INTERSECTION THEORY CLASS 1

RAVI VAKIL

CONTENTS

1. WELCOME!

Hi everyone — welcome to Math 245, Introduction to intersection theory in algebraic geometry. Today I'd like to give you a brief introduction to the subject, and then I'd like to hit the ground running.

Course webpage: http://math.stanford.edu/~vakil/245/. I intend to post notes for most lectures. I make no promises as to their quality; they are basically my notes to myself, slightly prettified when I have time.

Scheduling. We are tentatively considering switching the times of the course to something like Mondays and Wednesdays 9:30-10:50.

About the subject. We'll be using Fulton's book *Intersection Theory*. There are currently copies available at the bookstore, and it's about $45, which is cheap. I should point out that it's one of the few paperbacks you'll see in that Springer series; the author is one of the few people with both the stature and the interest to force Springer to put out a cheaper paperback version.

Intersection theory deals loosely with the following sort of problem: intersect two things of some codimension, get something of expected codimension. We'll basically construct something that looks like homology, and later something that looks like cohomology.

In algebraic geometry: can deal with singular things. Also over other fields. Even in holomorphic category, can get more refined information: 2 points on elliptic curve are the same in homology, but not in "Chow".

Before the 1970's, the field was a mess — there were a great deal of painful ad hoc constructions that people used. Fulton and MacPherson understood the right way to

Date: Monday, September 27, 2004.

describe it, and the result is in this book. In particular, the first 8 chapters are the heart of the subject, and subsume volumes and volumes of earlier work. Each of the subsequent chapters is a different important application; although not every application is important to everyone, every application is important to someone.

I'd like to suggest two readings for you for next day, and they are both light. First, read the MathReview for this book, to get a sense of the context in which these ideas appeared. www.mathscinet.org. Second, take a look at the introduction.

What you need to know: ideally: a first course in algebraic geometry, such as much of Hartshorne Chapter II, plus flatness. However: we can make do, if you're willing to work. The reason he can move so quickly is because of the power embedded in certain algebro-geometric ideas. So we're not going to be able to avoid notions such as flatness, or Cartier divisors. It doesn't matter if you haven't seen Chern classes before; you'll get a definition here.

2. Examples

Before I dive into the subject, let me start with some examples.

The first will get across some notion of *intersection multiplicity*. Also, if you haven't seen schemes or varieties before, this will get your feet wet. Admittedly, I'll let you dip in your toe here, and throw you in the deep end. (Talk to me!)

Picture of parabola $x = y^2$. Project it to t-line. Call the plane $\mathbb{A}^2$. $(x, y) \to x = t$.

Algebra: $K[x, y]$ surjects into $K[x, y]/(x - y^2)$. Affine schemes correspond to rings (the categories are the "same" with the arrows reversed). I'm being agnostic about my field. (I'll try to call it K throughout the course.) Closed immersions correspond to surjections. Ideals correspond to closed subschemes. $K[t]$. Map of rings in the opposite direction $t \mapsto x$.

Let's intersect this with $t = 1$, or equivalently $x = 1$. $K[x, y]/(x - 1)$. Intersecting these schemes corresponds to taking the union of the deals. (Unions of schemes correspond to intersections of ideals.)

$$K[x, y]/(x - y^2, x - 1) \cong K[y]/(y^2 - 1) \cong (K[y]/(y + 1)) \oplus (K[y]/(y - 1))$$

(the latter by the Chinese remainder theorem; here I'm assuming $\operatorname{char} K \neq 2$). Notice how we can see this in the picture. Great, the line meets the parabola at 2 points. If I replace 1 by something else nonzero, then I still get 2 points (assuming K is algebraically closed!).

Complication 1 is not important: for most of this course I'll assume the field is algebraically closed. But for those of you willing to think about nonalgebraically closed fields, like $\mathbb{Q}$, consider

$$\mathbb{Q}[x, y]/(x - y^2, x - 2) \cong \mathbb{Q}[y]/(y^2 - 2).$$

But now $y^2 - 2$ is irreducible! So we get a *single* point, but we want it to count for 2. *What to do?*

Complication 2 is more serious, but you've seen it before: what happens when you intersect with $x = 0$? Then even if you only care about the complex numbers, you definitely only get 1 point. In this case, you also want to say that the multiplicity is 2. Here's the algebra:

$$\mathbb{Q}[x, y]/(x - y^2, x) \cong \mathbb{Q}[y]/(y^2).$$

(In complication 1, the ring is a domain; in the second case it isn't; the ring has a nilpotent y. Those of you who have seen the geometry will know how to draw this.)

Okay, how do we get a consistent answer of 2 no matter which value of x we pick? Answer: we measure the size of the fiber by counting the *dimension* of the ring as a vector space. (That was an important observation that will come up later!) Then in complication 1 we get 2, and in complication 2 we get 2 as well.

Interpretation of complication 2: we get 1 point of multiplicity 2.

Interpretation of complication 1: we get 1 point of multiplicity 1, but that point counts for 2. Bizarre, isn't it?!

Here's a picture that might (or might not) help you deal with complication 1. (Give a triple cover picture.)

So: we have a good way of intersecting things of complimentary codimension meeting at a bunch of points: we intersect the schemes, and count the *length* at those points, which in these cases is a *dimension*.

2.1. Fundamental theorem of algebra. Given a nonzero polynomial $f(x)$ of degree n, the number of zeros, counted appropriately, is n.

Idea of "proof": $K[x]/(f(x))$ is a dimension n vector space.

Solving things corresponds to breaking $f(x)$ into prime factors. Examples:

- $\mathbb{C}[x]/(x(x-1)(x+2)) \cong \mathbb{C}[x]/x \oplus \mathbb{C}[x]/(x-1) \oplus \mathbb{C}[x]/(x+2)$. (using Chinese remainder)
- $\mathbb{C}[x]/x^2(x+1) \cong \mathbb{C}[x]/x^2 \oplus \mathbb{C}[x]/(x+1)$. (One point with multiplicity 2, and one point with multiplicity 1.)
- $\mathbb{R}[x]/x^2(x^2+1) \cong \mathbb{R}[x]/x^2 \oplus \mathbb{R}[x]/(x^2+1)$.
- $\mathbb{Q}[x]/x^2(x^2-2) \cong \mathbb{R}[x]/x^2 \oplus \mathbb{R}[x]/(x^2-2)$.

2.2. Bezout's Theorem. Given two curves in $\mathbb{P}^2$ of degree d and e with no common components, they meet in de points, counted properly. More generally, given n curves in $\mathbb{P}^n$ of degree $d_1, \ldots, d_n$, such that their intersection is zero-dimensional, they meet in $d_1 \cdots d_n$ points, counted properly.

We haven't proved this, but these things were known long ago, certainly before the twentieth century. But here's an example of why people got nervous.

Let X_1 and X_2 be two random planes in $\mathbb{P}^4$. They meet in a point. We can even give them co-ordinates. Say the coordinates on $\mathbb{P}^4$ are $[v; w; x; y; z]$. We'll say that X_1 corresponds to $w = x = 0$ and X_2 corresponds to $y = z = 0$. Then $X_1 \cap X_2$ meet at the point $[1; 0; 0; 0; 0]$. (Remember how projective space works: $[1; 0; 0; 0; 0] = [17; 0; 0; 0; 0]$.)

Let X be the union $X_1 \cup X_2$ in $\mathbb{P}^4$. Then this reasonably has degree 2. Let P be third random plane. Then P meets X_1 in one point, X_2 in another, and misses their intersection, to it meets X in 2 points. So far so good. If we move P around we should always get 2 (so long as $\dim X \cap P = 0$).

But: we get something strange if we put P through $X_1 \cap X_2$: we'll get 3. Here's the calculation. We'll work locally on projective space, on the open set where $v \neq 0$, so we can set $v = 1$. Coordinates on this 4-space are given by w, x, y, z. The open set corresponds to the ring $K[w, x, y, z]$. (You can let $K = \mathbb{C}$.)

X_1 corresponds to the ideal (w, x), and X_2 corresponds to the ideal (y, z). So $X_1 \cup X_2$ corresponds to $(w, x) \cap (y, z)$ (Ask.) $= (wy, wz, xy, xz)$.

Let's say P is given by $w = y$, $x = z$. This indeed meets X in one point. (Where does it meet X_1? $w = y$, $x = z$, $w = x = 0$, so $w = x = y = z = 0$. Where does it meet X_2? similar.)

But let's put our machine to work, and work out the scheme-theoretic intersection. Ideal:

$$K[w, x, y, z]/(wy, wz, xy, xz, w - y, x - z) \cong K[y, z]/(y^2, yz, z^2, yz) \cong K[y, z]/(y^2, yz, z^2).$$

This is a dimension 3 vector space (with basis $1, y, z$).

This was very alarming; Serre figured out what to do. I want to write down the answer, but modern intersection theory bypasses this, so you shouldn't worry about it much — this may even begin to give you a warm feeling in your stomach for the new version of the subject.

Suppose you wanted to intersect two things of complementary dimension in $\mathbb{A}^n$, corresponding to $K[x_1, \ldots, x_n]/I_1$ and $K[x_1, \ldots, x_n]/I_2$ respectively. (Let $R = K[x_1, \ldots, x_n]$ for convenience.) Interpretation of old formula

$$\dim_k \frac{R}{I_1} \otimes \frac{R}{I_2}.$$

Now $\otimes$ is a slightly weird thing. For example, it is right exact. If $0 \to A \to B \to C \to 0$ is exact then we only know that $M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact.

There is something that could go to the left that would make this look like a long exact sequence in cohomology:

$$
\begin{array}{l}
\longrightarrow \operatorname{Tor}^2(M,A) \longrightarrow \operatorname{Tor}^2(M,B) \longrightarrow \operatorname{Tor}^2(M,C) \longrightarrow \\
\longrightarrow \operatorname{Tor}^1(M,A) \longrightarrow \operatorname{Tor}^1(M,B) \longrightarrow \operatorname{Tor}^1(M,C) \longrightarrow \\
\longrightarrow M \otimes A \longrightarrow M \otimes B \longrightarrow M \otimes C \longrightarrow 0
\end{array}
$$

$\operatorname{Tor}^0(M, A)$ should be interpreted as $M \otimes A$.

General philosophy h^0 shouldn't behave well in families, but $\sum(-1)^i h^i = \chi$ should. Serre says that the right thing is:

$$\sum(-1)^i \dim_k \operatorname{Tor}^i(\frac{R}{I_1}, \frac{R}{I_2})$$

and he proved it. Magically, in all of our previous examples, all the "higher" Tor's vanished. ("Cohen-Macaulay".) But we were just lucky.

3. Strategy

Here's the strategy we're going to use. Here are things that homology satisfies in "usual" topology in "good circumstances". We have cycles in homology, and cycle classes, which are cycles modulo homotopy.

(1) Two points on a curve are homotopic.
(2) There sometimes a pullback on homology, when the map is a submersion. $\pi : X \to Y$, $\dim X = \dim Y + d$, then $\pi^* : H_n(Y) \to H_{n+d}(X)$.
(3) There is a pushforward in homology by proper morphisms. Proper: image of closed sets is closed. $X \to Y$, $\pi_* : H_n(X) \to H_n(Y)$.

Homology satisfies lots of other things. (In order to make this precise, Robert suggests: allow locally finite chains.)

We'll *define* our version of homology groups, which we'll call *Chow groups*, using this.

(1) Two points on $\mathbb{P}^1$ are defined to be *rationally equivalent*.
(2) If $X \to Y$ is *flat* then there is a pullback. $\pi : X \to Y$, $\dim X = \dim Y + d$, then $\pi^* : H_n(Y) \to H_{n+d}(X)$.
(3) If $X \to Y$ is *proper* (new definition!) then we have a pushforward: $X \to Y$, $\pi_* : H_n(X) \to H_n(Y)$.

We'll require that rational equivalences pullback and pushforward. This will turn out to give an amazing theory! (For example, it will give 2 in that $\mathbb{P}^4$ example without needing Tor's etc.)

E-mail address: vakil@math.stanford.edu