}E}(1).$$

In other words, we have a nowhere vanishing section of $(p^*E) \otimes \mathcal{O}_{\mathbb{P}E}(1)$. Note that $(p^*E) \otimes \mathcal{O}_{\mathbb{P}E}(1)$ has a filtration with quotient line bundles $p^*L_i \otimes \mathcal{O}_{\mathbb{P}E}(1)$. Thus our previous lemma implies that

$$\prod_{i=1}^{r} c_1(p^*L_i \otimes \mathcal{O}_{\mathbb{P}E}(1)) = 0.$$

We'll now unwind this to get the result. Let $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ for convenience. Let σ_i be the ith symmetric function in $c_1(L_1), \dots, c_1(L_r)$. Let $\tilde{\sigma}_i$ be the ith symmetric function in $c_1(p^*L_1), \dots, c_1(p^*L_r)$.

We want to show that $(1 + \sigma_1 t + \sigma_2 t^2 + \cdots + \sigma_r t^r) = c_t(E)$.

We know that $c_1(p^*L_i \otimes \mathcal{O}_{\mathbb{P}E}(1)) = c_1(p^*L_i) + c_1(\mathcal{O}_{\mathbb{P}E}(1)) = c_1(p^*L_i) + \zeta$. Hence we know:

$$\zeta^r + \tilde{\sigma}_1 \zeta^{r-1} + \cdots + \tilde{\sigma}_r = 0.$$

(We feel like turning ζ into $1/t$ and using injectivity. That's in spirit what we'll do.) Multiply by ζ^{i-1} for some i. Pick any $\alpha \in A_*X$, and cap the equation with $p^*\alpha$. Then pushforward:

$$p_*(\zeta^{e+i} \cap p^*\alpha) + p_*(\tilde{\sigma}_1 \zeta^{e+i-1} \cap p^*\alpha) + \cdots + p_*(\tilde{\sigma}_r \zeta^{i-1} \cap p^*\alpha) = 0.$$

Thus these are Segre classes:

$$s_i(E) \cap \alpha + \sigma_1 s_{i-1}(E) \cap \alpha + \cdots + \sigma_r s_{i-r}(E) \cap \alpha = 0. \tag{1}$$

Multiply this by a formal variable t^i, and add up over all i to get:

$$(1 + \sigma_1 t + \cdots + \sigma_r t^r) s_t(E) = 0.$$

Oops, that wasn't quite right! Equation (1) holds for $i > 0$, so in fact

$$(1 + \sigma_1 t + \cdots + \sigma_r t^r) s_t(E) = \text{constant}.$$

But that constant is 1. Thus by the definition of $c_t(E)$, we get our desired result: $c_t(E) = 1 + \sigma_1 t + \cdots + \sigma_r t^r$. $\square$

I'm now finally ready to prove (a) and (e) of the Chern class theorem. It suffices to prove (a) assuming that E is filtered. But then $c_t(E) = \prod_{i=1}^{r}(1 + c_1(L_i)t)$ is clearly a polynomial of degree at most r — we've proved (a).

(e) is also easy. Given an exact sequence of vector bundles as in the statement, pullback to a flat $f : X' \to X$ so that both the (pullback of the) kernel E' and the (pullback of the) cokernel E'' split into line bundles. Then the pullback of E also splits. Thus by the lemma,

$$c_t(f^*E) = c_t(f^*E')c_t(f^*E'').$$

$\square$

Notation. If X is a pure-dimensional scheme, and P is a polynomial in Chern classes (or Segre classes) of various vector bundles of total codimension $\dim X$, then $\deg P \cap [X]$ is a number. This is denoted $\int_X P$. Example 1: Suppose X is a compact projective manifold (i.e. nonsingular complex projective variety) of dimension n, and T_X is the tangent bundle. Then $c_n(T_X)$ is a codimension n Chern class. Fact: $\int_X c_n(T_X) := c_n(T_X) \cap [X] = \chi(X)$, where $\chi(X)$ is the (topological) Euler characteristic. Example 2: Suppose $i : X \hookrightarrow \mathbb{P}^N$ is a projective variety of dimension n. Then $i^*\mathcal{O}_{\mathbb{P}^N}(1)$ is a line bundle on X. Then

$$\int_X c_1(i^*\mathcal{O}_{\mathbb{P}^N}(1))^d := c_1(i^*\mathcal{O}_{\mathbb{P}^N}(1))^d \cap X = \deg X.$$

(Reason: we can interpret each factor $c_1(i^*\mathcal{O}_{\mathbb{P}^N}(1))$ as intersecting with a randomly chosen hyperplane.)

2.1. **Fun with the splitting principle.** Thanks to the splitting principle, given the Chern classes of a vector bundle, you can find the Chern classes of other related vector bundles.

The way I think about it: imagine that the Chern polynomial factors (even though it doesn't!). Imagine that the bundle splits (even though it doesn't!).

Example 1: Dual bundle. Suppose E is a vector bundle, and $E^\vee$ is the dual bundle. Then $c_i(E^\vee) = (-1)^i c_i(E)$. (Reason: $c_t(E) = c_{-t}(E)$. The reason for this in turn is that if you assume that E is filtered (which we may do by the splitting principle) then $E^\vee$ is filtered too. Do you see why?

Example 2: Tensor products. I'll do a specific example, in the hope that you'll see the general pattern. Suppose E and F are rank 2 bundles. Then $E \otimes F$ is a rank 4 bundle. We can compute its Chern classes in terms of those of E and F. Suppose E has Chern roots e_1 and e_2, and suppose F has Chern roots f_1 and f_2. (Translation: assume that both E and F can be filtered. Let e_1 and e_2 be the line bundle quotients of the filtration of E, and similarly for f_1 and f_2.) Thus from

$$1 + c_1(E)t + c_2(E)t^2 = (1 + e_1 t)(1 + e_2 t)$$

we get $e_1 + e_2 = c_1(E)$ and $e_2 = c_2(E)$, and similarly for F. Then

$$\begin{aligned} c_t(E \otimes F) &= (1 + (e_1 + f_1)t)(1 + (e_1 + f_2)t)(1 + (e_2 + f_1)t)(1 + (e_2 + f_2)t) \\ &= 1 + (2e_1 + 2e_2 + 2f_1 + 2f_2)t + \cdots \\ &= 1 + (2c_1(E) + 2c_1(F))t + \cdots \end{aligned}$$

from which we get $c_1(E \otimes F) = 2c_1(E) + 2c_1(F)$, and similarly we can compute formulae for higher Chern classes of $E \otimes F$.

To justify that first equality for $c_t(E \otimes F)$, we need to give a filtration of $E \otimes F$ using the filtrations of E and F. I'll leave that for you.

Example 3: Exterior powers. I'll again do a specific example to illustrate a general principle. Suppose E is rank 3, with Chern roots e_1, e_2, e_3. In other words, as assume we have a specific filtration of E. The $\wedge^2 E$ is also rank 3, with Chern roots $e_1 + e_2$, $e_1 + e_3$, $e_2 + e_3$. Again, we do this by producing a filtration of $\wedge^2 E$ induced by that filtration on E.

Thus we can find the Chern classes of $\wedge^2 E$ in terms of those of E. We know $e_1+e_2+e_3 = c_1(E)$, $e_1e_2 + e_2e_3 + e_3e_1 = c_2(E)$, and $e_1e_2e_3 = c_3(E)$. Thus

$$\begin{aligned} c_t(\wedge^2(E)) &= (1 + (e_1 + e_2)t)(1 + (e_1 + e_3)t)(1 + (e_2 + e_3)t) \\ &= 1 + (2e_1 + 2e_2 + 2e_3)t + \cdots . \end{aligned}$$

In general, if E is rank n and we want to compute the Chern classes of $\wedge^k E$, the roots are sums of k distinct Chern roots of E.

Exercise: if E is rank n, then you can check that $\wedge^n E = \det E$. Show that $c_1(E) = c_1(\det E)$. This gives a different interpretation of c_1 of a vector bundle — as c_1 of the determinant bundle.

Exercise: what about symmetric powers? If E is rank 2, can you compute the Chern classes of $\operatorname{Sym}^4 E$?

Homework (due Nov. 1.) Suppose E is a bundle of rank r on a scheme X, p is the projection $\mathbb{P}E \to X$, and $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$. Show that $\zeta^r + c_1(p^*E)\zeta^{r-1} + \cdots + c_r(p^*E) = 0$. (Hint: consider the exact sequence of vector bundles on $\mathbb{P}E$: $0 \to \mathcal{O}_{\mathbb{P}E}(-1) \to p^*E \to Q \to 0$.)

Example: Chern classes of the tangent bundle to projective space:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0.$$

For convenience let, $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence $c_t(T_{\mathbb{P}^n}) = (1+Ht)^{n+1}$. (Note that $\deg c_n(T_{\mathbb{P}^n}) = n + 1$, which is indeed the topological Euler characteristic of $\mathbb{P}^n$.)

Example: Chern classes of the tangent bundle of a hypersurface in Y in X:

$$0 \to T_Y \to T_X|_Y \to N \to 0.$$

$(N \cong \mathcal{O}_X(Y))$.

Suppose next that $X = \mathbb{P}^n$, and Y is a degree d hypersurface. Let H denote the restriction of $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ to Y. (Equivalently, it is c_1 of the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ to Y: we've shown that c_1 commutes with any pullback.) Then as operators on A_*Y, we get

$$c_t(T_Y) = (1 + Ht)^{n+1}(1 + dHt)^{-1} = (1 + Ht)^{n+1}\left(1 - dHt + (dHt)^2 - (dHt)^3 + \cdots\right)$$

You can use this to compute the topological Euler characteristic of a hypersurface, or inductively, of a complete intersection. (Fun exercise: use this to work out the genus of a degree d plane curve.)

2.2. **The Chern character and Todd class.** The Chern character ch is defined by $\mathrm{ch}(E) = \sum_{i=1}^{r} e^{\alpha_i}$. Then if $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of vector bundles, $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$. (You should immediately see the corresponding long exact sequence!) Also, $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E)\mathrm{ch}(E')$.

The Todd class is defined by $\mathrm{td}(E) = \prod_{i=1}^{r} Q(\alpha_i)$ where

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}x^{2k}.$$

Again, $\mathrm{td}(E) = \mathrm{td}(E')\mathrm{td}(E'')$.

Sample application. Let X be an n-dimensional abelian variety lying in projective space $i : X \hookrightarrow \mathbb{P}^m$. Then $m \geq 2n$, and if equality holds, then $\deg X = \binom{2n+1}{n}$. Fact: for an abelian variety, T_X is a trivial bundle. (Reason over $\mathbb{C}$, $X = \mathbb{C}^n$ modulo a lattice.) Hence T_X has all Chern classes 0 (except c_0).

The first two cases are relative straightforward: if $n = 1$, then this corresponds to curves in planes; the only way for a genus 1 curve to lie in $\mathbb{P}^2$ is if it is degree 3.

If $n = 2$: there is no way for an abelian surface to be a hypersurface in $\mathbb{P}^3$. Reason: we've computed Chern classes of hypersurfaces.

It can sit in $\mathbb{P}^4$, but we'll see that it can only sit as a degree 10 hypersurface, and there is a famous such example called the Horrocks-Mumford abelian variety.

Here's the proof. $0 \to T_X \to i^*T_Y \to N \to 0$. $c_i(i^*T_Y) = c_i(N)$. Now the rank of N is $m - n$. $c_i(i^*T_Y) = \binom{m+1}{i}H^i$. If $i \leq n$, this is non-zero, as $H^n = \deg X[pt] \in A_0X$. On the other hand, $c_i(N) = 0$ for $i > \operatorname{rank} N$, and $\operatorname{rank} N = n - m$. Thus $m > n$.

2.3. **Looking forward to next day: Rational equivalence on bundles.** I stated a couple of things that we'll do on Wednesday.

Theorem Let E be a vector bundle of rank $r = e + 1$ on a scheme X, with projection $\pi : E \to X$. Let $\mathbb{P}E$ be the associated projective bundle, with projection $p : \mathbb{P}E \to X$. Recall the definition of the line bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$.

(a) The flat pullback $\pi^* : A_{k-r}X \to A_kE$ is an isomorphism for all k.

(b) Each $\beta \in A_k\mathbb{P}E$ is uniquely expressible in the form

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i,$$

for $\alpha \in A_{k-e+i}X$. Thus there are canonical isomorphisms

$$\theta_E : \oplus_{i=0}^{e} A_{k-e+i}X \xrightarrow{\sim} A_k\mathbb{P}E.$$

$\theta_E : \oplus\alpha_i \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i p^*\alpha_i$.

Intersecting with the zero-section of a vector bundle. We can already intersect with the zero-section of a line bundle (i.e. an effective Cartier divisor); we get a map $A_k X \to A_{k-1} D$, which we've called the Gysin pullback.

Definition: Gysin pullback by zero section of a vector bundle. Let $s = s_E$ denote the zero section of a vector bundle E. s is a morphism from X to E with $\pi \circ s = id_X$. By part (a) of the Chern class theorem allows us to define *Gysin homomorphisms* $s^* : A_k E \to A_{k-r} X$, $r = \operatorname{rank} E$, by $s^*(\beta) := (\pi^*)^{-1}(\beta)$.

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