MATH 245C (AN INTRODUCTION TO ALGEBRAIC STACKS)
PARTIAL LECTURE NOTES SPRING 2022

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CONTENTS

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Part 1. Getting started

1. Introduction

1.1. Basic information about the course.

Spring 2022, Mondays, Wednesdays, Fridays 9:45-10:45 in 380-Y. Mon. March 28 through Wed. June 1.

The spring's topics course in algebraic geometry, "An introduction to algebraic stacks", will be an attempt at a direct and straightforward route to orbifolds, algebraic spaces, Deligne-Mumford stacks, and Artin stacks. We will develop the topic from the beginning, trying to motivate and rigorously define everything we do.

To follow this class, you'll have to be prepared to ask questions, and reserve some time outside of class for thinking, reading, and proving.

You don't have to be enrolled to attend.

1.2. *Email list.* To keep you informed, I'll have a low-volume email list for the class. You don't have to be enrolled to be on the email list. Indeed, if you might occasionally attend, you may want to be on the email list. I'll collect email addresses in the first class. After that, just let me know if you want to be added or removed from the list.

1.3. Appendix: Facts invoked from last quarter.

We're using facts in algebraic geometry from the Rising Sea / Hartshorne/ etc. without shame, including in particular the Cohomology and Base Change Theorem (for which I advertise Eric Larson's proof).

From last quarter, we are using the existence/representability of the Hilbert scheme, Isom scheme, Mor scheme. There is a Mumford-esque statement about where two line bundles on a proper family are fiberwise isomorphic, and what that means.

2. Motivation

We started by discussing why people wanted to know about algebraic stacks, and what they wanted to know. Ideas that came up included: orbifolds, quotient stacks, Deligne-Mumford stacks, algebraic stacks, algebraic spaces.

One idea that will guide us repeatedly: we understand one notion of geometric space, and we are trying to invent a more general notion, because we have some ideas of things want to be this more general kind of space.

Things people wanted to hear about: why are $\mathcal{M}_{1,1}$ Deligne-Mumford stacks/orbifolds? $\mathcal{M}_g$? Why is $\mathfrak{m}_0$ (the space of nodal genus 0 connected curves) a smooth algebraic stack? Why is $\dim BGL(n) = -n^2$? Representability of Pic. Statement of Alper-Hall-Rydh.

I tried to make some philosophical points: hard vs. easy; properties not definitions; the use of well-defined and clearly demarcated black boxes.

What makes algebraic stacks tricky is that there are a number of separate complications that can be turned on, like toggles. What can make this simpler is turning them on one at a time. I will try to do that.

3. Your favorite "geometric spaces", i.e., your category $\mathcal{G}$

3.1. *Non-Definition.* You undoubtedly like to think about a type of *geometric space* (informally). Precise definition later.

Big list of examples:

- (Balls in $\mathbb{R}^n$, or open subsets of $\mathbb{R}^n$), and (continuous, or differentiable, or C^∞) maps
- (Balls in $\mathbb{C}^n$, or open subsets of $\mathbb{C}^n$), and holomorphic maps
- continuous, differentiable, or C^∞ real manifolds
- holomorphic complex manifolds
- (affine, or general) complex analytic varieties

- (affine, or general) varieties over an algebraically closed field
- (affine, or general) complex analytic schemes
- (affine, or general) schemes
- algebraic spaces (eventually)
- (orbifolds and stacks will *not* be on this list!)

Pick your favorite one, and follow it through! Objects: geometric spaces. Definition: *Your geometric category.*

3.A. EXERCISE. What is your geometric category?

We call it $\boxed{\mathcal{G}}$.

3.2. **Morphisms.**

These should form a category: we know what the objects are, and what the maps are.

3.B. EXERCISE. What is the final object in your category? (Implicit in the exercise is that I am guessing your category *has* a final object!) In most cases your final object deserves the name of *point*.

3.3. **"Reasonable" classes of morphisms.** Classes of morphisms in $\mathcal{G}$ that you care about tend to have 3 properties. (i) they compose; (ii) they are preserved by pullback; and (iii) they are local in the target

3.4. *Aside.* Very likely your $\mathcal{G}$ has fibered products (pullbacks). I'm not sure if we need this, or if we need something slightly less: that fibered products with open embeddings exist. So for now: *we're not assuming that fibered products exist in* $\mathcal{G}$. I'll try to include this as an explicit hypothesis when we use it.

3.5. **New kinds of spaces from old.**

Recall from §2: *One idea that will guide us repeatedly: we understand one notion of geometric space, and we are trying to invent a more general notion, because we have some ideas of things want to be this more general kind of space.*

Your $\mathcal{G}$ likely are locally ringed spaces with some additional properties. A natural way to get a new kind of spaces are to consider locally ringed spaces that locally look like things in your $\mathcal{G}$. In other words:

$$\mathcal{G} \subset \underline{\mathbf{Loc}}_{\mathcal{G}} \subset \underline{\mathbf{LRing}}$$

This is one way in which we create schemes out of affine schemes, and manifolds (minus the Hausdorff and 2nd countable condition) from open subsets of balls (or even from balls).

We will deliberately make a worse choice, in order to create something which generalizes in the way that we want. In particular, we will not even need to know what "points" are on our spaces, and this will be a good thing.

However, the things we create will often be locally ringed spaces, see §15.

4. Revisiting sheaves

First, let us revisit sheaves.

What makes a notion worthy of a name? It should be an abstraction of a number of useful examples, and it should be useful to make this abstraction.

How do you define a mathematical notion to someone for the first time? Ideally, first you show them a number of examples that they are familiar with, so they can see that the notion is one they are already familiar with. They you draw out what these examples have in commmon. Then you make this a definition.

How do you define a sheaf on a topological space to someone for the first time? (I'm assuming you're already happy with sheaves!) Take a familliar topological space. I'll describe the sheaf of continuous functions on it. To each open set, we have a ring of continuous functions. When one open set is contained in another, we have a restrction map from the ring on the bigger space to the ring on the smaller space. We have now described they players; they will satisfy some properties. If you have one open set containing another containing a third then if you restrict then restrict, that's the same as restricting.

We observe that we can identify continuous functions locally (saying this precisely, in terms of a cover — two continuous functions on some union of open sets that happen to be the same on each of each of these open sets must have been the same to begin with). W can glue continuous functions together (if we have a bunch of open sets, and continuous functions on each of them, and the contintuous function on each open set agrees with that on each other open set on the overlap, then they all are restrictions of one continuous function on the union.

We then observe that the collection of differentiable functions on open subsets have the same structure; and ditto for smooth functions. We then abstract the definition of sheaf.

The fact that the restriction of a restriction is the restriction is our first property. So far, we have defined a **presheaf**. I'll call this a(n element of) $\boxed{\underline{\mathbf{PSh}}}$ for short. (This is our private notation).

Then we have two axioms, the **identity axiom** and the **gluability axiom**. If a presheaf satisfies the identity axiom (which I won't repeat for you here), people sometimes say it is a *separated presheaf*, but I don't care; we'll call it an (element of) $\boxed{\underline{\mathbf{PSh}}^+}$. If it further satisfies the gluability axiom, it is a *sheaf*, although we'll call it a $\boxed{\underline{\mathbf{PSh}}^{++}}$ or a $\boxed{\underline{\mathbf{Sh}}}$.

First (but not final) reformulation of presheaves: we define the **category of open sets** $\boxed{\underline{\mathbf{Top}}_X}$ of a topological space X (the objects are the open sets, and the morphisms are the inclusions). Then a presheaf on X is the same information as a contravariant functor $\underline{\mathbf{Top}}_X \to \underline{\mathbf{Sets}}$. For this we just care about the category, and nothing further about the topology; the identity and gluability axioms involve more about the topology

I didn't say it in class, but morphisms of presheaves translate in this language to "natural transformations of functions". So the category of presheaves on X is the same as the category you get where the objects are "contravariant functors $\underline{\mathbf{Top}}_X \to \underline{\mathbf{Sets}}$" and the morphisms are natural transformations of functors. More generally, if $\mathcal{T}$ is any category, we will define $\underline{\mathbf{PSh}}_{\mathcal{T}}$, which we call the category of **presheaves on** $\mathcal{T}$ (but others call the *functor category for* $\mathcal{T}$) to be the category where the objects are "contravariant functors $\mathcal{T} \to \underline{\mathbf{Sets}}$" and the morphisms are natural transformations of functors.

Part 2. Presheaves (the functor category)

5. The (category of) presheaves (on a category $\mathcal{C}$)

5.1. Presheaves on $\mathcal{C}$ = functor category of $\mathcal{C}$.

Suppose $\mathcal{C}$ is *any* category. (It needn't have a topology or anything like that.) You should think of your category $\mathcal{G}$ of spaces.

For each $Y \in \mathcal{C}$, we have a contravariant functor $\boxed{h_Y} : \mathcal{C}^{opp} \to \underline{\mathbf{Sets}}$. This is the functor of maps to Y. (You may prefer to call it $\boxed{\mathcal{M}aps^{\to Y}}$.) This is also called the *functors of points of* Y.) (Why subscript Y even though it is contravariant? Answer: this is "covariant in Y". But maybe I'll make it h^Y. See comment below on why I prefer it to be Presheaves.)

5.2. **Remark: Baby Yoneda.** Baby version of Yoneda: If you know maps to Y, you know Y up to unique isomorphism, by universal property nonsense. Explain.

Perhaps motivated by this:

5.3. *Definition.* A **presheaf on** $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to **Sets** (i.e., a covariant functor $\mathcal{C}^{opp} \to$ **Sets**). I might use the symbol $\text{PSh}(\mathcal{C})$ to mean a presheaf on $\mathcal{C}$.

5.4. **Running examples.**

- $G(k, n)$
- h_Y is in this bigger category.
- Families of curves up to isomorphism.
- Vector bundles up to isomorphism.
- Hilbert functor Hilb
- $\mathcal{M}_3^a := \mathcal{M}_3^{\text{aut-free}}$ (mentioned in previous section); family of curves up to automorphism.

5.5. **The category of presheaves on $\mathcal{C}$.** Define the category $\boxed{\underline{\textbf{PSh}}(\mathcal{C})}$ of presheaves on $\mathcal{C}$ to be the category of contravariant functors from $\mathcal{C}^{opp}$ to **Sets**). Ask: What are morphisms in this category? I.e., what are maps of presheaves? Alternative notation that I like less (see below): $\underline{\textbf{Fun}}(\mathcal{C})$.

Answer: Morphisms are *natural transformations of functors*.

5.A. EXERCISE. Given a morphism $Y \to Z$ in $\mathcal{C}$, we have an induced morphism of presheaves (which might be called $h_{Y\to Z}$) in $\underline{\textbf{PSh}}(\mathcal{C})$, $h_Y \to h_Z$.

To show that you have to be careful, I will confuse you. The following exercise seemingly contradicts the previous ones.

5.B. EXERCISE. These h_Y are contravariant functors. So any morphism $Y \to Z$ induces $h_Z \to h_Y$. This seems to contradict Exercise 5.A. What has gone wrong?

This is why we're better off calling these *presheaves rather than functors* — because otherwise we are confused when talking about a covariant functor from the original category to the functor category (of contravariant functors from the original category to sets).

Here's another example (say it in terms of functors): $G(k, n) \to \mathbb{P}^{\binom{n}{k}-1}$.

In Exercise 5.A, we have proved:

5.6. **Proposition. —** *We have a (covariant) functor* $\boxed{\text{Yo} : \mathcal{C} \to \underline{\textbf{PSh}}(\mathcal{C})}$.

We call this functor the **Yoneda functor** Yo. More on this soon, in §6.

5.7. *Definition.* **subpresheaf** (or *subfunctor*)

5.8. *Fibered products in the presheaf category.*

5.C. EASY EXERCISE. Show that $\underline{\mathbf{PSh}}_{\mathcal{G}}$ has fibered products. (Explain!) (Reason: because $\underline{\mathbf{Sets}}$ has fibered products. Caution: you need to check that that the naive fibered product is indeed a fibered product in $\underline{\mathbf{PSh}}_{\mathcal{G}}$.) Notation: $\boxed{F \times_H G}$ in $\underline{\mathbf{PSh}}_{\mathcal{G}}$.

5.9. *Definition.* If $F, S \in \underline{\mathbf{PSh}}(\mathcal{C})$, we define $\boxed{F(S)} := \mathrm{Hom}(S, F)$.

5.10. *Subpresheaves.* Define **subpresheaf-on-**$\mathcal{G}$ or a *subfunctor-on-*$\mathcal{G}$ (because we know what a subset is). (Do we need this?)

Define **subpresheaf**. (Dispreferred notation : subfunctor.) Really, this is a quality of a morphism of presheaves.

5.D. EXERCISE. Show that subpresheaves are monomorphisms in $\underline{\mathbf{PSh}}$.

Presumably not every monomorphism is a subpresheaf.

6. REPRESENTABLE PRESHEAVES (ON $\mathcal{C}$), AND YONEDA

(To place later: We start by trying to extend geometric notions to objects in $\underline{\mathbf{PSh}}_{\mathcal{G}}$, and then also do the same to morphisms in $\underline{\mathbf{PSh}}_{\mathcal{G}}$. Probably say: start with them actually being things we understand. Then even if not, they might have line bundles; and morphisms can look like them. Later: even locally look like them.)

We return to the Yoneda morphism $\mathrm{Yo} : \mathcal{C} \to \underline{\mathbf{Psh}})\mathcal{C}$.

6.A. EXERCISE (YONEDA'S LEMMA, IN SOME SENSE). Suppose $z \in \underline{\mathbf{PSh}}(\mathcal{C})$, and $Y \in \mathcal{C}$. Describe a bijection between the morphisms $Y \to z$ in $\underline{\mathbf{PSh}}(\mathcal{C})$ and the elements of $z(Y)$. Show that this bijection is natural, i.e. is functorial in both z and Y. (First: What does this mean?)

So basically, whenever you see $Y \to z$, you should immediately think $z(Y)$; and vice versa.

We have a number of immediate consequences.

6.1. **Translation (Yoneda's lemma). —** *First, we have a natural correspondence between maps* $\mathrm{Yo}(X) \to \mathrm{Yo}(Y)$ *and* $\mathrm{Mor}(X, Y)$. *In other words,* Yo *is a fully faithful functor.*

Important question: does fully faithful functor include injectivity on objects? No! Perhaps 2-surjective would give "essential injectivity"?

(This formalizes the usual "universal property" abstract nonsense of Remark 5.2 on baby yoneda.)

Based on this, we define **representable presheaf** (anything isomorphic to $\mathrm{Yo}(X)$ for some $X \in \mathcal{C}$; and we know that it is uniquely isomorphic), and denote this full subcategory of $\underline{\mathbf{PSh}}(\mathcal{C})$ by $\boxed{\underline{\mathbf{Rep}}(\mathcal{C})}$. (Normal people would call these **representable functors**, which is short for *representable contravariant functors from* $\mathcal{C}$ *to* $\underline{\mathbf{Sets}}$.)

We get a bit more than traditional Yoneda: *presheaves are determined by maps from representable presheaves.* But that's kind of tautological.

We think of $\mathcal{C}$ as a (full) subcategory of $\underline{\mathbf{PSh}}(\mathcal{C})$ (although really, it is equivalent to a full subcategory). $\underline{\mathbf{Rep}}(\mathcal{C})$ is the *essential image* of Yo. (This is in the appendix on category theory.)

6.2. *Examples (in whatever geometric category they like).* products. Moduli spaces. $\mathcal{O}$. $\mathcal{O}^*$.

6.B. EASY EXERCISE (ASK IN CLASS). Assume $\mathcal{C}$ has fibered products. The fibered product of representable functors is a representable functor.

So we start with our spaces, and we have a more general notion. We have moduli spaces, which are in $\underline{\mathbf{PSh}}(\mathcal{C})$, and we wonder if they are in (the (essential) image of) $\mathcal{C}$.

We say this as: does the moduli space exist? Is the moduli functor representable?

7. REPRESENTABLE MORPHISMS (OF PRESHEAVES ON $\mathcal{C}$)

We now know which *objects* in $\underline{\mathbf{PSh}}_{\mathcal{G}}$ we can think of very geometrically. We can do the same with *morphisms*.

7.1. *Definition.* A morphism $F \to G$ in $\underline{\mathbf{Psh}}_{\mathcal{G}}$ is **representable** if for all elements $Y \in \mathcal{G}$, we have that the fibered product $Y \times_G F$ is representable.

7.A. EXERCISE (IN $\underline{\mathbf{Sch}}$). We have a Hilbert functor (or presheaf) $\mathcal{H}ilb$ (see examples above). Define the universal family (functor) $\mathcal{U}$, and show that $\mathcal{U} \to \mathcal{H}$ is representable.

7.B. EXERCISE. Define $M_3^{\text{aut-free}}$ as you'd expect. Show that the universal curve is $UM_3^{\text{aut-free}} \to M_3^{\text{aut-free}}$ representable.

7.C. EXERCISE. Representable morphisms compose and pullback.

7.2. **Properties of representable morphisms (of presheaves on $\mathcal{G}$).**

7.D. EXERCISE. Properties of morphisms on $\mathcal{G}$ that are preserved by pullback extend to definitions for representable morphisms of functors.

Examples: in Schemes: Zariski open, affine, etale, etc.

Example: open subpresheaves (or subfunctors), closed subfunctors.

Observe: if such morphisms are preserved by composition in $\mathcal{G}$, then they are preserved by composition in $\underline{\mathbf{Psh}}$.

7.E. IDLE QUESTION. Is it true that $\underline{\mathbf{PSh}}(\mathcal{C}) \to \underline{\mathbf{PSh}}(\underline{\mathbf{PSh}}(\mathbb{C}))$ is an equivalence of categories? In other words, is $\underline{\mathbf{PSh}}(\cdot)$ a "closure" operation? I doubt it.

7.3. *Definition.* Define **open subpresheaf**, **closed subpresheaf**, and give some examples. Some how open subpresheaf, and closed subpresheaf, don't sound right to me.

7.4. **Review at the start of class 3.**

Where we are: you have a category of geometric spaces you understand $\mathcal{G}$, which in particular has a class of morphisms, called "opens" or "open immersions" (perhaps we mark such morphisms with an o, and describe the subcategory as $\mathcal{G}^o \subset \mathcal{G}$).

Our plan is to start with $\mathcal{G}$, and invent a new kind of space from it.

For example, we will invent manifolds, by starting with $\mathcal{G}$ as open subsets of $\mathbb{R}^n$ (with, perhaps, smooth maps). We will later have to add in the second countable and Hausdorff axioms by hand, but only when we are forced to.

We discussed *Yoneda's Lemma*: $\mathcal{G}$ is a full subcategory of the "functor" category $\mathsf{PSh}_{\mathcal{G}}$, which is a fancy painful way of saying something more concrete.

Perhaps to be careful, we should describe those "representable" presheaves (or "representable functors") as $\mathsf{Rep}_{\mathcal{G}}$, and write:

$$\mathcal{G} \xrightarrow{\sim} \underline{\mathbf{Rep}}_{\mathcal{G}} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}$$

but it won't be confusing (I hope!) to just write

$$\mathcal{G} \subset \underline{\textbf{PSh}}_{\mathcal{G}}.$$

Now for a geometric space $X \in \mathcal{G}$, we've discussed the "category of open sets of X", perhaps denoted $\boxed{\underline{\textbf{Top}}_X}$, and we can see that we have a natural "inclusion" $\underline{\textbf{Top}}_X \subset \mathcal{G}^o$.

The two next sections make sense of geometry on presheaves — some geometry on all presheaves, and more geometry on other non-representable presheaves.

8. Geometry on the category of presheaves on $\mathcal{G}$

We discussed how sometimes there are line bundles or cohomology classes on elements of $\mathsf{PSh}_{\mathcal{G}}$.

If we do not know a moduli functor is representable, we can still think about geometry. I discussed how to make sense of some geometric notions on some of these objects of $\mathsf{PSh}_{\mathcal{G}}$ (some of these presheaves on $\mathcal{G}$). My motivating example was the presheaf of "rank 2 vector bundles up to isomorphism", where $\mathcal{G}$ are the complex manifolds, or complex analytic spaces. Call this presheaf P. Then P has something which deserves the name $c_1 \in H^2(P, \mathbb{Z})$, $c_2 \in H^4(P, \mathbb{Z})$, which are the Chern classes of the corresponding vector bundle. We discussed what this means.

We similarly discussed the meaning of the Hodge bundle on $\mathcal{M}_g^a$, which is a rank g vector bundle on this presheaf.

There are some objects in $\mathsf{PSh}_{\mathcal{G}}$ that have lots of geometry. If $\mathcal{G}$ are open subsets of $\mathbb{C}^n$, then any manifold M is in $\mathsf{PSh}_{\mathcal{G}}$ (as the presheaf of maps to M). Indeed, the category of complex manifolds is a subcategory of $\mathsf{PSh}_{\mathcal{G}}$, and we should be able to define it complexly from this point of view, and this is what we will do!

8.A. Exercise. Think this through.

Warning: cohomology classes can be more complicated.

9. Presheaves (on $\mathcal{G}$) can be quite geometric without being representable

Here's another reason to look in $\underline{\textbf{PSh}}(\mathcal{C})$ — there may be bigger categories that we want to think of in the same way.

To see this, consider this example: **From open subsets of $\mathbb{C}^n$ to complex manifolds.**

Example: Do example from complex, extending balls to manifolds. We have the category of balls, the category of presheaves, and the manifold category.

9.A. INTERESTING NONTRIVIAL EXERCISE. Show that the manifold category is in the functor category of balls, as a full subcategory.

Hence we have a new definition of complex manifolds: a complex manifold is $\underline{\mathbf{PSh}}_{\mathcal{G}}$ on $\mathcal{G} = \underline{\textbf{affine complex manifolds}}$, that satisfies some additional properties.

$$\mathcal{C} \subset \underline{\mathbf{Man}}_{\mathbb{C}} \subset \underline{\mathbf{PSh}}(\mathcal{C})$$

Big question: What is missing? What is in the [some additional properties]? (Discuss with them: Secret answer: locally look like balls.)

Notice: we understand line bundles on manifodls, if we are given line bundles on $\mathcal{C}$ (you can almost believe). Relate this to an earlier section (link to be added later).

How would we "discover" manifolds, if we only knew about open subsets of $\mathbb{C}^n$?

Or how can we discover these functors-on-$\mathcal{G}$ that are actually quite geometric?

We can fit this with moduli spaces together — maybe some moduli space is representable. But if not, maybe we should be considering some bigger category.

((Foreshadowing: we want things we can think about geometrically, extending our notion of space.))

We even get the *right* notion of line bundles on manifolds in this way — they are the line bundles on the functors/presheaves!

10. FIBERED PRODUCTS = PULLBACKS

We haven't yet required that $\mathcal{G}$ have fibered products, and I'm not sure if we want to require it yet. For now, no.

We will want $\mathcal{G}^o$ to have fibered products, but we'll get to this later, when discussing Grothendieck topologies next day.

10.1. **Lemma. —** $\underline{\mathbf{PSh}}_{\mathcal{G}}$ *has fibered products (regardless of whether* $\mathcal{G}$ *does)*

We discussed this; you should think through the proof. The reason is that $\underline{\mathbf{Sets}}$ has fibered products.

10.2. *Definition.* A morphism $a : F \to G$ is sais to be a **representable morphism** in $\underline{\mathbf{PSh}}_{\mathcal{G}}$ if for $X \in \mathcal{G}$, and all maps $X \to G$ (I really mean $h^X \to G$, but hopefully you are getting the hang of identifying X with h^X), the fibered product $F \times_G X$ is representable (i.e., in $\mathcal{G}$).

$$\begin{array}{ccc} F \times_G X & \longrightarrow & X \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

10.A. EXERCISE. Show that the notion of representable morphism is closed under fibered product.

10.B. EXERCISE. If $\mathcal{G}$ is *not* closed under fibered product, show that not every morphism in $\mathcal{G}$ is representable! So maybe this terminology isn't great if $\mathcal{G}$ isn't closed under fibered product.

10.C. EXERCISE. If $X \to Z$ and $Y \to Z$ are morphisms in $\mathcal{G}$, and they have a fibered product in $\mathcal{G}$, then their fibered product in $\underline{\mathbf{PSh}}_{\mathcal{G}}$ is the same as in $\mathcal{G}$.

Any property of morphisms in $\mathcal{G}$ (e.g., open immersion) can then be made a property of representable morphisms in general. We say $F \to G$ in $\underline{\mathbf{PSh}}_{\mathcal{G}}$ is purple if it is representable, and for all $X rightarrow G$ from $X \in \mathcal{G}$, we have $X \times_G F \to X$ is purple.

So for example, $\mathcal{M}^a_{g,1} \to \mathcal{M}^a_g$ is smooth of relative dimension 1.

11. GROUP ACTIONS, GROUP QUOTIENTS

I'll put some of our discussion here on group actions and group quotients.

We know what a **group object** G is in $\mathcal{G}$ (e.g., a Lie group is a group object in case where $\mathcal{G}$ are manifolds).

11.A. EXERCISE. Write out what this means! (I should do it myself, but haven't gotten around to it.) Let pt be a final object in $\mathcal{G}$.

We know what a **group action** is in $\mathcal{G}$, of G on X: A map $G \times X \to X$ satisfying some properties (write them out!).

If we are in the category of topological spaces or manifolds, we know what it means for a group action to be **free**: for each point $p \in X$, if $gp = p$ then $g = e$. Translation: $G \times X \to X \times X$ should be an injection.

11.1. *Definition: free group action in the algebraic category.* How to make sense of the "freeness" of a group action in the case of schemes (or, perhaps, varieties over fields that are not algebraically closed)? Answer: an action of G on X is **free** if $G \times X \to X \times X$ should be an injection when evaluated on T-valued points: $(G \times X)(T) \to (X \times X)(t)$ should be an injection, i.e., $\mathrm{Hom}(T, G \times X) \to \mathrm{Hom}(T, X \times X)$ should be an injection. Translation: $G \times X \to X \times X$ should be a "subpresheaf" = "subfunctor".

Part 3. **Sheaves, especially locally representable sheaves**

12. MIXING IN THE TOPOLOGY ON YOUR $\mathcal{G}$

Quick reminder at the start of class 4: Suppose $F \in \underline{\mathbf{PSh}}_{\mathcal{G}}$, and $X \in \mathcal{G}$. we think of maps $X \to F$ as the same as elements of the set $F(X)$.

In your favorite geometric category $\mathcal{G}$, you will note that for any of your spaces Y, "maps to Y glue", or more precisely, "maps to Y from open subsets of X form a sheaf". I translated what this meant into something concrete.

(Aside: although this happens to be true in your particular well-chosen category, we will want to make this a requirement of our set-up. The fancy statement is that your topology is "subcanonical", but this word won't help us, so I'll try not to use it.)

Also, many of the moduli spaces that we had in mind, that were indeed "reasonable" and were "presheaves on $\mathcal{G}$, were also sheaves.

So if we want to invent new kinds of spaces, a natural place to look are sheaves on $\mathcal{G}$. These are the presheaves which are sheaves on every one of your geometric spaces $X \in \mathcal{G}$. They form a category, $\underline{\mathbf{Sh}}_{\mathcal{G}}$. I want to abstract from this, so we can use this notion in more general situations.

12.1. **Abstracting: a topology on a category.**

We want to see how the sheaf axioms translate into our language of geometric spaces, and abstract them.

First, let's see how the topology axioms translate, and see what we need in order to discuss sheaves.

Suppose we have a set X. Then a topology on the space is a collection of subsets, which we call **open subsets**, satisfying a few axioms:

(i) $\varnothing$ and X are open
(ii) finite intersections of open are open
(iii) unions of opens are opens

Let's see what we need for sheaves (or better, for the identity and gluability properties).

Some of our morphisms in $\mathcal{G}$ are called "opens" or "open immersions" (or "admissible opens", as Ben suggested by email after class 4). Perhaps we might write this as a subcategory $\mathcal{G}_X^o \subset \mathcal{G}_x$, which has the same objects as $\mathcal{G}_X$, but only some of the morphisms (the "opens").

(i) $\varnothing \to X$ is in $\mathcal{G}_X^o$. $X \to X$ is in $\mathcal{G}_X^o$. Fairly clearly, we would like to say that all isomorphisms $Y \to X$ are in $\mathcal{G}_X^o$.

We also need an axiom that fibered products in $\mathcal{G}$ with any open are also open: "opens pullback". Translation: given any map $X \hookrightarrow Z$ in $\mathcal{G}^o$, and any map at all $Y \to Z$, the pulled back map $X \times_Z Y \to Z$ is also in $\mathcal{G}^o$:

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{o} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{o} & Z \end{array}$$

(ii) the (finite) fibered products in $\mathcal{G}_X^o$ exists, and is the same as the fibered product in $\mathcal{G}$. We call this *intersection*.

(iii) We seem to need the notion of unions of open sets (e.g., perhaps uions are pushouts of some sort). But *(insight!)*: we really need less than (iii) — we just need to know which things are covers.

Example: convex open sets in $\mathbb{R}^n$ They don't form a topology, but they for a base.

Example 2: Distinguished opens in an affine scheme.

Example 3: Affines in a separated scheme.

We have some axioms for covers.

(i) isomorphisms are covers
(ii) covers pullback
(iii) covers of covers are covers

12.2. *Definition.* For the purposes of this course, we will call this data, satisfying these conditions, a **topology on the category** $\mathcal{G}$.

12.3. *Linguistic note.* This isn't standard notation, but isn't so terrible. The data of a category with topology is called a *site*, although we won't need this language. The data of a topology on a category is more commonly known as a *pretopology*, but I don't see any advantage of this word, at least right now. You may have heard the phrases "big étale site" or "small étale site on X", and similar phrases. We don't need this, but in case you are curious: the "étale" in the name means that is the topology we are working with. So the only question is what category we are working with. For "small" things, the category is that of étale maps to X. For example, the small etale site on a scheme X is denoted $X_{\text{ét}}$. For "big" things, it is all maps. For example, the big smooth site will be denoted $\underline{\mathbf{Sch}}_{sm}$.

12.4. **Sheaves on a category (with a topology).**

We call this a **topology on** $\mathcal{G}$, or a *Grothendieck topology on* $\mathcal{G}$. A category with topology is called a *site* (we won't use this).

12.A. EXERCISE. Verify that this induces topologies on $\mathcal{G}_X$ for all X, and these are "traditional" topologies. (Actually, they aren't quite! But we want them to be...) So this really is a generalization of the notion of a topology on X.

We can then define a **sheaf on** $\mathcal{G}$. Also, in just same way, sheaves on $\mathcal{G}$ form a category $\underline{\mathbf{Sh}}_{\mathcal{G}}$. We thus have a full embedding $\underline{\mathbf{Sh}}_{\mathcal{G}} \hookrightarrow \underline{\mathbf{PSh}}_{\mathcal{G}}$. (A category of sheaves on a site is called a *topos*, but I don't want to use this word.)

12.B. EXERCISE. Show that $Sh_{\mathcal{G}}$ has fibered products.

12.C. EXERCISE. $h_Y : \mathcal{C} \to \underline{\mathbf{Sets}}$ form a sheaf.

Words: "a space **is** a sheaf". Translation "Maps$^{\to Y}$ is a sheaf". Better way to say it: an element of $\mathcal{G}$ **is** a sheaf on $\mathcal{G}$. (Unimportant remark: A topology is *subcanonical* if all the objects are sheaves.)

We now have:

$$\begin{array}{ccccccccc}
\mathcal{G}_X^o & \longrightarrow & \mathcal{G}^o & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\mathcal{G}_X & \longrightarrow & \mathcal{G} & \xrightarrow{\sim} & \underline{\mathbf{Rep}}_{\mathcal{G}} & \xhookrightarrow{\text{full}} & \underline{\mathbf{Sh}}_{\mathcal{G}} & \xhookrightarrow{\text{full}} & \underline{\mathbf{Pre}}_{\mathcal{G}} \quad = \underline{\mathbf{Fun}}_{\mathcal{G}}
\end{array}$$

Everything here is a subcategory.

12.D. EXERCISE. Mixing Yoneda and sheaves: Show that $\mathcal{G} \to \underline{\mathbf{Sh}}_{\mathcal{G}}$ is a full embedding.

12.E. EXERCISE. Mark the arrows in that diagram above that are full embeddings.

(Mental tongue twister: say this five times fast: *Sheaves on a category form a category.* Objects in your geometric category have a topology, which is a category. The objects are these topological spaces with sheaves. The objects themselves are sheaves. Sheaves on the category form a bigger category. This is way more confusing than it should be.)

12.5. **Upshot.** If a moduli space is representable, it had better be a sheaf-on-$\mathcal{G}$ (i.e., in $\underline{\mathbf{Sh}}_{\mathcal{G}}$)! Or if a functor-on-$\mathcal{G}$ (element of $\underline{\mathbf{PSh}}_{\mathcal{G}}$) should be considered geometric, it had better be a sheaf-on-$\mathcal{G}$ (in $\underline{\mathbf{Sh}}_{\mathcal{G}}$)!

13. EXAMPLES (AND NON-EXAMPLES) OF SHEAVES (ON $\mathcal{G}$)

13.1. **Example:** h_Y**.** is a representable sheaf. (Refer back to that earlier exercise.)

13.2. **Example: every complex manifold forms a locally representable sheaf on the category of open subsets of complex affine spaces.**

13.3. **Example: every scheme forms a sheaf on the category of affine schemes.**

13.4. **Example:** $\mathcal{O}$**,** $\mathcal{O}^*$**,** Ω**,** $\wedge^3\Omega$ **on the category of complex manifolds.**

13.5. **Example: The Grassmannian functor** $G(k, n)$**.**

(where we haven't worried about representability)

We then have a map of sheaves $G(k, n) \to \mathbb{P}^n$.

Ask: Hilbert functor of $\mathbb{P}^n$, or indeed of anything.

13.6. **Example:** $\mathcal{M}_3^{aut-free}$**.**

13.7. **Non-example: vector bundles.**

Here we could potentially get a functor-on-$\mathcal{G}$, if we say "up to isomorphism". Otherwise, we don't even have a functor-on-$\mathcal{G}$ in any obvious way!

And even this functor sucks — it isn't even a separated presheaf on $\mathcal{G}$.

13.8. **Non-example:** $\mathcal{M}_3$**.**

Same caveat as with vector bundles!

14. Locally representable sheaves $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$

Perhaps $\mathcal{G}$ has fibered products; we're staying agnostic about that.

Definitely we have the notion which morphisms in $\mathcal{G}$ are opens, which we might describe as: $\mathcal{G}^o \subset \mathcal{G}$.

We know that maps to any element of $\mathcal{G}$ form a sheaf.

Examples: balls in $\mathbb{R}^n$? Maybe. We discussed how Cech cohomology works particularly well when you have a "good" cover. Complex analytic varieties and complex analytic spaces.

We now have a new kind of space!

14.1. *Definition.* We define **locally representable sheaves** $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$. I might want to call them **locally representables** for short, to deliberately elide the word "sheaves", so I can think of them as "spaces".

$$\mathcal{G} \subset \underline{\mathbf{LRepSh}}_{\mathcal{G}} \subset \underline{\mathbf{Sh}}_{\mathcal{G}} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}^{+} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}$$

What are these things? What have we built?

$\underline{\mathbf{LRepSh}}_{\mathcal{G}}$ has fibered products.

If we started with open subsets of $\mathbb{R}^n$, we have something more general than manifolds — we have the definition minus the "second countable+Hausdorff condition".

Define complex analytic spaces, complex manifolds, schemes (all without "Hausdorff, second countable" conditions).

If we start with the category of affine schemes $\boxed{\underline{\mathbf{Aff}}}$, we will see later that we will get some schemes, but perhaps not all of them. A great question: do we get the affine plane with doubled origin? The "obvious" cover of it (by two copies of $\mathbb{A}^2$) isn't by "representable opens".

I am guessing that this will not be locally representable, but my attempt in class 5 to show this didn't work! But we still learned something.

15. If $\mathcal{G}$ consists of locally ringed spaces, or something like that, so does $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$

15.1. **Topological spaces.**

Your objects of $\mathcal{G}$ are probably topological spaces. (Possible translation: you have in mind a forgetful functor $\mathcal{G} \to \underline{\mathbf{Top}}$.) Your "opens" probably give open subsets of topological spaces: $\mathcal{G}^o \to \underline{\mathbf{Top}}^o$.

15.A. Exercise. Then your locally representable sheaves are also topological spaces: you can extend $\mathcal{G} \to \underline{\mathbf{Top}}$ to $\underline{\mathbf{LRepSh}}_{\mathcal{G}} \to \underline{\mathbf{Top}}$.

15.2. **Ringed spaces.**

Are your elements of $\mathcal{G}$ a ringed space (or even a locally ringed space)? Probably, although we haven't required it. What are the functions on $X \in \mathcal{G}$? They are the (allowed) maps to $\mathbb{R}^1$ (if you are dealing with real manifolds), or $\mathbb{C}^1$, or more generally $\mathbb{A}^1$.

15.B. Exercise. Suppose $\mathbb{A}^1$ is a "ring object" in $\mathcal{G}$. We call maps to $\mathbb{A}^1$ "functions". Then if you have a forgetful map $\mathcal{G} \to \underline{\mathbf{Top}}$, you even have a map to ringed spaces.

15.C. Exercise. Explain how, if your $\mathcal{G}$ is a category of locally ringed spaces (ie has a functor to $\underline{\mathbf{LRepSp}}$), this extends to $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$.

16. Examples of locally representable sheaves

16.A. Exercise. If we start with $\mathcal{G}$ = open subsets of balls, we get things *including* manifolds (and even manifolds-minus-Hausdorff-and-second-countability).

16.1. *Question (rhetorical for now).* Do we get anything else?

16.2. **Starting with $\mathcal{G} = \underline{\mathbf{Aff}}$, the category of affine schemes.** If we start with $\mathcal{G} = \underline{\mathbf{Aff}}$, it isn't completely clear what we get. (We'll see soon... I'll add a ref forward...)

Excellent question that will motivate us: is the plane with the doubled origin in $\underline{\mathbf{LRepSh}_{\mathbf{Aff}}}$? (Answer will be: "no".)

So let's try to see what we can build in this way.

17. Revisiting sheafification

We can sheafify a presheaf on a topology on a category — There is a sheafification functor. We don't necessarily have "points", so we can't describe it using "compatible stalks". Here is an alternate take, which is useful.

Identity axiom: the notion of "equals" glues. Translation: if you have two sections s and s' over $X \in \mathcal{G}'$, and over every "open" in a given open cover of X, if s equals s' on that open, then $s = s'$ on X.

Gluability axiom: objects glue.

Define $\boxed{\underline{\mathbf{PSh}}_{\mathcal{G}}^{+}}$, the presheaves satisfying the identity axiom. (These are also called *separated presheaves*, although we won't use this.) This is a full subcategory of $\underline{\mathbf{Psh}}_{\mathcal{G}}$.

So here is the functor $\underline{\mathbf{PSh}}_{\mathcal{G}} \to \underline{\mathbf{PSh}}_{\mathcal{G}}^{+}$.

It is left-adjoint to the forgetful functor $\underline{\mathbf{PSh}}_{\mathcal{G}}^{+} \to \underline{\mathbf{PSh}}_{\mathcal{G}}$, and you can readily check this.

You just enforce that "equals" glue! Any time you have two sections over some U that are locally equal, you declare them to be the same! Any question? Do you see what the functor is? Do you see why it has the universal property?

Next we define the functor $\underline{\mathbf{PSh}}_{\mathcal{G}}^{+} \to \underline{\mathbf{PSh}}_{\mathcal{G}}^{++} = \underline{\mathbf{Sh}}_{\mathcal{G}}$, which is adjoint to the forgetful functor. You just enforce that "objects" glue. This means that you now have new objects. What is an object? It is just a bunch of objects on an open cover, which agree on overlaps. Equality if... So we have a colimit.

What are maps between such things?

17.1. *Fun aside.* What is a (-1)-sheaf? It is a property of open sets, that restricts to smaller open sets, and is preserved by union.

17.A. Easy exercise. Show that the sheafification is the composition of the two functors mentioned above. (This is just adjoint nonsense.)

18. DESCRIBING LOCALLY REPRESENTABLE THINGS, BY GLUING TOGETHER REPRESENTABLE THINGS

18.1. **Motivation: Gluing manifolds from balls, in the language of sheafification.**

This next bit of exposition is messed up and disorganized, but contains what I said in class.

Suppose we build a manifold M by gluing together some balls U_i. How do we understand the "presheaf" M in terms of the "presheaves" U_i? How do we build locally representable sheaves?

What is a map to X?

To first approximation: maps to $\coprod U_i$ modulo some equivalence relation. But this is not a sheaf; it is not the right thing. But it is a $\underline{\mathbf{PSh}}^+$, so to build M, we sheafify.

I'll give this example, but then move on to discussing sheafification.

A map X to M is: you cut up X into opens X_i; you have maps X_i to U_i; and you glue together.

Or you have maps $X_j \to \coprod U_i$, and two maps are the same if they agree up to some equivalence relation.

18.2. *Remark.* Compare this to the following situation. Suppose G acts on Y, say freely. What are maps from X to Y/G? Answer: cut X into pieces, X_i; choose maps $X_i \to Y$; but on overlaps they should lie in G (they should be "equivalent").

18.3. **Sheafification take two.**

Say you have a cover of a manifold by pieces. And you have overlaps. How do you recover the original space?

Sheafify $\prod F(U_i)/\prod F(U_{ij})$!

18.A. EXERCISE. Show that $\prod F(U_i)/\prod F(U_{ij})$ is a separated presheaf already.

This will grow up to be an exercise below.

So we just need to apply "+" once.

How do we define a manifold M? We give a cover U_i, and figure out how to glue it together:

$$\begin{array}{ccc} U_i \times_M U_j & \xrightarrow{o} & U_j \\ \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle o} \\ U_i & \xrightarrow{o} & M \end{array}$$

And we have this condition on the triple overlaps.

So to build something, we can take: $U_i \in \mathcal{G}$ (i runs through some index category).

We want $U_{ij} \in \mathcal{G}$ $(i, j \in I)$

$$\begin{array}{ccc} U_{ij} & \xrightarrow{o} & U_j \\ \downarrow{\scriptstyle o} & & \\ U_i & & \end{array}$$

symmetric in i and j. Thse are "relations".

We have a condition on triple overlaps. What is it?

$$\begin{array}{ccc} U_{ijk} & \xrightarrow{o} & U_{jk} \\ \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle o} \\ U_{ij} & \xrightarrow{o} & U_j \end{array}$$

(Best to draw this as a cube balanced on a vertex, with the bottom vertex removed.)

How do we get M from this data? Here is the answer.

Define $\boxed{M^-} := \left(\coprod_i h^{U_i}\right)/R$ where R is the equivalence relation from U_{ij}.

18.B. EXERCISE. Show that $M^- \in \underline{\mathbf{PSh}}^+_{\mathcal{G}}$. Translation: M^- "already" satisfies the identity axiom. (Easy!)

Then define $\boxed{M} := (M^-)^+$.

18.C. EXERCISE. Show/describe:

(a) $U_i \to M$ is a representable open.
(b) These $U_i \to M$ form an open cover of M.

(c) Show that the following is a fibered square:

$$\begin{array}{ccc} U_{ij} & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & M \end{array}$$

(All of these morphisms are "representable opens" from the above discussion.)

Thus we have successfully described an element of $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$!

18.4. **Corollary. —** *Any* cover produces a description of this sort!

18.5. *Definition.* Let's call this construction $\boxed{\left(\coprod U_i / \coprod R_{ij}\right)^{+}}$ or $\boxed{\left(\coprod U_i / \coprod R_{ij}\right)^{sh}}$.

18.6. **Theorem. —** *Conversely,* any *element* M *of* $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$ *arises in this way (as* $(\coprod U_i / \coprod R_{ij})^{+}$*)*

Proof. M has a cover by representable opens $U_i \to M$, and we create $R_{ij} = U_i \times_M U_j$ — basically this was our motivation for our construction the first place! ☐

Class 7 starts here. One lesson about plane with doubled origin: the issue wasn't whether it was a sheaf. It was whether it was "locally representable", ie, whether it had a "cover".

19. Examples in familiar situations

19.1. **Open subsets of balls.** If $\mathcal{G}$ = open subsets of balls (where maps are smooth, differentiable, or whatever we have picked in advance), we get manifolds (of the corresponding sort), minus the Hausdorff and second countable condition.

19.2. **Affine schemes.** If $\mathcal{G} = \underline{\mathbf{Aff}}$ (the category of affine schemes), we get schemes with affine diagonal.

19.A. Exercise. Prove this.

So algebraic geometers get the Hausdorff condition for free! Also: This class is good for Cech cohomology (refer back to desire for covering a compact manifold with contractable open sets all of whose multiple intersections are empty or contractable).

19.3. **Schemes with affine diagonal.** If $\mathcal{G}$ = schemes with affine diagonal, *and we take as the opens, the scheme-theoretic notion of open embeddings*, we get *all* schemes. But to be clear: we can't just take "open embeddings which are affine morphisms" (which are the ones that you might have naively guessed from §19.2). We needed some additional insight to know that we wanted a bigger class of things to be called "opens", which indeed restricts to the previusly expected class of opens for maps of affine schemes to affine schemes. (Thanks to Spencer Dembner for pointing this out. Ben Church had some ideas of why one should have "predicted" this larger class, and in fact he may have the "best explanation"; I'll have to talk to him more.)

19.B. EXERCISE. Prove this.

19.4. **Schemes.** If $\mathcal{G} = \underline{\textbf{Sch}}$ (and opens are "open embeddings" in the usual, Zariski, sense), then you get nothing new.

19.5. **Complex analytic spaces.** Tentatively define **complex analytic varieties** and **complex analytic schemes** by doing this construction with the appropriate atomic $\mathcal{G}$ (both are ringed spaces, with sets things cut out in open subsets of $\mathbb{C}^n$ by holomorphic equations; for the functions, take either the restrictions of holomorphic functions to these subsets, to get varieties, or else quotient out by the ideal sheaf, to allow yourself nilpotents and get schemes).

Either of these might be called *complex analytic spaces* (I think Griffiths and Harris used this for the first, while Grothendieck used it for the second). So to be safe, we might use the bold-faced phrases to disambiguate.

20. THE DIAGONAL MORPHISM MAKES AN APPEARANCE

Many of the examples above have a property that does not always hold. This also gives us an excuse to revisit the important lesson that the "diagonal morphism" is key.

20.1. **Important Lemma. —** *Suppose $\mathcal{G}$ has fibered products, and $F \in \underline{\textbf{PSh}}_{\mathcal{G}}$. Then all $X \to F$ is representable for all $X \in \mathcal{G}$ if and only if the diagonal morphism $\delta : F \to F \times F$ is representable.*

The representability of *all of these morphisms* is captured by the representability of this one morphism. We say that such F has **representable diagonal**.

Proof. Suppose first that $\delta : F \to F \times F$ is a representable morphism. We wish to show that $X \times_F Y$ is a representable object. But it is obtained by:

$$\begin{array}{ccc} X \times_F Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ F & \xrightarrow{\delta} & F \times F \end{array}$$

Next, suppose $X \times_F Y$ is represnetable for all $X \to F$, $Y \to F$, $X, Y \in \mathcal{G}$. We wish to show that $\delta : F \to F \times F$ is representable, i.e. for all $U \to F \times F$, from

$$\begin{array}{ccc} U \times_{F \times F} F & \longrightarrow & U \\ \downarrow & & \downarrow \\ F & \xrightarrow{\delta} & F \times F, \end{array}$$

$U \times_{F \times F} F$ is representable.

But consider this diagram:

$$\begin{array}{ccc} ? & \longrightarrow & U \\ \downarrow & & \downarrow \\ U \times_F U & \longrightarrow & U \times U \\ \downarrow & & \downarrow \\ F & \xrightarrow{\delta} & F \times F \end{array}$$

□

Notice that we proved that $U \times_F U = (U \times U) \times_{F \times F} F$.

20.2. *Remark.* We can interpret $U \times_{F \times F} F$ as "Isom"! And $(X \times Y) \times_{F \times F} F$ as Isom! I'll explain this at the board; it is harder to explain this in writing.

20.A. EXERCISE. Show that everything in $\underline{\mathbf{Psh}}_{\mathbf{Aff}}$ has representable diagonal (i.e., affine diagonal). Ditto for complex analytic varities and complex analytic schemes.

20.3. **Question.** What is the property of $\mathcal{G}$ that makes everything in $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$ have representable diagoanal? You collectively told me after class 6. (Because for more general $\mathcal{G}$, this is not the case.) Have we thought about the case of manifolds, which does *not* have fibered products?

Added here May 7 2022

20.B. EXERCISE. Show that if $\mathcal{G}$ has fibered products, so does $\underline{\textbf{LRepSh}}_{\mathcal{G}}$. (This is proved in Proposition 39.4, so I'll probably move that here.)

Part 4. Extending the notion of a topology on a set (topological space) to a topology on a category

21. INTERESTING NEW TOPOLOGIES

This section is disorganized, but contains within it the exposition I gave in class 6.

Recall: **Definition: topology on a category.** Given a category $\mathcal{C}$, a *topology* on $\mathcal{C}$ is a collection of sets of arrows $\{V_i \to U\}$ for each $U \subset \mathcal{C}$ (called "coverings") such that:

(i) Any isomorphism is a covering.
(ii) Coverings pull back: If $\{V_i \to U\}$ is a covering and $W \to U$, then $V_i \times_U W$ exists for each i, and $\{V_i \times_U W \to W\}$ is a covering.
(iii) Coverings compose: If $\{W_{ij} \to V_i\}$ are coverings, and $\{V_i \to U\}$ is a covering, then $\{W_{ij} \to U\}$ is a covering.

Example: Sheaf is a contravariant functor satisfying two additional axioms involving the topology.

Zariski topology: open sets are "too big". The implicit function theorem fails. Let's fix it.

Define the etale topology. Explain why the implicit function theorem holds.

The issue is that we need an inverse function for things where the Jacobian condition says there should be one.

The simplest things we can do is to use covering spaces themselves as covers! But in algebraic geometry, these aren't subsets.

Discuss $\sqrt{t}$ and $\sqrt[3]{t-2}$ near $t = 1$, in usual topology.

Where is it well-defined? On big open sets; there may not be a "biggest". Where is their sum defined? Certainly on their "intersection".

Do etale open set example.

Consider $(\sqrt{t} + \sqrt[3]{t-2}) - \sqrt{t}$.

Define Grothendieck topology. "Intersection" is fibered product. So $U \cap U$ is not U! But this isn't tragic; $\sqrt{t} + \sqrt{t}$. Etale topology is a Grothendieck topology.

Implicit function is clearly true.

$$\begin{array}{ccc} X \times_Y X & \longrightarrow & X \\ \downarrow \uparrow \Delta & & \downarrow \text{et} \\ X & \longrightarrow & Y \end{array}$$

Put differently, there are local sections!

Let's do sheaves in this topology.

Let $U \to X$ be an etale cover.

For convenience, we can take a single cover!

Even in the more usual topologies this is interesting.

$$\mathcal{F}(X) \longrightarrow \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times U) \ .$$

Let's try this out, with double cover of x-line branched over 0, with the 0 removed.

Example 1.

Then the cover is $\operatorname{Spec}(k[x,y]/(x-y^2))_x \to \operatorname{Spec} k[x]_x$.

$U \times_k U = \operatorname{Spec} k[x,y,z]/(x-y^2, y-z^2) = \operatorname{Spec} k[x,y,z]/(x-y^2, y-z) \coprod \cdots$.

When does a polynomial in y "descend" = "glue" to a polynomial in x? Answer: even exponents only.

Side remark: we didn't need to toss out 0 here! So etaleness isn't what matters. (It will be flatness.)

Example 2. $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$.

Example 3. (finite) Galois extension.

Example 4. (finite) separable extension.

Fun things you can do with etale topology: smooth = locally $\mathbb{A}^n$. (Sketch proof...)

Double covers := finite flat degree 2.

Isotrivial things become locally trivial. $\mathbb{P}^n$-bundles in the etale topology.

—

Define: *Etale topology:* etale open sets. (Analytic version too!) Intersection = fibered product.

Examples: etale opens of X. The etale site on the category of schemes.

Examples: $\sqrt{1+x}$. cube root of $x-1$. Add them together.

—

21.1. **Caution.** By this point, we have discussed the fact that for sheaves on a topologized category, we have to deal with self-intersections of opens, because they give us something nontrivial!

22. Other topologies on the category of schemes

These are less weird.

Warning: we will want our spaces to be sheaves in these topologies! We'll worry about this later.

22.1. *Definition.*

In the following, I'm being a bit sloppy; I want to make these definitions simultaneously for real manifolds of various flavors (topological, smooth, analytic), and complex spaces of various types (holomorphic manifolds, complex analytic varieties, complex analytic schemes). So apply these separately to different $\mathcal{G}$.

(0) We have decided to call the "classical" topology the analytic topology (abbreviated "an").

(1) put an open cover into *one* open set. things glue! In other words, a cover in this topology is the disjoint union of covers in the classical topology. I won't bother giving this topology a name.

Explain how self-intersections now have everything; do a double cover.

(2) Classical: source-local. locally on the source an isomorphism! We have decded to call this the **analytic-étale topology** (abbreviated **an.et**).

Example: $\mathbb{R} \to S^1$. picture, including relations.

(4) Analytic-smooth. ("Smooth morphism" in algebraic geometry means "submersion". I really should say "submersion", but I won't be able to stop myself from saying "smooth". I apologize.) Covers are surjective submersions. We have decided to call this the **analytic-smooth topology** (abbreviated **an.sm**).

22.A. EXERCISE. Check that each of these are indeed topologies on the appropriate $\mathcal{G}$.

22.2. *Definition.* We don't need these words, but a topology is *finer* than it if it has more opens. The opposite is *coarser*.

By the way, I like this: `https://www.math3ma.com/blog/comparing-topologies`.

23. IF YOU HAVE DIFFERENT TOPOLOGIES ON THE SAME CATEGORY, THE "SHEAVES" MAY STILL BE THE SAME

If we have several different topologies on $\mathcal{G}$, they may still have the same sheaves. I could jump right to the statement and proof, but I think some warm-up examples will help motivate a super-useful theorem.

23.1. *Examples.*

if $\mathcal{G}$ are open subsets of $\mathbb{R}^n$ (with various notions of maps, e.g. smooth or continuous or differentiable), or real manifolds:

$$\underline{\mathbf{Sh}}_{\mathcal{G}}^{an.sm} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}^{an.et} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}^{an} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}$$

if $\mathcal{G}$ are holomorphic closed subsets or closed subschemes of open subsets of $\mathbb{C}^n$, or complex manifolds:

$$\underline{\mathbf{Sh}}_{\mathcal{G}}^{an.sm} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}^{an.et} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}^{an} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}$$

I will refer to all such $\mathcal{G}$ as "analytic" examples.

23.A. EASY EXERCISE. Prove this. (Basically, the requirements for a presheaf to form a sheaf in one topology are a *subset* of the requirements in a finer topology.)

23.2. **Sheaves in these topologies (in this analytic setting) are the same.**

We will see why these categories have the same sheaves. (In particular, our spaces are sheaves in all of these topologies.)

(I may have discussed the case of moduli of curves here, but probably didn't.)

As a warm-up:

23.3. **Theorem. —** *Sheaves in the analytic topology are same as in sheaves in analytic-étale topology.*

Reason: any cover in analytic-étale can be "refined" to cover in analytic, and vice versa.

The following Lemma is a general "warm-up" tool that will be the start of proving the "warm-up" Theorem above.

23.4. **Lemma. —** *Suppose $\mathcal{F} \in \underline{\mathbf{PSh}}_{\mathcal{G}}$. (Here $\mathcal{G}$ is any of category, not just these special geometric analytic ones.) Given*

$$\begin{array}{ccc} \coprod V_j & \longrightarrow & \coprod U_i \\ & \underset{identity}{\overset{+}{\searrow}} \quad \swarrow & \\ & X & \end{array}$$

Then the identity axiom for the cover V implies the identity axiom for the cover U.

In that diagram, we are requiring nothing about the map from the V's to the U's — the V_j's aren't required to cover the U_i's in any sense.

Proof.

$$\begin{array}{ccc} \prod \mathcal{F}(V_j) & \longleftarrow & \prod \mathcal{F}(U_i) \\ & \nwarrow \quad \nearrow & \\ & \mathcal{F}(X) & \end{array}$$

□

23.5. *Quick break to introduce sloppy but convenient notation.* I'm just going to write $\{U\}$ for $\coprod U_i$. In class I wrote U (and stated that although our $\mathcal{G}$ might not have arbitrary disjoint unions — e.g. an uncountable disjoint union of manifolds isn't a manifold, because it doesn't have the second countable property — it should be clear what I mean). But perhaps $\{U\}$ is a good compromise.

23.6. I then explained how, if $\{U\} \to X$ is an analytic étale cover, you could complete the above diagram with an analytic cover $\{V\} \to X$.

Similarly, if $\{U\} \to X$ is an analytic smooth cover, you could complete the above diagram with an analytic cover $\{V\} \to X$, becuase $\{U\}$ "locally" looksl ike the base $\times$ a ball in $\mathbb{R}^n$ (or $\mathbb{C}^n$), so (using Spencer's trick) you just take that bit of the base times $0 \in \mathbb{R}^n$.

23.7. **Corollary. —** *The notion of "separated presheaves" is the same in the analytic, analytic-étale, and analytic-smooth topologies.*

This is going to set up something super fundamental...

24. The Very Cool Theorem 24.3 (which needs a better name)

To set up the rest of our argument, I posed a type of question which we can ask in various circumstances.

24.1. **Important Rhetorical Question.** Given $\pi : \{U\} \to X$ in $\mathcal{G}$, and $F \in \underline{\mathbf{PSh}}_{\mathcal{G}}$, does F satisfy the two sheaf conditions for $\{U\}$? In other words:

- (†) (identity) Is $F(X) \to F(\{U\})$ injective? (More precisely, the morphism is $F(X) \to \prod_i F(U_i)$, but I am trying to practice this new notation).
- (‡) (gluability) Given $s \in F(\{U\})$, such that $\pi_1^* s = \pi_2^* s \in F(\{U\} \times_X \{U\})$, can we write $s = \pi^* t$ for some $t \in F(U)$?

As usual, if the identity statement (†) is satisfied, the t in the gluability statement (‡) is unique.

24.A. Exercise. Explain what I mean in the gluability statement (‡). For example, $F(\{U\} \times_X \{U\})$ really means $\prod_{i,j} F(U_i \times_X U_j)$, where i and j are running over the index set of $\{U\}$. And certainly we are not excluding $i = j$.

So for example, if the answer to Question 24.1 is "yes" for all covers $\{U\} \to X$ in your topology, then F is a sheaf in your topology.

24.2. **Theorem. —** *In the situation of Question 24.1, if π has a section σ, then the answer is* **always** *yes.*

24.B. Exercise (for those feeling like it). Do the same if F is a fibered category (in which case F will satisfy the three gluing conditions for a stack).

Proof. Let $\sigma : X \to \{U\}$ be the section. (Important: Do you see what $X \to \{U\}$ means?)

$$\begin{array}{c}\{U\} \\ {\scriptstyle\pi}\downarrow\uparrow{\scriptstyle\sigma} \\ X\end{array}$$

Then the commutativity of

$$\begin{array}{ccccc} & & F(\{U\}) & & \\ & \overset{\pi^*}{\nearrow} & & \overset{\sigma^*}{\searrow} & \\ F(X) & & \xrightarrow[\mathrm{id}]{\qquad\qquad} & & F(X) \end{array}$$

shows that π^* must be injective.

Now for gluability (‡). Given $s \in F(\{U\})$, with $\pi_1^* s = \pi_2^* s$, how do we find t such that $s = \pi^* t$? There is only one obvious choice: we guess $t = \pi^* s$. These satisfy $\pi_1^* s = \pi_2^* s = \pi_{12}^* t$, where $\pi_{12} : \{U\} \times_X \{U\} \to X$ is the projection.

The following diagram is useful.

$$\begin{array}{ccc} \{U\} \times_X \{U\} & \underset{\pi_1}{\overset{\sigma_1}{\rightleftarrows}} & \{U\} \\ \sigma_2 \uparrow\downarrow \pi_2 & & \downarrow \pi \\ U & \xrightarrow{\ \pi\ } & X \end{array}$$

Here σ_1 and σ_2 are the sections of π_1 and π_2 respectively, pulled back from σ (let me know if it isn't clear what I mean). In particular, $\sigma_2 = (\sigma \circ \pi, \mathrm{id})$, and $\sigma_1 = (\mathrm{id}, \sigma \circ \pi)$.

To show that this $t = \sigma^* s$ works, we are required to show that $s = \pi^* \sigma^* s$.

Spencer then pointed out that $\sigma \circ \pi = \pi_1 \circ \sigma_2$, basically by the definition of σ_2. Thus $\pi^* \sigma^* s = \sigma_2^* \pi_1^* s$. But from our hypothesis $\pi_1^* s = \pi_2^* s$, we have that

$$\pi^* \sigma^* s = \sigma_2^* \pi_1^* s = \sigma_2^* \pi_2^* s = (\mathrm{id}_{\{U\}})^* s = s$$

as desired. Thanks Spencer! □

24.3. **Very Cool Theorem. —** *Suppose* $F \in \underline{\mathbf{PSh}}_{\mathcal{G}}$ *is a sheaf in a topology* T *on* $\mathcal{G}$*, and* $\{V\} \to X$ *is a cover in* T *(which I indicate by decorating the arrow with a "*T*"). Then given any commuting diagram*

$$\begin{array}{ccc} \{V\} & \xrightarrow{\qquad\qquad} & \{U\} \\ & {}_{T}\searrow \quad \swarrow & \\ & X & \end{array}$$

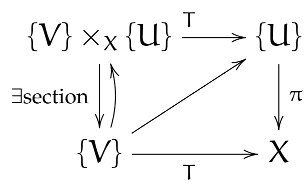
F *satisfies the sheaf axioms for* $U \to X$.

24.C. EXERICSE. Prove this helpful and essential equivalent formulation of the extremely useful theorem: the hypotheses are the same as saying that $\{U\}$ has a section T-locally.

Proof. By the easy Lemma 23.4 before, we see that the identity axiom is satisfied (which would also come out in the wash in this argument anyway). I will outline how gluability should work.

The idea: If $\{U\} \to X$ had a section, we would win. It might not — but it *does* T-locally (the neighborhood $\{V\} \to U$ will work), and that will be all we need, because we can glue sections of F along T-covers.

In more detail: consider the following diagram, where F satisfies the sheaf axioms for the three arrows marked "T" or "∃section".



Given $s \in F(U)$ "with gluing data", we get $s' \in F(\{V\} \times_X \{U\})$ "with gluing data over $\{V\}$ and $\{U\}$", so we get a section $s'' \in F(\{V\})$, "with gluing data over X", so we get a section $s''' \in F(X)$.

I claim that $\pi^* s''' = s$ in $F(\{U\})$ (completing the argument). How do I know that? Informally: we have two sections of F on $\{U\}$ that agree T-locally. More precisely: when I pull both back to $\{V\} \times_X \{U\}$, I get the same things, with the same gluing data for $\{U\}$; and I have the identity axiom for these. □

24.D. EXERCISE. Fill in the details. (It should actually be quite reasonable! Let me know if it isn't.)

25. REVISITING THE VERY COOL THEOREM 24.3

25.1. **Step 1.**

Suppose you are in your category $\mathcal{G}$ (which I am assuming is a very familiar one to you), and you have a nice open cover of X by $\{U_i\}$. For convenience of this motivating exposition, let's assume the index set is ordered.

Now suppose you have a vector bundle on $\mathcal{F}$ on X, and you have two sections t_1 and t_2 of it on $\mathcal{F}$. Then to see if they are the same, you need to check only that they are the same after pulling back to the U_i. To understand "sameness" of sections, you just need to check on a cover.

Next, suppose you want to build a section t, and you have sections s_i on the U_i. What do you need to be able to build t? Well, you need the s_i's to agree on the pairwise overlaps, $U_i \cap U_j$. If that holds, then you can glue them together. And by the previous paragraph, when you glue them together, they can be glued together in only one way. Summary: to build a section, you need data on the cover, along with a condition on the overlaps (or, if you want, the data of "equalities" on the pairwise overlaps). All we've done is restate the sheaf conditions. Indeed, this works just fine $\mathcal{F}$ is any sheaf – the only reason I said "vector bundle" is for psychological comfort.

Now, suppose you want to *build* a vector bundle (or, of course, a sheaf) on X, and you have sheaves $\mathcal{F}_i$ on the U_i. What data do you need? As you know, you need the further data of isomorphisms $\phi_{ij} : \mathcal{F}_i \to \mathcal{F}_j$ on the pairwise overlaps (you'll realize I'm being sloppy in notation). And you'll need further data of equality on the triple overlaps. Then you can build $\mathcal{F}$. And furthermore, the $\mathcal{F}$ you build is unique up to unique isomorphism. Why? Becuase if you build two different sheaves, $\mathcal{F}$ and $\mathcal{F}'$, then they come with isomorphisms on the U_i that agree on the pairwise overlaps. Translation: it is the previous paragraph, applied to build a section of the sheaf $\mathrm{Isom}(\mathcal{F}, \mathcal{F}')$ on X. (Think about this!)

25.2. **Step 2.**

Now all of the above could describe the double covers as $U_{ij} = U_i \times_X U_j$, and the triple covers as $U_{ijk} = U_i \times_X U_j \times_X U_k$. And the first time you learn about such things, you imagine that $i < j < k$. But when you are older you realize that you needn't require $i < j < k$; you could just take i, j, k distinct, and then you don't have to worry about an ordering on the index set (or even if there has to be one). Then when you are older still, you realize that you needn't even have i, j, and k distinct!

With this maturity, we now see that in our classical situations, we can build things on X using things on a cover $U \to X$, sometimes with a little more information. To check if two sections of $\mathcal{F}$ on X are the same, we just check on the cover. To have a section on X, we have our section on U, but we need further data on the pairwise intersection $U \times_X U$. To have a sheaf on X, we have a sheaf on U, but we need further data on the pairwise intersection $U \times_X U$, and also on the triple intersection $U \times_X U \times_X U$.

You can imagine that to have a stack on X, you need a stack on $\{U\}$, along with further data all the way up to quadruple intersections.

25.3. **Step 3.**

But sometimes given $\pi : \{U\} \to X$, we can build things even when it isn't a cover in your classical setting. In fact, if there is a section $\sigma : X \to \{U\}$, then all of the above works.

Remember why this isn't hard: we build up to it one step at a time.

Sections of a sheaf are determined by their pullback to $\{U\}$ via the fact that the composition $\mathcal{F}(X) \to \mathcal{F}(\{U\}) \to \mathcal{F}(X)$ is the identity. (Fancy way of saying it: we "pull back" the equality from $\{U\}$ to X by σ.) That's our identity axiom.

If you have $s \in \mathcal{F}(\{U\})$ satisfying the double-cover (cocycle) condition, how do you know it comes from some $t \in \mathcal{F}(X)$, and uniquely? First we construct t; then we show it works (the pullback is inded s); then we show it is unique (there is no other t). Construction is easy: just pull back the section s by t. Showing that it works is easy: the doublecover condition ensures that when you pull back t you get s. (This is Spencer's trick.) How do you know there is no other t? That's just the *previous paragraph*!

Next, if you have a sheaf $\mathcal{G}$ on $\{U\}$ satisfying the triple-cover (cocycle) condition, how do you know it comes from some $\mathcal{F}$ on X, and uniquely (i.e. it is unique up to unique isomorhpism, which is what "unique" means in this sense)? First, we construct $\mathcal{F}$; then we show it works (the pullback is indeed $\mathcal{G}$); then we show it is unique up to unique isomorphism (there is no other $\mathcal{F}'$). Construction is easy: just pull back $\mathcal{G}$ by σ. Showing that it works is easy: the triple cover condition ensures that when you pull back $\mathcal{F}$, you get $\mathcal{G}$, along with all the additional data. If you have another $\mathcal{F}'$ with the same pullbacks $\mathcal{G}$, how do you get a unique isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}'$? You apply the *previous paragraph* to $\mathrm{Isom}(\mathcal{F}, \mathcal{F}')$, using the same cover!

(If you want, you can do the same thing with a stack on $\{U\}$; but I don't think we need it.)

26. Sieves explain the Very Cool Theorem 24.3 in a very nice way

Thanks to Daniel Kim for explaining sieves to us, really well. I feel that they are very well motivated at this point.

The sheaf condition for a presheaf $\mathcal{F}$ and a cover $U \to X$ can be rewritten as:

$$\mathcal{F} \to \lim \left(\mathcal{F}(U) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \mathcal{F}(U \times_X U) \right)$$

is an isomorphism.

Let's rewrite this in terms of maps maps in $\underline{\mathbf{Psh}}$.

The left side is simply $\mathrm{Hom}_{\underline{\mathbf{Psh}}}(h_X, \mathcal{F})$, by Yoneda's Lemma.

For the right side:

$$\begin{aligned}\lim\left(\mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U) \right) &= \lim\left(\mathrm{Hom}_{\underline{\mathbf{PSh}}}(h_U, \mathcal{F}) \rightrightarrows \mathrm{Hom}_{\underline{\mathbf{PSh}}}(h_U \times_{h_X} h_U, \mathcal{F}) \right) \\ &= \mathrm{Hom}_{\underline{\mathbf{PSh}}}\left(\mathrm{colim}\left(h_U \leftleftarrows h_U \times_{h_X} h_U \right), \mathcal{F}\right)\end{aligned}$$

Thus we have rephrased the sheaf condition for a presheaf $\mathcal{F}$ and a cover $U \to X$ as:

$$\mathrm{Hom}_{\underline{\mathbf{PSh}}}(h_X, \mathcal{F}) \to \mathrm{Hom}_{\underline{\mathbf{PSh}}}\left(\mathrm{colim}\left(h_U \leftleftarrows h_U \times_{h_X} h_U \right), \mathcal{F}\right)$$

is an isomorphism.

This map is induced by the map in $\underline{\mathbf{Psh}}$:

$$\mathrm{colim}\left(h_U \leftleftarrows h_U \times_{h_X} h_U \right) \to h_X$$

This begs the question: what is this creature $\mathrm{colim}\left(h_U \leftleftarrows h_U \times_{h_X} h_U \right)$?

(For convenience of notation, define $\boxed{S_{U\to X} \in \underline{\mathbf{PSh}}}$ to be $\mathrm{colim}\left(h_U \leftleftarrows h_U \times_{h_X} h_U \right)$. We will call this a *sieve* for $U \to X$, see Definition 26.4.)

Let's find out, by evaluating $S_{U\to X}$ on $T \in \mathcal{G}$. $S_{U\to X}$ evaluated on T is precisely the data of a map $T \to U$ such that the two maps $T \to X$ induced by the two maps $U \to X$ are the *same* map $T \to X$. Translation:

26.1. **Proposition.** — $S_{U\to X}$ *are precisely the maps* $T \to X$ *that factor through* $T \to U$. *In particular,* $S_{U\to X} \to h_X$ *is a subfunctor.*

This perhaps motivates why $S_{U\to X}$ is called a sieve: $S_{U\to X}$ is a subfunctor of h_X; of all the maps to X (i.e., h_X), we consider just the maps through U (we are "blocking" some of the maps).

We have proved:

26.2. **Proposition.** — $\mathcal{F} \in \underline{\mathbf{Psh}}$ *satisfies the sheaf conditino for* $U \to X$ *if and only if* $\mathrm{Hom}_{\underline{\mathbf{PSh}}}(h_X, \mathcal{F}) \to \mathrm{Hom}(S_{U\to X}, \mathcal{F})$ *is a bijection.*

Equipped with this, we very easiy prove the things in the Very Cool Theorem 24.3.

Recall that the first part was this: If $U \to X$ has a section, then *every* presheaf $\mathcal{F}$ automatically satisfies the sheaf condition. This is because $S_{U\to X}(T) = h_X(T)$.

For the second part, given $V \to U \to X$, where $\mathcal{F}$ is a sheaf in topology T, and V is a cover in topology T, then T satisfies the sheaf condition for $U \to X$. The reason for this is that we have:

$$S_{V\to X} \subset S_{U\to X} \subset h_X$$

We then have:

$$S^{++}_{V\to X} \subset S^{++}_{U\to X} \subset h^{++}_X$$

26.3. **Claim. —** $S^{++}_{V\to X} \subset h^{++}_X$ *is an equality*

Proof. Both are sheaves. So to check this, we can check this by applying Hom to an arbitrary sheaf $\mathcal{G}$ (basically, Yoneda's Lemma). But then

$$\begin{aligned} \mathrm{Hom}(S^{++}_{V\to X}, \mathcal{G}) &= \mathrm{Hom}(S_{V\to X}, \mathcal{G}) \quad \text{(univ. prop. of sheafification)} \\ &= \mathrm{Hom}(h_X, \mathcal{G}) \quad (\mathcal{G} \text{ satisfies sheaf property for } V \to X) \\ &= \mathrm{Hom}(h^{++}_X, \mathcal{G}) \quad \text{(univ. prop. of sheafification)} \end{aligned}$$

Thus

$$S^{++}_{V\to X} = S^{++}_{U\to X} = h^{++}_X$$

from which

$$\mathrm{Hom}(S_{U\to X}, \mathcal{F}) = \mathrm{Hom}(S^{++}_{U\to X}, \mathcal{F}) = \mathrm{Hom}(h^{++}_X, \mathcal{F}) = \mathrm{Hom}(h^X, \mathcal{F})$$

as desired.

We can put these pieces together to introduce some terminology useful in other contexts.

26.4. **Definitions. —** A **sieve** over X is a subpresheaf of h_X. Example: $S_{U\to X}$ for any $U \to X$; we call this a **sieve for** $U \to X$ (as promised earlier. Given a topology T, a sieve is called a **covering sieve** if it contains some cover.

We have thus proved the following.

26.5. **Theorem. —** *$\mathcal{F}$ is a sheaf iff* $\mathrm{Hom}(S, \mathcal{F}) \leftarrow \mathrm{Hom}(h_X, \mathcal{F})$ *s bijective for every covering sieve.*

26.6. *Remark.* Given a topology, the covers give covering sieves, which in turn determine the sheaves. So what really matters for the notion of a sheaf is the notion of the covering sieves, not the notion of the covers. For example, we have proved that the smooth and etale topologies have the same covering sieves.

27. APPLYING THE VERY COOL THEOREM 24.3 TO OUR ANALYTIC SETTING

As a corollary, we have a bunch of equalities in our analytic settings

$$\underline{\mathbf{Sh}}_{\mathcal{G}}^{an.sm} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}^{an.et} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}^{an} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}$$

27.1. **Corollary. —** *We have, in the analytic setting:*

$$\underline{\mathbf{Sh}}_{\mathcal{G}}^{an.sm} = \underline{\mathbf{Sh}}_{\mathcal{G}}^{an.et} = \underline{\mathbf{Sh}}_{\mathcal{G}}^{an} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}.$$

Similarly, in the algbraic setting:

$$\underline{\mathbf{Sh}}_{\mathcal{G}}^{an.sm} = \underline{\mathbf{Sh}}_{\mathcal{G}}^{an.et} =\subset \underline{\mathbf{PSh}}_{\mathcal{G}}.$$

Proof. We've seen how to get from an analytic-smooth cover (to an analytic-etale cover) to an analytic cover. We've seen how to get from a smooth cover to an étale cover. □

27.2. **Locally representables are the same in these topologies (analytic case).**

But what about the locally representables? We quickly see in each case:

$$\mathcal{G} \subset \underline{\mathbf{LRepSh}}_{\mathcal{G}}^{an} \subset \underline{\mathbf{LRepSh}}_{\mathcal{G}}^{an.et} \subset \underline{\mathbf{LRepSh}}_{\mathcal{G}}^{an.sm} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}$$

(Translation: any analytic cover is itself already an analytic-étale cover. Any analytic-étale cover is automatically an analytic-smooth cover.)

Each of these incusions are full subcategories.

Now I claim that each of these three "LRepSh's" are the same as well in each instance.

27.3. **Theorem. —**

$$\mathcal{G} \subset \underline{\mathbf{LRepSh}}_{\mathcal{G}}^{an} = \underline{\mathbf{LRepSh}}_{\mathcal{G}}^{an.et} = \underline{\mathbf{LRepSh}}_{\mathcal{G}}^{an.sm} \subset \underline{\mathbf{Sh}}_{\mathcal{G}}$$

To do this, I need to show you that in any of these settings, if you have an objects of $\underline{\mathbf{LRepSh}}^{an.sm}$ it is an object of $\underline{\mathbf{LRepSh}}^{an}$. (You don't have to worry about morphisms.) So suppose you have $M \in \underline{\mathbf{LRepSh}}^{an.sm}$, along with a bunch of representable morphisms $U_i \to M$ from elements of $\mathcal{G}$, which are smooth (i.e., submersions), that cover M. We just need a bunch of other representable morphisms $U_i' \to M$ that are open embeddings, where $U_i' \in \mathcal{G}$.

The next two statements have proofs of two quite different flavors, and this will lead to some interesting things when we try to make them algebraic.

The main thing to note: these are the *same* constructions we used to show that categories of sheaves are the same.

27.4. **Proposition. —** $\underline{\mathbf{LRepSh}}^{an.et}_{\mathcal{G}} = \underline{\mathbf{LRepSh}}^{an.sm}_{\mathcal{G}}$

Proof. We have made this construction earlier! It suffices to prove the following. If $M \in \underline{\mathbf{Sh}}_{\mathcal{G}}$, and $U \in \mathcal{G}$ and $U \to M$ is a smooth (submersive) representable morphism of relative dimension $n > 0$, then I'll give you a bunch of maps $V_i \to U$, all in $\mathcal{G}$, where $V_i \to U$ are collectively surjective onto U, and $V_i \to M$ (a priori representable, as this is the composition of two representables) is a submersion of relative dimension $n-1$.

(Explain: in this way, we can start with a "smooth" cover, and do this repeatedly to get a "smooth cover of relative dimension 0", i.e. an étale cover.)

How might we do this? Slice? Given any point in $p \in U$, I'm going to take a small slice. (Discuss.) □

27.5. **Proposition. —** $\underline{\mathbf{LRepSh}}^{an}_{\mathcal{G}} = \underline{\mathbf{LRepSh}}^{an.et}_{\mathcal{G}}$

Proof. We have made this construction earlier! It suffices to prove the following. If $M \in \underline{\mathbf{Sh}}_{\mathcal{G}}$, and $U \in \mathcal{G}$ and $U \to M$ is an étale representable morphism, and $p \in U$, then I'll give an open neighborhood V of p in U so that $V \to M$ is even "classical-open".

Here is how. Let $R = U \times_M U$. Then $R \subset U \times U$. $R \to U$ is étale. We have a small neighborhood of $(p, p) \in R$ that is isomorphic to a small neighborhood of p in U. Then because $U \times U$ has the product topology (!!!!!), are two small neighborhoods $V', V'' \subset U$ of p, where $(V' \times V'') \cap R$ is a small neighborhood of U (on the diagonal, picture needed). Then take $V = V' \cap V''$. Then show that this works. □

In just the same way, we have:

27.6. **Theorem. —** *In the algebraic setting:* $\underline{\mathbf{LRepSh}}^{et}_{\mathcal{G}} = \underline{\mathbf{LRepSh}}^{sm}_{\mathcal{G}}$.

27.A. EXERCISE. Think this through. (We've basically done this in class — it is the same argument as in the analytic setting!)

27.7. **Taking stock.**

Let's quickly see what this has bought us in the analytic setting. Here $\mathcal{G}$ might be over $\mathbb{R}$, in which the objects might be manifolds, or else open subsets of $\mathbb{R}^n$, and the morphisms might be continuous, differentiable, smooth, or real-analytic maps. Or over $\mathbb{C}$, the objects might be open subsets of $\mathbb{C}^n$, or complex manifolds, or things cut out in open subsets of $\mathbb{C}^n$ by holomorphic equations (either set-theoretically or "scheme-theoretically"), or complex varieties, or complex schemes. In this setting we have

$$\mathcal{G} \subset \text{manifolds} \subset \underline{\mathbf{LRepSh}}_{\mathcal{G}} \subset \underline{\mathbf{Sh}}_{\mathcal{G}} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}$$

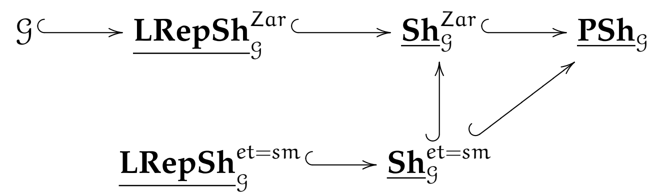
Here with both $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$ and $\underline{\mathbf{Sh}}_{\mathcal{G}}$ I didn't say which topology to take, as we've established that each of our three topologies of choice (an, an.et, an.sm) all give the same LRepSh's and sheaves. I think of $\underline{\mathbf{LRepSh}}_{\mathcal{G}}$ as "manifolds" (with scare quotes), because I haven't hadded in the "second countable" and "Hausdorff" conditions. Those two conditions are about the size of the cover/atlas needed (second countable means countable cover by real balls), and a property of the diagonal (Hausdorff means the diagonal is a closed embedding); and we have good reasons for wanting these (as Eric pointed out). We'll see analagous hypotheses in the algebraic setting.

(In the complex analytic space setting, I shouldn't say "manifold" in the above paragraph; I should say "complex analytic space", since our objects aren't "smooth".)

So what have we gained? In some sense, almost nothing — we haven't discovered any new kind of space. Perhaps we have noticed that we can check things on "smooth" covers (we can work in the "topology of submersions"). That's not too exciting, and is just a rewording/reworking of things that geometers know well. But at least we are used to thinking in terms of these exotic topologies that will serve us well soon, when we leave "sheaves" and go to more general "stacks". Plus they are helping us think cleanly in the algebraic setting.

28. Taking our analytic success into the algebraic setting as much as we can

The argument getting from an analytic-étale cover to an analytic cover didn't work in the algebraic setting — we have not shown that every sheaf in the Zariski topology is also a sheaf in the étale topology. Similarly, we don't konw that every locally representable sheaf in the Zariski topology isn't yet known to be locally representable in the étale topology. Thus our current understanding is:



We don't even see (yet!) whether $\mathcal{G}$ sits inside anything in the bottom row!

28.1. *Side Question.* What is an easy example of something that is a sheaf in the Zariski topology that is not a sheaf in the étale topology? There should be some super-easy example. I think we'll see an example later, and if I remember, I will refer to it in the notes.

28.2. So what now? Remember that at the very start of thinking about topologies, we wanted our geometric spaces $\mathcal{G}$ to be sheaf in our topologies — basically, for any object X in $\mathcal{G}$, we wanted "maps to X to glue". (Ben pointed out that in this case we say the topology is *subcanonical*, but we won't use this phrase.) We knew this in the analytic case because sheaves in the an.sm topology are the same as in the analytic topology, so it seems a little much to hope for.

But there is a huge miracle: our objects in our algebraic setting will be sheaves not just in the smooth topology, but even more! This is known as the **Miracle of Descent**, and one of the great holidays in the algebraic geometry calendar commemorates the day that Grothendieck received his vision of this miracle, and shared it with humankind.

There is a variant of this you may have seen before if you've done some number theory. If you want to prove something over a field k, you might prove it over its algebraic closure $\overline{k}$, and show that this property "descends" to k. This looks very similar to what we're doing, and $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ looks like it could be called a "cover", but it isn't in general smooth (let alone étale or Zariski). But it will fit into what we are building.

29. Using our Very Cool Theorem once more, to get the single algebra fact behind descent

The key single algebra fact is the following, which I could prove quickly. But I want to explain why it is really yet another example of the Very Cool Theorem 24.3. So I will deliberately do this in slow motion.

29.1. **Theorem. —** *Suppose* $\phi : A \to B$ *is a faithfully flat morphism of rings. Then the sequence of* A*-modules*

$$(1) \qquad 0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\phi\otimes 1 - 1\otimes\phi} B \otimes_A B$$

is exact. More generally, for any A*-module* M*, the exact sequence remains exact upon tensoring with* M*:*

$$(2) \qquad 0 \longrightarrow M \longrightarrow M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B$$

is exact.

I'll make some quick remarks before getting into the proof.

First, Here B might be replaced with a coproduct of many B's. This adds more annoying notation, and I ignore it for now.

Second, clearly (1), and hence (2), is a complex, so the issue is exactness.

Third, it is helpful to draw the situation in terms of a cofiber diagram of rings, or a fiber diagram of schemes.

$$\begin{array}{ccc} B \otimes_A B & \xleftarrow{b \mapsto 1\otimes b} & B \\ \uparrow{\scriptstyle b\mapsto b\otimes 1} & & \uparrow{\scriptstyle \phi} \\ B & \xleftarrow{\phi} & A \end{array}$$

i.e., geometrically

$$\begin{array}{ccc} \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B & \xrightarrow{\pi_2} & \operatorname{Spec} B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} B & \xrightarrow{\pi} & \operatorname{Spec} A \end{array}$$

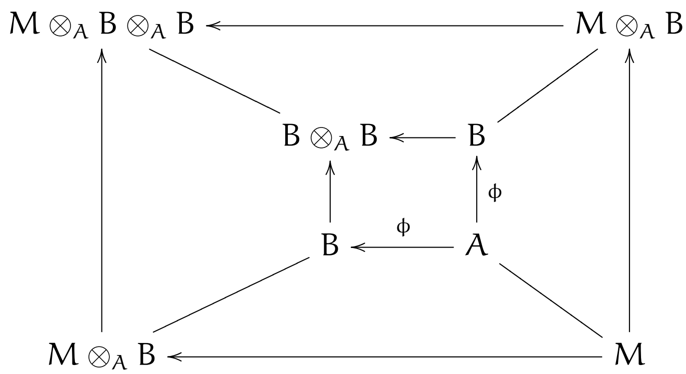
which looks very like our very cool theorem 24.3.

Also very importantly: if $\operatorname{Spec} B \to \operatorname{Spec} A$ is a smooth (or étale or Zariski) cover then it is faithfully flat — this is precisely the situation we want to be in! We could have required that π is smooth, but we will see that *precisely* what we need is that it is faithfully flat, which is what will tell us precisely what general topology we may want to consider. (This may cause us to utter an expletive, fpqcK!)

We really see (1) as an equializer diagram:

$$A \longrightarrow B \rightrightarrows B \otimes_A B$$

We can readily mix in the A-module M to the above discussion:



and

$$M \longrightarrow M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

Furthermore, we expect that if there is a map $\rho : B \to A$, so that the composition

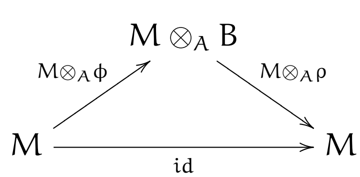
$$A \xrightarrow{\phi} B \xrightarrow{\rho} A$$

is the identity, the result should be quite easy.

So as a warm-up, we use this trick to prove just the injectivity part of Theorem 29.1, under the assumption that such a ρ exists. (Then we'll do it without such a ρ; then we'll prove the full statement of Theorem 29.1 *with* such a ρ; then we'll prove the full statement *without* such a ρ.)

29.2. **Warm-up. —** *Suppose* $\phi : A \to B$ *is a ring morphism, and* $\rho : B \to A$ *is a ring morphism, such that* $\rho \circ \phi = \mathrm{id}_A$. *Then for any* A*-module* M, $M \to M \otimes_A B$ *given by* $m \otimes 1$ *is injective.*

Proof. This follows (as before) from:



□

29.3. **Next warm-up. —** *Suppose* $\phi : A \to B$ *is a faithfully flat ring morphism. Then for any* A*-module* M, $M \to M \otimes_A B$ *given by* $m \otimes 1$ *is injective.*

Proof. We wish to prove that $0 \to M \to M \otimes_A B$ is exact. By faithful flatness of $A \to B$, it suffices to prove that $0 \to M \otimes_A B \to M \otimes AB \otimes_A B$ is exact. But if we take $M' = M \otimes_A B$, $A' = B$, $B' \otimes_A B$, then we can rewrite this as:

$$0 \to M' \to M' \otimes A'B$$

and this time, we have $\phi' \to A' \to B''$ (given by $B \to B \otimes AB$ with $b \mapsto 1 \otimes b$) has $\rho' : B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$ which is a one-side inverse to ϕ', so by Warm-up 29.2 this is true. □

We are ready for the full case, in the case where there is "section" ρ' (I guess a "cosection"?).

29.4. **Warm-up. —** *Suppose* $\phi : A \to B$ *is a ring morphism, and* $\rho : B \to A$ *is a ring morphism, such that* $\rho \circ \phi = \mathrm{id}_A$. *Then for any* A*-module* M, (2) *is exact.*

Proof. We have proved the injectivity part in Warm-up 29.2. And we've observed it is a complex. So we have the following question. Given $s \in M \otimes_A B$ such that $\pi_1^* s = \pi_2^* s$, is it true that $s = phi(t)$? We guess as usual that $t = \rho(s)$ will do the trick. We wish to show that $s = \phi(\rho(s))$. But $\phi(\rho(s)) = \rho_1'(\pi_2^*(s))$ by the definition of ρ_1' (Spencer's trick — I want to write more here). Then $\rho_1' \pi_2^*(s) = \rho_1' \pi_1^*(s) = (id^*)s = s$ as desired. □

29.5. *Finishing the proof of Theorem 29.1.*

You may want to try this yourself, given the steps above! Basically the argument is:

$$\begin{array}{ccc} \text{Proof of Thm. 29.1} & \longrightarrow & 29.4 \\ \downarrow & & \downarrow \\ 29.3 & \longrightarrow & 29.2 \end{array}$$

□

29.6. **For experts: a random fact from Danny Krashen.** Danny Krashen says: "In standard descent, if one of the morphisms from an fpqc cover is a completion $R \to \hat{R}$, then part of the descent data will involves an isomorphism of structures over $\hat{R} \otimes_R \hat{R}$, which is a difficult ring to work with (for example, often non Noetherian). The result of Beauville-Laszlo says in effect, you get an equivalence or categories even when ignoring these self intersections." My guess is the corresponding algebra result will not be too hard, so good value for money.

30. Topologies in the algebraic setting: Zariski, étale, smooth, fp+lpf, fp+qc, fpqcK

Here $\mathcal{G}$ will be any reasonable algebraic setting. For us now, it can be **Aff** or **Sch**. Later, it can reasonably the the category of schemes with affine diagonal, or algebraic spaces, or anything else — rather than memorizing what works, you should just *see* how little is needed to make it work, and realize how widely it applies.

Here are some toplogies we've seen. We would like our objects to be sheaves in each of these topologies. The boldfaced ones are ones you should remember. The italicized ones are ones that sound impressive, and you might want to know about them.

30.1. **Zariski topology.** Covers of X are collections of maps $\{U\} \to X$ that are "jointly surjective" (every point of X is in the image of some $U_i \to X$), and where $U_i \to X$ is Zariski-open.

30.2. **Etale topology.** Covers of X are collections of maps $\{U\} \to X$ that are "jointly surjective" (every point of X is in the image of some $U_i \to X$), and where $U_i \to X$ is étale.

30.3. **Smooth topology.** Covers of X are collections of maps $\{U\} \to X$ that are "jointly surjective" (every point of X is in the image of some $U_i \to X$), and where $U_i \to X$ is smooth.

We've seen the above before. Notice that in each case, they are jointly surjective — but there is even more! They are all "faithfully flat" (flat and jointly surjective), and this will be (as you'll see) the key thing. $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ is also faithfully flat. In the original french, faithfully flat is "fidèlement plant", abbreviated "fp", so when you see "fp", that's what it means. (Hence I prefer to abbreviate finitelypresented as fpr, or to use the french "pf", presentation fini. Apologies for french spelling and accents pronunciation)

30.4. *(ignore this for now) Faithfully flat and locally finitely presented.* Covers of X are collections of maps $\{U\} \to X$ that are faithfully flat and locally finitely presented. If you prefer (so it looks like the earlier definitions): Covers of X are collections of maps $\{U\} \to X$ that are "jointly surjective" (every point of X is in the image of some $U_i \to X$), and where $U_i \to X$ is flat and locally finitely presented. A reasonable abbreviation is "fp+lpf". Unfortunately people call it "fppf", which makes you think that the "locally" isn't in the definition. (Spencer gave a possible reason why, which I won't repeat here.) Good news about it: it generalizes (is "finer than") the Zariski, étale, and smooth topologies. Bad news: $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ isn't such a cover. Also bad news: it hides what makes gluing work. So we'll mostly ignore it, but now you know what Ben means when he says a morphism is fppf (although I'm personally not sure if he means fp+pf or fp+lpf).

30.5. *(ignore this for now) Faithfully flat and quasicompact.* Covers of X are collections of maps $\{U\} \to X$ that are faithfully flat and quasicompact. If you prefer (so it looks like the earlier definitions): Covers of X are collections of maps $\{U\} \to X$ that are "jointly surjective" (every point of X is in the image of some $U_i \to X$), and where $U_i \to X$ is flat and quasicompact, and there are finitely many U_i. A reasonable name for this is "fpqc" for "fp+qc". Unfortunately, I *think* people call the fpqcK topology (intorduced shortly) fpqc sometimes too — for example, the Stacks Project does; I just get confused. Good news: it includes $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$. Bad news: in the Zariski, étale, and smooth topologies, we didn't have any quasicompactness assumptions. We could patch this but we don't need to.

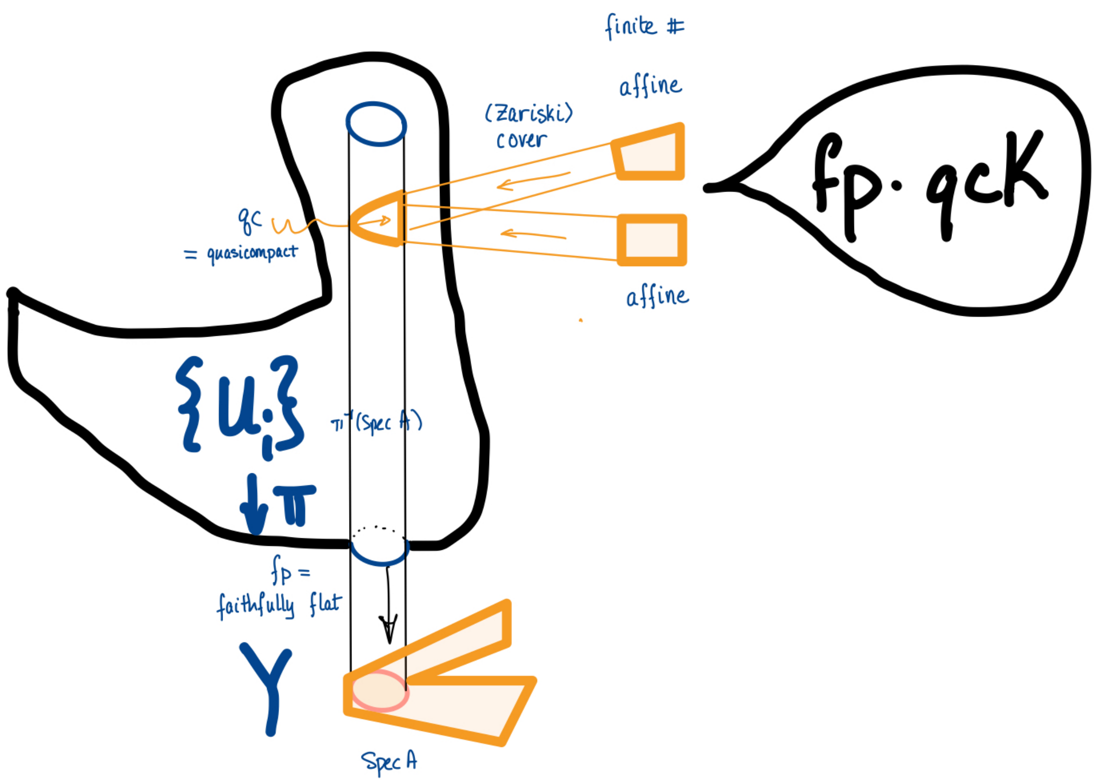


FIGURE 1. An fpqcK cover

30.6. **One topology to rule them all (Figure 1).** We now come to a topology that is great because it is finer than the ones we care about, and easy to work with (we'll see that our arguments are short and sweet). The only downside is the definition seems unmotivated. But it is very well motivated to you now, when you see how well the arguments work and how it easily generalizes the ones we care about. This is a variant of fp+qc due, I think, to Steve Kleiman. So I will call it the **fpqc(K)** or **fpqcK** topology, which I will pronounce "fp-quack" topology, just because it pleases me. No one else will call this topology this.

(People often talk about the "flat" topology. I've neer bothreed checking what this means. Probably that refers to this, but I don't know if this is completely standard. Maybe one of you will tell me!)

30.7. **The fp-quack topology.** Covers of X are collections of flat maps $\pi : \{U\} \to X$ such that for any affine open subset $\operatorname{Spec} A \subset X$, there is a quasicompact open subset $\{V_i\} \hookrightarrow \{U_i\} \to X$ that is surjective (and of course, flat) onto $\operatorname{Spec} A$.

30.A. EXERCISE. Check that this indeed gives a topology on the category of schemes, and (hence) on the category of affine schemes.

Let's unpack this definition a bit. Every fpqcK cover is certainly flat, surjective (hence faithfully flat).

30.8. *Easy fact. Every cover in the smooth topology is also an fpqcK cover.*

This will use an easy fact: if $\pi : Y \to X$ is a smooth morphism, and $U \subset Y$ is an open subset, then $\pi(U)$ is an open subset of X. Translation: smooth morphism are "open maps" (in the topological sense). This is clear in the analytic setting (with a little bit of experience with manifolds, I guess). In the algebraic setting, a more general fact is true.

30.9. **Fact. —** *If $\pi : Y \to X$ is flat and locally finitely presented, and $U \subset Y$ is an open subset, then $\pi(U)$ is an open subset of X.*

That's an algebra fact. I should give a reference. (Beginning of proof: reduce to the case where Y and X are both affine, so then π is finitely presented. Then (main hard fact used, I guess) show that it is pulled back from the Noetherian situation. Then prove it here.)

So we may as well prove a better result.

30.10. **Theorem. —** *Every fp+lpf cover is fpqcK.*

Proof. Suppose $\{U_i\} \to X$ is fp+lpf. We need to check this over an arbitrary open $\operatorname{Spec} A \subset X$, so we restrict there. We have thus reduced to the case $\{V_j\} \to \operatorname{Spec} A$. It suffices to deal with the case $V := \coprod V_j \to \operatorname{Spec} A$. We build our desired quasicompact open subset as follows. For each $p \in \operatorname{Spec} A$, we do the following. The point p is in the image of V, so pick a preimage q of p, and take an affine neighborhood V_p of q, in V. The image of V_p, call it U_p, is open in $\operatorname{Spec} A$ (using our fact), so $\operatorname{Spec} A = \cup U_p$. But $\operatorname{Spec} A$ is quasicompact, so we can find a finite set S of p so that $\operatorname{Spec} A = \cup_{p \in S} U_p$. Then our quasicompact open in V is $\cup_{p \in S} V_p$. $\square$

Also, clearly fp+qc covers are fpqcK.

"fp+pf", which makes you think that the "locally" isn't in the definition. (Spencer gave a possible reason why, which I won't repeat here.) Good news about it: it generalizes (is "finer than") the Zariski, étale, and smooth topologies. Bad news: $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ isn't such a cover. Also bad news: it hides what makes gluing work. So we'll mostly ignore it, but now you know what Ben means when he says a morphism is fppf (although I'm personally not sure if he means fp+pf or fp+lpf).

How I remember things: all are faithfully flat (i.e. flat and surjective). We need another finiteness property. Locally finitely presented, or qc opens are images of qc opens.

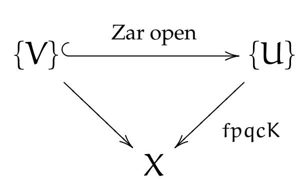
30.11. *Interesting example of a fpqcK cover.* $\operatorname{Spec} \hat{A} \to \operatorname{Spec} A$ when $A \hookrightarrow \hat{A}$ maps Noetherian local ring to its completion, or a field to its algebraic or separable closure. Then $\hat{A} \times_A \hat{A}$ is not Noetherian. When doing descent, we need to make sure algebraic geometry works over not-necessarily (locally) Noetherian schemes.

(Example I might add later: Also, if Y is Noetherian, and G finite, Y/G need not be. But we need Y to not be finite type over a field for this work.)

30.12. **How we will use the fpqcK topology.**

The reason the fpqcK topology will be the right definition is actually quite simple.

30.B. Easy Exercise. Suppose X is affine, and $\{U\} \to X$ is an fpqcK cover. Explain how to find a finite number of affine schemes $\{V\}$ such that we have a commutative diagram



where $\{V\} \to X$ is surjective and flat (i.e., faithfully flat).

Even better: if $V := \{V\}$, we have a map of affine schemes $V \to X$ that is faithfully flat. We have basically (or close enough) "refined" the fpqK cover over each affine so that it is actually just a single (faithfully flat) map of one ring to another!

31. Lots of things glue ("descend") in the fpqcK topology

Lots of things glue in the fpqcK topology. Of course anything gluing under fpqcK covers automatically glues for any smooth covers (because smooth covers *are* fpqcK), which is what we want.

There is a really big list of things which will glue in the fpqcK topology, but I'd like to do things that are easy that we'll use.

So we did the following.

31.1. **Proposition. —** *Sections of quasicoherent sheaves glue in the fpqcK topology. Translation: if* $\pi : \{U\} \to X$ *is fpqcK, and* $\mathcal{F}$ *is a quasicoherent sheaf on* X*, and* $s \in \mathcal{F}(\{U\}$ *"agrees with itself on the double overlaps", then* $s = \pi^* t$ *for a unique* $t \in \mathcal{F}(X)$*.*

31.A. Exercise. State this proposition more precisely

Proof. Here is the outline of how to break it up into steps, each individually easy.

Step 1: do it in the case when X is affine, and $\{U\}$ is a single affine. That is just Amritsur's Lemma 29.1.

Step 2: do it when X is affine, and $\{U\}$ is a finite number of affines.

Step 3. do it when X is affine.

Step 4. Do it in general. □

31.2. **Corollary. —** *Functions (i.e., sections of* $\mathcal{O}_X$*) glue in the fpqcK topology.*

Reason: $\mathcal{O}_X$ is a quasicoherent sheaf.

31.3. **Corollary. —** *Maps to* $\mathbb{A}^1$ *glue in the fpqcK topology.*

Reason: these are the same as functions.

More generally:

31.4. **Corollary. —** *For any ring* R*, maps to* $\operatorname{Spec} R$ *glue in the fpqcK topology.*

Reason: maps $X \to \operatorname{Spec} R$ are in precise bijection wit maps $R \to \Gamma(X, \mathcal{O}_X)$, i.e., maps from R to the ring of functions. In other words, this corresponds to a choice of a bunch of functions (one for each element of R), subject to a bunch of conditions.

31.5. **Theorem. —** *For any scheme Z, maps to Z glue in the fpqcK topology.*

Sketch of the proof I gave: we are given an fpqcK cover $\{U\} \to X$, and maps $\{U\} \to Z$ agreeing on the pairwise overlaps. We want to see that this comes from a unique map $X \to Z$.

Now choose an affine (Zariski-)open cover of Z, say $\{Z_i\}$. Let $\{U_i\} = Z_i \times_Z \{U\}$.

Show how to get X_i which is "supposed to" map to Z_i; then use fpqcK gluing to get the map $X_i \to Z_i$; then explain why these glue together to give a map $X \to Z$ (basically this is gluing in the Zariski topology). □

31.6. **More gluing.**

31.B. EXERCISE. Show that maps of quasicoherent sheaves glue in the fpqcK topology. Hint: Proposition 31.1 applied again.

31.7. **Theorem. —** *Quasicoherent sheaves glue in the fpqcK topology.*

Translation: if $\pi : \{U\} \to X$ is a fpqcK cover, and $\mathcal{F}$ is a quasicoherent sheaf on $\{U\}$, and we have gluing information (i.e., isomorphisms $\phi_{12} : \pi_1^* \mathcal{F} \xrightarrow{\sim} \pi_2^* \mathcal{F}$ on $\{U\} \times_X \{U\}$, that agrees on the "triple overlaps", then there is a quasicoherent sheaf $\mathcal{G}$ on X, along with an isomorphism $\pi^* \mathcal{G} \xrightarrow{\sim} \mathcal{F}$ on $\{U\}$, inducing the isomorphism(s) ϕ_{12}. Furthermore, $\mathcal{G}$ is unique up to unique isomorphism.

31.C. EXERCISE. Prove this. (Hint: it is just Amritsur's Lemma gussied up. The "furthermore" is basically Exercise 31.B.)

31.D. EXERCISE. Fill in the blanks to give the complete precise statement that "affine morphisms glue in the fpqcK topology": Suppose $\pi : \{U\} \to X$ is an fpqcK cover, and $\{Y\} \to \{U\}$ is an affine morphism. Suppose you have further an isomorphism $\psi_{12} : \pi_1^* \{Y\} \xrightarrow{\sim} \pi_2^* \{Y\}$ on [blank] where [blank]. Show that there is an affine morphism $Z \to X$ along with an isomorphism $\psi : \pi^* Z \xrightarrow{\sim} \{Y\}$ over $\{U\}$ such that [blank].

31.E. EXERCISE. Prove that affine morphisms glue in the fpqcK topology.

31.F. EXERCISE. Show that closed embeddings glue in the fpqcK topology.

31.G. EXERCISE. Projective morphisms with the right definition (those proper $\pi : Y \to X$ with a line bundle $\mathcal{L}$ on Y, with the natural isomorphism $Y \cong \underline{\mathbf{Proj}}\ \oplus_n \pi_*(\mathcal{L}^{\otimes n})$) glue in the fpqcK topology.

31.H. EXERCISE. The condition of a morphism of schemes being affine glues in the fpqcK topology. Translation: suppose $\rho : Y \to Z$ is a morphism of X-schemes. To check if ρ is affine, it suffices to check on an fpqcK cover of X.

31.I. EXERCISE. The condition of a morphism being an open embedding glues in the fpqcK topology.

31.8. **Theorem. —** *Quasiaffine morphisms glue in the fpqcK topology.*

(Remider: a **quasiaffine morphism** is a morphism $\pi : Y \to X$ that is quasicompact, and admits an open embedding into $Z \to X$ that is affine over X. Equivalently (not immediate, but not hard) a morphism $\pi : Y \to X$ is quasiaffine if it is quasicompact and quasiseparated, and $\pi : Y \to \underline{\mathbf{Spec}}\ \pi_* \mathcal{O}_Y$ is an open embedding. This latter is the one you need.)

I want to make this an exercise. It isn't an exercise yet, because I've not yet shown that qcqs morphisms glue in the fpqK topology.

Part 5. **Algebraic spaces**

32. DEFINING ALGEBRAIC SPACES

Thus we now finally have

$$\mathcal{G} \subset \underline{\mathbf{Sh}}^{\mathrm{fpqcK}} \subset \underline{\mathbf{Sh}}^{\mathrm{sm}} = \underline{\mathbf{Sh}}^{\mathrm{et}} \subset \underline{\mathbf{Sh}}^{\mathrm{Zar}} \subset \underline{\mathbf{PSh}}_{\mathcal{G}}$$

where we can take $\mathcal{G} = \underline{\mathbf{Aff}}$ or $\mathcal{G} = \underline{\mathbf{Sch}}$. We're not going to use $\underline{\mathbf{Sh}}^{\mathrm{fpqcK}}$ any more.

To avoid saying which $\mathcal{G}$ I am using, let me write this as:

$$\underline{\mathbf{Aff}} \subset \underline{\mathbf{Sch}} \subset \underline{\mathbf{Sh}}^{\mathrm{sm=et}} \subset \underline{\mathbf{Sh}}^{\mathrm{Zar}}.$$

Notice that the categories of sheaves are the same whether we are thinking about affine schemes or schemes. (So I will deliberately be vague about this in the future.) You might imagine that we want some sort of name for this notion; but I don't want to distract you with more names.

But I should now be careful about what I mean about "representable morphisms" as I am being coy about what $\mathcal{G}$ is. (Probably I'll pick $\mathcal{G}$ to be $\underline{\mathbf{Sch}}$ for now, if I had to choose.)

Morphisms in $\underline{\mathbf{Sh}}^{Zar}$ can be "affine" (representable if we had taken $\mathcal{G} = \underline{\mathbf{Aff}}$), or **schematic** (representable if we had taken $\mathcal{G} = \underline{\mathbf{Sch}}$).

To be concrete, and to use terminology in a conventional way, let me define a morphism to be **strongly representable** iff it is *schematic*.

Now we have locally representables (where $\mathcal{G} = \underline{\mathbf{Sch}}$). (Would you prefer locally schematic?)

Things that are locally schematic in the Zariski topology are just schemes — we get nothing new. But there a priori might be new things that are locally representable in the étale topology! (And these will be just the same things that are locally representable in the smooth topology.

32.1. *Definition.* The category of **algebraic spaces**, which we might call $\boxed{\underline{\mathbf{AlgSp}}}$ is the subcategory of $\underline{\mathbf{Sh}}^{et=sm}$ that are locally representable by schemes in the étale (or equivalently, smooth) topology. In other words, we have a (schematic) étale (or even smooth) cover of any algebraic space M from schemes $\{U\} \to M$. I may as well replace $\{U\}$ by their disjoint union: $U \to M$ is smooth (even etale), and schematic. (We actually don't yet know what "smooth" or "etale" means if it weren't schematic of course...)

We can similarly define $\boxed{\underline{\mathbf{AlgSp}}_k}$ of algebraic spaces over a field k (or indeed over any scheme or indeed algebraic space S).

32.2. **Important question.** You should be asking at this point: why should we care about algebraic spaces? You might ask: is there some algebraic space that I want to tell you about right now that is clearly not a scheme? (My answer: no!)

Let me explain why you should care, using a metaphor. We will be able to geometry with algebraic spaces almost as easily as we can with schemes, so there is very little cost. And there will be a number of things we can construct that will be cheaply algebraic spaces, that are not obviously schemes. I'd like to argue that as long as we can work happily with algebraic spaces, we shouldn't care much if they are schemes.

For comparison: you can define compact complex algebraic variety. Basically any you can think of will be projective. And projective varieties are psychologically easier to deal with. But if nature hands you a compact (proper) complex algebraic variety, you shouldn't lose too much sleep trying to prove that it is projective (although you may later have reasons to).

I gave another example about real manifolds, and how humanity used to think of them as sitting in $\mathbb{R}^n$, but there is really very little advantage in sticking them in $\mathbb{R}^n$, other than psychological, so you should give that up.

We will see soon (Proposition 39.4) that algebraic spaces have all (finite) fibered products. [Actually, we this is now basically Exercise 20.B.]

33. PROPERTIES OF ALGEBRAIC SPACES, MORPHISMS THEREOF, AND QUASICOHERENT SHEAVES THEREON

33.1. **Properties of objects.** Any characteristic of schemes that is étale-local in nature clearly extends to algebraic spaces. (Question for you: why didn't I say "smooth-local"?) In this way, we can define **dimension**, **smooth over** k, **normal**.

33.A. EXERCISE. Explain how why **reducedness** makes sense in the same way. (Translation: explain why you can check reducedness étale-locally.)

The meaning of "locally" is different than for schemes, as locally means in our new topology (the étale topology), not the Zariski topology. So now we can define **locally of finite type**, **locally of finite presentation**, and **locally Noetherian**.

33.B. EXERCISE. Define **quasicompact** in your own way. Show that this is the same as showing that it has an étale cover by finitely many affine schemes.

Then with this definition of *quasicompact*, we can define **Noetherian**, **finite type**, **finite presentation**.

33.2. **Properties of morphisms of algebraic spaces.**

Any property of morphisms of schemes immediately extends to schematic morphisms of algebraic spaces. But we can also extend definitions further, because we have extended notions for objects.

33.C. EXERCISE. Define what it means for a morphism $\pi : X \to Y$ of algebraic spaces to be **quasicompact**, **finite type**, **étale**, **smooth of relative dimension** n. Show all reasonable versions of these definitions are the same.

33.3. **Separatedness.** It looks like there might be a subtlety here, but there won't be. We want the definition to be "the diagonal morphism is a closed embedding", or possibly something about the diagonal morphism being closed. Perhaps we're unsure because we haven't required the diagonal to be representable. I'll just take as a definition that the diagonal morphism is a (representable) closed embedding. Because the diagonal turns out to be representable (see Theorem 34.3) this will never come back to bite us.

33.4. **Question with an embedded lesson.** Are algebraic spaces the "locally representables" in (a) $\underline{\mathbf{Sh}}^{et}_{\underline{\mathbf{Aff}}}$, (b) $\underline{\mathbf{Sh}}^{et}_{\underline{\mathbf{Sch}}}$, (c) $\underline{\mathbf{Sh}}^{sm}_{\underline{\mathbf{Aff}}}$, or (d) $\underline{\mathbf{Sh}}^{sm}_{\underline{\mathbf{Sch}}}$? Answer: it doesn't matter! These are all the same (really, equivalent) categories. (This may make you want to have a new word — topos. But we won't need it.)

33.5. *Extending our notion of algebraic spaces.*

added May 14

We have now extended our notion of étale morphism — we have étale morphisms of algebraic spaces that are not schematic. This suggests a possible way of getting a more general notion than algebraic spaces: (i) a sheaf (in the category of schemes, with the étale topology), which admits an "étale algebraic-spatial cover". (ii) a sheaf in the category of algebraic spaces, with the étale topology, which admits and "étale algebraic-spatial cover".

We get nothing new in either case, but I have to think through whether we need something hard for this.

(This discussion will be continued at §33.11.)

33.6. **The analytification functor.** In the case of algebraic spaces locally of finite type over $\mathbb{C}$, I explained the analytification functor to complex analytic schemes. This restricts to sending reduced complex algebraic spaces to complex analytic varieties; and smooth complex algebraic spaces to complex manifolds.

The idea was this: we have a functor from the category of local models of one to local models of the other: given a scheme of the form $\operatorname{Spec}\mathbb{C}[x_1,\dots,x_n]/I$, we have closed analytic subscheme of $\mathbb{C}^n$ cut out by the holomorphic equations in I. (At the level of topological spaces, this is even a continuous map!) So this functor extends to functors of locally representables. Perhaps I'll type more in here.

33.7. **Points of algebraic spaces.** Algebraic spaces have points just as schemes do. What is a **point of an algebraic space** X? It is a map $\operatorname{Spec} k \to X$, except that we consider two points $p \to X$ and $q \to X$ the same if $p \times_X q$ is nonempty.

33.D. EXERCISE. Show that this gives you the usual definition of points on a scheme, in the case where X is a scheme.

A **geometric point** of an algebraic space X has the same definition as uusal: it is a morphism $\operatorname{Spec} K \to X$ where K is an algebraically closed field.

33.8. **Quasicoherent sheaves on an algebraic space.**

A **quasicoherent sheaf** on an algebraic space X is defined in any number of equivalent ways. One easy one: it is an $\mathcal{O}_X$-module, such that when pulled back to any scheme Y, it is a quasicoherent sheaf on Y.

Or it is the data of, for every $\operatorname{Spec} A \to X$, of an A-module M_A that is "functorial": given $\operatorname{Spec} B \to \operatorname{Spec} A \to X$, we have an isomorphism $M_A \otimes_B B \xrightarrow{\sim} M_B$ such that for any $\operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A \to X$, ...

33.E. EXERCISE. Finish that sentence. (Hint: it is better to make the statement more precise, by giving those maps some names.)

Then the definition of pullback of quasicoherent sheaves, and pushforward under quasicompact separated morphisms, applies without change.

We can make sense of Cech cohomology of a quasicoherent sheaf on an algebraic space X in the case where X is quasicompact with affine diagonal, just as usual.

33.F. EXERCISE. Show that the cohomology is independent of the topology (Zariski, smooth, étale) you take in the cover (assuming uch a cover exists).

33.G. EXERCISE. Define **finite type quasicoherent sheaf**, **finitely presented quasicoherent sheaf**, and **coherent sheaf**. (Hint: same as for schemes.)

33.9. **Derived functor cohomology.** Derived functor cohomology works as before, once you know you have enough injectives. A little caution is needed — the usual cheap way of producing enough injectives involves using stalks at points, and we haven't discussed stalks of étale sheaves at points (for good reason).

33.10. **Vector bundles.** It is not yet clear that the notion of when a coherent sheaf is locally free is independent of topology. More explicitly: if $\mathcal{F}$ is a coherent sheaf on a scheme X, and there is an étale cover $\pi : Y \to X$ for which $\pi^*\mathcal{F}$ is a free sheaf of rank n, is there a *Zariski cover* $\psi : Y' \to X$ for which $\psi^*\mathcal{F}$ is free of rank n? The answer is "yes", but this is not obvious. This is often called *Hilbert 90*, because it is in some sense a generalization of Hilbert's original Hilbert 90 theorem.

33.11. **In the definition of algebraic space, a smooth cover suffices.**

Added May 14

(This is in some sense an extension of the discussion at §33.5.)

We proved (some time while talking about stacks, before May 13, but after we were talking about algebraic spaces) that a sheaf in the étale or smooth topology that has a representable *smooth* cover by a scheme is an algebraic space.

add link later

This conceptually is nice: you could replace étale by smooth in the definition of algebraic space and you would get the same thing. (An algebraic space is a sheaf in the smooth topology, that has a schematic smooth cover by a scheme.)

I record here a useful result from the Stacks Project, that Ben told us about on May 14: tag 04S6. I'm going to state a simpler version, whose proof should be a "subset" of that of the stacks project.

33.12. **Expected theorem (based on Stacks Project tag 04S6). —** *Suppose F is a sheaf in the category of schemes (perhaps over a base scheme S), with the smooth topology. Suppose there exists a map of sheaves $\pi : X \to F$, where X is an algebraic space, and π is a "relative algebraic space" (it is "algebraic spatial"), and smooth and surjective. Then F is an algebraic space.*

The Stacks Project has a better result, where "smooth" is replaced by "fp(l)pf" throughout. More precisely: in the *definition* of algebraic space, he works in the fp(l)pf topology. And he shows that you get the same notion whether you work in the fp(l)pf topology, or the étale topology. We've mostly shown that you get the same notion whether you work in the smooth topology or the étale topology.

Update (May 14): this is a consequence of Theorem 33.12.

34. FANCY FACTS ABOUT ALGEBRAIC SPACES WE WON'T REALLY USE

34.1. **An algebraic space that is not a scheme.** Ben Church says: "This gives an example of a genus 1 relative curve over $\mathbb{A}^1$ that is not a scheme:

`https://stacks.math.columbia.edu/tag/0D5D`

However, this should not happen if it is a torsor for a relative elliptic curve."

This example will be relevant when we discuss how the moduli stack of genus 1 curves is algebraic — we need to say what the stack actually *is*!

I am hoping Ben will explain this to the class at some point, without simply invoking the magical name of Raynaud.

34.2. **The diagonal map of algebraic spaces is actually schematic.** If X is an algebraic space, we have shown that its diagonal map $X \to X \times X$ is "locally representable by schemes", i.e. "algebraic spatial". (All morphisms of algebraic spaces are algebraic spacial!)

But in fact you can show that it is schematic. Put differently, if $Y \to X$ and $Z \to X$ are both morphisms to X from schemes, then $Y \times_X Z$ is always a scheme. (This equivalence was Lemma 20.1.)

This is not easy, and so I will try not to use it! But I'll state it formally here, and give the start of the proof, as it is similar to arguments we've just seen. (For a full proof, see any of the three standard references: the Stacks Project (tag 0265), or [Olsson], or [Alper].)

34.3. **Accidental Theorem. —** *If X is an algebraic space (in the sense defined here), then the diagonal morphism is schematic.*

I am calling this "accidental" because it is accidental to our setting of schemes and the etale topology, and isn't a formal consequence of our robust set up of "A [blah] is a locally representable [blah] in the [blah] topology over our geometric category $\mathcal{G}$ = [blah]."

The argument starts as follows. We use the diagrams in the proof of Proposition 39.4. We know that Z, Z_1, and Z_2 are schemes, and we want to show that Z_3 is a scheme (not just an algebraic space). Notice that $Z_2 \to Z_1$ is the base change of $R \to U \times U$.

34.A. EXERCISE. Show that $\rho : R \to U \times U$ is a monomorphism. (Hint: all diagonals are, for sheaves.) Equivalently, the diagonal δ_ρ of ρ is an isomorphism.

This means that ρ is separated (which means that δ_ρ is a closed embedding — but of course it is an isomorphism), and the preimage of any point is either a point or the empty set, so it has finite fibers. Also, $R \to U$ is the base change of $U \to M$, which is etale or smooth, hence locally finite type, so $R \to U \times U$ is locally of finite type.

Thus $\rho : Z_2 \to Z_1$ is *separated, locally finite type, with finite fibers*. It turns out that this is glues (in the fpqcK topology), so $Z_3 \to Z$ is separated, locally finitely type, with finite fibers, and schematic. That's the hard fact!

It's not overwhelmingly hard. (Method of proof: reduce to the case where ρ is quasicompact. Then ρ is finite type, separated, with finite fibers, and is thus quasiaffine by Zariski's Main Theorem. Then affine things glue, and open subsets glue, so quasiaffine things glue.) But it is way more complicated given how little we care. (A reference: [Olsson, 5.2.5]; I think it would also be in Knutson Sr., Raynaud-Gruson.)

Another reason you shouldn't care: you're already working in the étale or smooth topology — why are you particularly hung up with the Zariski topology?

34.4. **Some low-dimensional algebraic spaces are automatically schemes.** If X is a quasiseparated algebraic space, and we have an epimorphism $\operatorname{Spec} K \to X$, then X is also the spectrum of a field. (See [Olsson].)

One-dimensional finite-type algebraic spaces over a field are schemes.

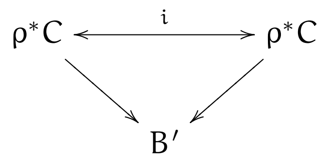
I think proper smooth surface algebraic spaces are schemes.

34.5. **Gluing (descent) of algebraic spaces.** Rather obviously, algebraic spaces glue in the smooth topology. I think it is true that they glue in the fppf topology, although there may be a quasiseparated hypothesis. We won't need this, so we won't care.

35. Example: moduli of asymmetric curves

Consider the following presheaf on $\mathcal{G} = \underline{\mathbf{Sch}}$, which I'll call $\mathcal{M}_g^a$ — the presheaf (soon to be moduli space) of asymmetric genus g curves.

Over $B \in \underline{\mathbf{Sch}}$, it is a family $C \to B$ of genus g curves (a proper family, smooth of relative genus g, whose geometric fibers are smooth irreducible genus g curves), such that under any pullback $\rho : B' \to B$, $\rho^* C \to B'$ has no nontrivial automorphisms. In other words, the only isomorphism i making the diagram



commute is the identity.

Fact: this is the same as requiring the geometric fibers (the case where B' is the spec of an algebraically closed field) to have no nontrivial automorphisms. Because this example is intended to be motivational, I'm not going to prove this; please take it as a black box.

Now $\mathcal{M}_g^a \in \underline{\mathbf{PSh}}_{\mathcal{G}}$. But it is not just a presheaf —it is a sheaf in the fpqcK topology.

To see why this is true, I need to make some quick comments on gluing (descent).

A quasicoherent sheaf is not just a presheaf in the etale topology — it is a sheaf, thanks to our earlier discussion.

That means an affine scheme is not just a presheaf in the etale topology — it is a sheaf. (We'd discussed this before.) We could describe this as "affine morphisms glue/descend under fpqcK covers". This is because they are just (relative) spec of a quasicoherent sheaf of algebras (a quasicoherent sheaf with multiplication and addition satisfying some properties).

In just the same way, projective schemes also glue/descend, once I say what I mean. By projective scheme over another scheme, I mean the (relative) proj of a quasicoherent sheaf of graded algebras. So the kind of geometric objects that glue/descend under this criterion are morphisms $X \to Y$, along with a line bundle $\mathcal{L}$ on X that is "relatively ample" over Y. (If you wish, for any affine $\operatorname{Spec} A \to Y$, under the pullback of the situation to $\operatorname{Spec} A$, $\mathcal{L}$ is ample.)

Now no smooth curve of genus 0 or 1 (or even 2) is asymmetric, so $g > 2$ in our situation. And in this situation, we have an $\mathcal{L}$ which is relatively ample — it is the relative canonical line bundle $\Omega_{C/B}$. (In fact, for asymmetric curves, it is relatively *very ample*.)

35.1. **Theorem. —** *$\mathcal{M}_g^a$ is an algebraic space.*

Proof. Becasue we've already proved that it is a sheaf, we need only prove that it is locally representable.

Now our functor/presheaf $\mathcal{M}_g^a$ sent B to families $C \to B$ of asymmetric curves over B (of genus g etc.).

Using an argument involving the Cohomology and Base Change Theorem, we showed that this functor was the same as that sending B to

$$\begin{array}{ccc} C & \xrightarrow{\quad\mathcal{K}\quad} & \mathcal{P} \\ & {\scriptstyle \mathbb{P}^{g-1}\text{-bundle}}\searrow \qquad \swarrow & \\ & B & \end{array}$$

where $\mathcal{K}$ is a canonical bundle, which in turn is the same as

$$\begin{array}{ccc} C & \xrightarrow{\text{nondegenerate}} & \mathcal{P} \\ & {\scriptstyle \mathbb{P}^{g-1}\text{-bundle}}\searrow \qquad \swarrow & \\ & B & \end{array}$$

Instead we consider a different functor $U \in \underline{\mathbf{PSh}}_{\mathcal{G}}$ which sends B to

$$\begin{array}{ccc} C & \xrightarrow{\text{nondegenerate}} & \mathbb{P}^{g-1} \\ & \searrow \qquad \swarrow & \\ & B & \end{array}$$

where $C \to B$ is as usual a family of asymmetric genus g curves.

Then we argued that (i) $U \to \mathcal{M}_g^a$ is a $\mathrm{PGL}(g)$-bundle (hence representable and smooth). (We did this by considering $\mathrm{Isom}(\mathcal{P}, \mathbb{P}^{g-1})$.)

(ii) U is a scheme, because it is an open subscheme of a Hilbert scheme. (We invoked results form last quarter as a black box. We start with the existence of the Hilbert scheme; we took the open subset where the fiber was smooth of dimension 1; then the open subest wehre it has connected fibers; then the open subset where $\mathcal{O}(1) \cong \mathcal{K}$ on fibers; then the open subset wehre it is not contained in a hyperplane)

□

As a bonus fact, we get the following.

35.2. **Theorem. —** *The diagonal morphism* $\delta : \mathcal{M}_g^a \to \mathcal{M}_g^a \times \mathcal{M}_g^a$ *is schematic.*

Proof. For convenience, I'll type $\mathcal{M} := \mathcal{M}_g^a$.

By our earlier discussion [I'll add the link later], given two maps $X \to \mathcal{M}$ and $Y \to \mathcal{M}$ from schemes (corresponding to families of asymmetric curves $C_X \to X$ and $C_Y \to Y$, say), then

$$\begin{array}{ccc} X \times_{\mathcal{M}} Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\delta} & \mathcal{M} \times \mathcal{M} \end{array}$$

.

But $X \times_{\mathcal{M}} Y = \mathrm{Isom}_{X \times Y}(\pi_X^* C_X, \pi_Y^* C_Y)$, which is a scheme by our work last quarter. □

35.A. EXERCISE (FOR THOSE COMFORTABLE WITH COHOMOLOGY AND BASE CHANGE). Describe/define the "Hodge bundle" on $\mathcal{M}_g^a$, showing that it is a rank g vector bundle. (I'll put some definition here soon.)

35.3. **Reasonable question.** Is $\mathcal{M}_g^a$ a scheme? In fact it is, but this is not clear. You can prove this using geometric invariant theory. But do you have a good reason to care? (You might, but you might not...)

Part 6. **Stacks (2-sheaves), especially locally representable stacks**

36. STACKS ON A TOPOLOGICAL SPACE

Moved here from near the start, because it logically fits

The notion of a "stack" by itself is generalization of a sheaf. In some sense, it is a **2-sheaf** (and a sheaf is a **1-sheaf**). (I'll try to have new definitions in **bold**, or boxed.)

36.1. **Stacks: a first stab.**

I'll now give a first definition of a stack on a topological space X. It will generalize the notion of a sheaf. As usual, I'll make up my own terminology and notation to try to keep things simpler.

We begin with the example that I'll call the stack of rank 2 vector bundles on (open subsets of) X. I'm assuming that you have some notion in your head of what a vector bundle is (and I don't mnind that your notion depends on what kind of "geometric space" you like to work with).

For each open subset $U \subset X$ consider the vector bundles on U. Notice: I'm not saying "vector bundles up to isomorphism". I mean "all the vector bundles". What do I mean? Well, what would *you* mean by this?

Now if $V \subset U$ is an open subset, then we can restrict any vector bundle on U to get a vector bundle on V. (What do I mean by "restrict a vector bundle form U to V"?)

Moreover, if $W \subset V$ is a smaller open subset, then if we restrict a vector bundle from U to V, and then from V to W, that's the "same thing" as restricting it from U to W all at once. (What do I mean by "same thing"?)

(So far I've described something which might be called a "2-presheaf" or **2-PSh** on X (or $\underline{\mathbf{Top}}_X$).)

So far this sounds a lot like a presheaf. But it's not going to be a sheaf. It fails the identity axiom: any vector bundle is locally trivial: given a rank 2 vector bundle $\mathcal{V}$ on U (trivial or not), there is an open cover of U, say $U = \cup U_i$, where $\mathcal{V}_{U_i}$ is a trivial bundle. But the trivial rank 2 bundle on U (that I'd denote $\mathcal{O}_U^{\oplus 2}$) also has this property, so we can have two *different* (non-isomorphic) vector bundles that are nontheless isomorphic on every open set in an open cover.

Nonetheless, we can still say something. We know how to glue vector bundles on an open cover to get a vector bundle on the union; it is just a bit more subtler than gluing continuous functions. If $U = \cup U_i$ is an open cover of U, and we have rank 2 vector bundles $\mathcal{V}_i$ on U_i, and we have specific isomorphisms $\phi_{ij} : \mathcal{V}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{V}_j|_{U_i \cap U_j}$ (with certainly $\phi_{ij} = \phi_{ji}^{-1}$) satisfying, on triple overlaps, $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$, then we can get a unique (up to unique isomorphism) vector bundle $\mathcal{V}$ on U whose restrictions to the U_i "are" the $\mathcal{V}_i$. (More precisely, we have isomorphisms $\psi_i : \mathcal{V}|_{U_i} \xrightarrow{\sim} \mathcal{V}_i$.)

So this will later give us an axiom for a stack (gluing of objects) which is akin to the gluability axiom for sheaves. There will be another axiom that will be an analogue of the identity axiom, which will ensure that the thing we get by gluing is unique (up to unique isomorphism). This is the statement that for any two rank two vector bundles $\mathcal{V}$ and $\mathcal{V}'$ on an open set $U \subset X$, the presheaf $\mathcal{I}som(\mathcal{V}, \mathcal{V}')$ on U is actually a sheaf.

Let me make this more precise. What is this presheaf? To each open set $V \subset U$, the sectons of $\mathrm{Isom}(\mathcal{V}, \mathcal{V}')$ over V are the set of isomorphisms $\mathcal{V}|_V \xrightarrow{\sim} \mathcal{V}'_V$. (Can you see why there are natural restriction maps? Can you see why this gives a presheaf? Can you see, in the case of vector bundles, why this indeed forms a sheaf?)

36.2. **Another example that looks similar: G-bundles.** Let G be a group. Recall that a G bundle on a topological space X is a map $Y \to X$ with a G-action over X, such that there is an open cover $X = \cup U_i$, and over U_i, $Y|_{U_i} \cong G \times U_i$.

Now I will describe the stack of G-bundles on a topological space X. To each open subset $U \subset X$, I associate the G-bundles on u. You can see how to restrict. You can see that to glue G-bundles on U_i to get a G-bundle on $U = \cup U_i$, you want to be given isomorphisms on pairwise overlaps, satisfying a triple overlap condition.

And you can think through why given two G-bundles Y and Y' on U, $\mathrm{Isom}_{Y,Y}$ is a sheaf on U.

The stack we have described is $\boxed{BG}$, and is called the **classifying stack** of G. We will meet it soon.

36.3. **Another example: families of Riemann surfaces of genus 3.** Same thing, except that we have families of genus 3 Riemann surfaces over X. (Perhaps in your case, X is a complex manifold. Or an algebraic variety.) You should think through this example as well.

36.4. **Abstracting the definition of stack (and "stack in groupoids") from these.**

We start with the analogue of presheaf.

We are going to build a covariant functor from some category $\mathcal{A}$ to $\underline{\mathbf{Top}}_X$. We picture the information in each of these cases as follows. The functor will be "vertical"; $\underline{\mathbf{Top}}_X$ is on the bottom. A horizontal line separates $\mathcal{A}$ from $\underline{\mathbf{Top}}_X$.

The open subset $\varnothing \in \underline{\mathbf{Top}}_X$ is on the bottom left, and the open subset $X \in \underline{\mathbf{Top}}_X$ is on the bottom right, and all the arrows in $\underline{\mathbf{Top}}_X$ are pointing roughly rightward.

Above each object of $U \in \underline{\mathbf{Top}}_X$ (each open subset $U \subset X$), we have our objects we are parametrizing.

For the arrows upstairs, we put in "pullbacks". Spencer may want to add in other morphisms too.

36.5. *Definition.* Given a base category $\mathcal{G}$, a **category fibered** over $\mathcal{G}$ is a covariant functor $\phi : \mathcal{F} \to \mathcal{G}$, such that "pullbacks exist": given $\alpha : X \to Y$ in $\mathcal{G}$, and $b \in \mathcal{F}$ with $\phi(b) = Y$, then there exists $\alpha' : a \to b$ in $\mathcal{F}$, with $\phi(\alpha') = \alpha$ (so in particular $\phi(a) = X$), satisfying the following universal property: for any other $c \in \mathcal{F}$, along with $\beta : c \to b$, and $\gamma : \alpha(c) \to Y$, there *exists* a *unique* $\delta : c \to a$ making the diagram commute (see figure, which I should make).

If *every* morphism in $\mathcal{F}$ is a pullback, we say that $\mathcal{F}$ is a **2-presheaf** on $\mathcal{G}$ (or a *category fibered in groupoids*).

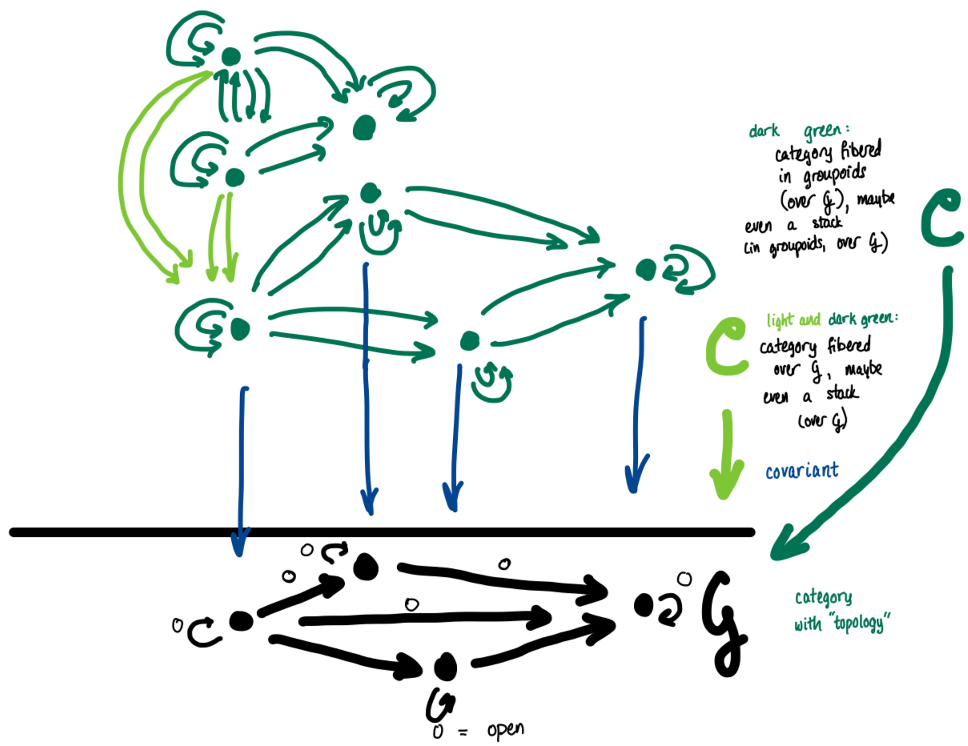


FIGURE 2. A stack (or category fibered) (perhaps in groupoids) over $\mathcal{G}$

36.A. EXERCISE. If $a : \mathcal{F} \to \mathcal{G}$ is a 2-presheaf, then for any $X \in \mathcal{G}$, the **fiber category** above X, denoted $\boxed{\mathcal{F}(X)}$ (defined the category whose objects are the objects of $\mathcal{F}$ mapping to X, and whose morphisms are morphisms in $\mathcal{F}$ mapping to id_X) is a groupoid. Hence the name "category fibered in groupoids", if you care.

Now to define a stack on a topological space, we restrict to the case where $\mathcal{G}$ is the category of open subsets of your topological space Z, call it $\mathcal{G} = \mathrm{Top}_Z$.

We require further that for any two elements $a, b \in \mathcal{F}$ mapping to the same $U \in \mathrm{Top}_Z$, $\mathrm{Isom}(a, b)$ is a sheaf on U. So far this is a $2\mathrm{PSh}^{++}$.

Finally, we require that "objects glue" given a bunch of objects over a bunch of open sets, along with isomorphisms over all pairwise intersections, that are compatible over triple intersections, then we there is an object over their union giving all of these data as restriction.

36.B. EXERCISE. Describe how the following are all the same.

- a stack where all of the fiber categories are equivalent to sets

- a stack where each object in the fiber categories has only the identity automorphism it is "asymmetric")
- a sheaf

37. (2-)Yoneda's Lemma

If $X \in \mathcal{G}$, in a topology where X is a sheaf, then h_X is a stack (as it is a sheaf).

And if $\mathcal{M}$ is another stack, then $\mathrm{Hom}(h_X, \mathcal{M})$ is identified with the fiber category $\mathcal{M}(X)$ over X. (The *fiber category* was defined earlier.) This the **(2-)Yoneda Lemma**.

So any map from X to a stack $\mathcal{M}$ should be labeled with an object α of the fiber category $\mathcal{M}(X)$:

$$X \xrightarrow{\alpha} \mathcal{M}$$

(Question: what is the meaning of the automorphisms of the object $\alpha \in \mathcal{M}(X)$, in terms of this map?)

We made the 2-Yoneda Lemma more precise. We have two categories: $\mathrm{Hom}(h^X, \mathcal{M})$, and $\mathcal{M}(X)$. We described functors from each to the other, which were easiest to describe using pictures. We then saw how the two compositions were both isomorphic to the respective identity morphisms. (One actually was the identity, I believe.)

38. (2-)Fibered products

Given that the right notion of "same" for categories isn't "isomorphism" but "equivalence", and two functors (with the same source and target) will never be the "same" but they may be "isomorphic", we are led to a slightly more sophisticated versino of fibered product. I'll first describe the *2-fibered product* (or *homotopy fibered product*) of categories, and then apply it to fibered categories.

Given a diagram of *catgories*

$$\begin{array}{ccc} & & \mathcal{Y} \\ & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Z} \end{array}$$

the 2-fibered product is a diagram

$$\begin{array}{ccc} \mathcal{W} & & \mathcal{Y} \\ & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Z} \end{array}$$

that "2-commutes" (meaning: the two composite functors $\mathcal{W} \to \mathcal{Z}$ aren't the same, but are isomorphic), and is universal with respect to this property.

38.A. EXERCISE. Show that the 2-fibered product $X \times_Y Z$ (if it exists) is unique up to equivalence, and this equivalence is unique up to isomorphism, and this isomorphism itself is unique.

38.1. **Construction of the 2-fibered product.** I then showed that the 2-fibered product existed, by construction. Given functors $F : X \to Z$ and $G : Y \to Z$, then consider the category W whose objects are:

$$\{(x \in X, y \in Y, \phi : F(x) \xrightarrow{\sim} G(y)\}$$

and whose morphisms $(x, y, \phi) \to (x', y', \phi')$ are the data of morphisms $\sigma : x \to x'$ in X, and $\tau : y \to y'$ in Y, such that the following diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{\phi} & G(y) \\ {\scriptstyle F(\sigma)}\downarrow & & \downarrow{\scriptstyle G(\tau)} \\ F(x') & \xrightarrow{\phi'} & G(y') \end{array}$$

We have "obvious" functors $F' : W \to Y$ and $G' : W \to X$.

38.B. EXERCISE. Show that

$$\begin{array}{ccc} W & \xrightarrow{F'} & Y \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} \\ X & \xrightarrow[F]{} & Z \end{array}$$

satisfies the universal property of the 2-fibered product.

38.C. EXERCISE. Show that the magic (Cartesian) diagram also works for 2-fibered products of categories: describe a 2-Cartesian diagram:

$$\begin{array}{ccc} U \times_M V & \longrightarrow & U \times V \\ \downarrow & & \downarrow \\ M & \xrightarrow{\delta} & M \times M \end{array}$$

where U, V, M are categories.

38.2. **The notion of 2-fibered product readily extends to categories fibered over $\mathcal{G}$.**

38.D. EXERCISE. Show that $\underline{\mathbf{CF}}_{\mathcal{G}}$ has 2-fibered products.

38.E. EXERCISE. Show that the 2-fibered product in $\underline{\mathbf{CF}}_{\mathcal{G}}$ induces 2-fibered products in $\underline{\mathbf{CFG}}_{\mathcal{G}}$.

38.F. EXERCISE. supopse that M in $\mathrm{CFG}_{\mathcal{G}}$, and $X \in \mathcal{G}$. Suppose we have two maps $\alpha : X \to M$ and $\beta : X \to M$, which are identified (by Yoneda) with objects of $M(X)$, which we also call α and β.

38.3. *[To be added later.* (a) I want to define the presheaf $\mathrm{Isom}_X(\alpha, \beta)$, on X. (b) I want to ask the exercise that

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{M}(X)}(\alpha, \beta) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \alpha,\beta} \\ \mathcal{M} & \xrightarrow{\ \delta\ } & \mathcal{M} \times \mathcal{M} \end{array}$$

is a (2-)fibered diagram. (c) Ask why $\mathrm{Aut}(\mathcal{M}(X))$ is a presheaf of groups in this case. (d1) Later, I want to ask the exercise that M is actually in $\mathrm{CFG}^{+}_{\mathcal{G}}$ if all such Isom presheaves satisfy the identity axiom; and (d2) M is actually in $\mathrm{CFG}^{++}_{\mathcal{G}}$ if all such Isom presheaves are actually sheaves. Hence in the case of stacks, (b) and (c) turn into : Isom is a sheaf, and Aut is a sheaf of groups.

38.4. **Mixing in the topology on $\mathcal{G}$.**

Fix now a topology on the category $\mathcal{G}$.

38.G. EXERCISE. Show that the 2-fibered product in $\underline{\mathbf{CF}}_{\mathcal{G}}$ induces 2-fibered products on $\underline{\mathbf{CF}}^{+}_{\mathcal{G}}$, $\underline{\mathbf{CF}}^{++}_{\mathcal{G}}$, $\underline{\mathbf{CF}}^{+++}_{\mathcal{G}}$, $\underline{\mathbf{CFG}}^{+}_{\mathcal{G}}$, $\underline{\mathbf{CFG}}^{++}_{\mathcal{G}}$, $\underline{\mathbf{CFG}}^{+++}_{\mathcal{G}}$. (I showed this in class. Basically, it is a one-liner, using the universal property of the functors $+$, $++$, and $+++$.)

38.5. **Linguistic note.** For psychological and linguistic simplicity, people always just say "fibered product" rather than "2-fibered product", because it is clear what is being meant. This helps avoid annoying situations such as: each scheme "is" a stack, and the fibered product of schemes "is" the same as the 2-fibered product of the corresponding stacks. Similarly, rather than saying we have functors between stacks (which are, of course, categories), we just call them "maps" or "morphisms".

38.H. EXERCISE (CF. EXERCISE 41.N). Suppose T is a scheme, and $\alpha : T \to \mathcal{M}_g$ is a map, which by Yoneda's Lemma corresponds to a family of genus g curves over T (defined appropriately). (a) What is the fibered product $T \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$?

$$\begin{array}{ccc} ? & \longrightarrow & \mathcal{M}_{g,1} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\alpha} & \mathcal{M}_g \end{array}$$

(b) Explain why $\mathcal{M}_{g,1}$ is called the **universal curve** over $\mathcal{M}_g$.

39. REPRESENTABLE MORPHISMS OF STACKS, AND LOCALLY REPRESENTABLE STACKS

We define **representable morphisms of stacks** (to avoid confusion with other unfortunate meanings of the phrase, we should probably say $\mathcal{G}$-representable morphisms of stacks), just as we did for sheaves.

39.1. *Definition.* A morphism $\pi : \mathcal{M} \to \mathcal{N}$ of stacks on $\mathcal{G}$ is **representable** (by $\mathcal{G}$) if for every morphism from an object of $\mathcal{G}$ (interpreted of course as a stack, through 2-Yoneda) $X \to \mathcal{N}$, the (2-)fibered product $X \times_{\mathcal{N}} \mathcal{M}$ is also in $\mathcal{G}$. (This definition doesn't involve the topology on $\mathcal{G}$, so this just as well could be used to define *representable morphisms of categories fibered in groupoids*.)

We can define properties of representable morphisms, just as we did for sheaves on $\mathcal{G}$, and in particular, we have the notion of a **locally representable stack**. We denote such things as $\underline{\mathbf{LRepSt}}_{\mathcal{G}}$. Keep in mind that each one is a fibered category; maps between them are functors; etc. So if you wanted, you could say they formed some sort of fully faithful 2-subcategory of $\underline{\mathbf{St}}_{\mathcal{G}}$. We don't need to define this, because all that matters is that they inherit all notions from $\underline{\mathbf{St}}_{\mathcal{G}}$.

39.2. **Lots of categories and 2-categories.** We thus have the following relations, where everything is "contained" in anything below it or to its right:

$\mathcal{G}$	$\underline{\mathbf{LRepSh}}_{\mathcal{G}}$	$\underline{\mathbf{Sh}}_{\mathcal{G}} := \underline{\mathbf{PSh}}_{\mathcal{G}}^{++}$	$\underline{\mathbf{PSh}}_{\mathcal{G}}^{+}$	$\underline{\mathbf{PSh}}_{\mathcal{G}}$	
	$\underline{\mathbf{LRepSt}}_{\mathcal{G}}$	$\underline{\mathbf{StG}}_{\mathcal{G}} := \underline{\mathbf{CFG}}_{\mathcal{G}}^{+++}$	$\underline{\mathbf{CFG}}_{\mathcal{G}}^{++}$	$\underline{\mathbf{CFG}}_{\mathcal{G}}^{+}$	$\underline{\mathbf{CFG}}_{\mathcal{G}}$
		$\underline{\mathbf{St}}_{\mathcal{G}} := \underline{\mathbf{CF}}_{\mathcal{G}}^{+++}$	$\underline{\mathbf{CF}}_{\mathcal{G}}^{++}$	$\underline{\mathbf{CF}}_{\mathcal{G}}^{+}$	$\underline{\mathbf{CF}}_{\mathcal{G}}$

39.3. **Fibered products of locally representable stacks.**

At this point, we have shown that everything in §39.2 except in the first two columns has (2-)fibered products.

Suppose now that $\mathcal{G}$ has fibered products.

39.4. **Proposition. —** *$\underline{\mathbf{LRepSh}}_{\mathcal{G}}$ have fibered products, and $\underline{\mathbf{LRepSt}}_{\mathcal{G}}$ have 2-fibered products.*

This is now in Exercise 20.B.

Proof. We prove both in the same way.

Suppose $M \in \underline{\mathbf{LRepSh}}_{\mathcal{G}}$. We wish to show that if $K, L \in \underline{\mathbf{LRepSh}}_{\mathcal{G}}$, and we have $K \to M$, and $L \to M$, then $K \times_M L \in \underline{\mathbf{LRepSh}}_{\mathcal{G}}$.

39.A. EXERCISE. Show that it suffices to show that if $X \in \mathcal{G}$ and $L \in \underline{\mathbf{LRepSh}}_{\mathcal{G}}$, and we have $X \to M$ and $L \to M$, then $X \times_M L \in \underline{\mathbf{LRepSh}}_{\mathcal{G}}$. Hint: consider:

$$\begin{array}{ccc} ? & \longrightarrow & X \\ \downarrow & & \downarrow \\ ? & \longrightarrow & K \\ \downarrow & & \downarrow \\ L & \longrightarrow & M \end{array}$$

39.B. EXERCISE. Similarly, show that it suffices to show that if $X, Y \in \mathcal{G}$, and we have $X \to M$ and $Y \to M$, then $X \times_M Y \in \underline{\mathbf{LRepSh}}_{\mathcal{G}}$.

Then from the magic diagram

$$\begin{array}{ccc} X \times_M Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ M & \xrightarrow{\delta_M} & M \times M \end{array}$$

we see that it suffices to show that if $Z \in \mathcal{G}$, then $M \times_{M \times M} Z$ is locally representable.

Now $R = U \times_M U$ is representable (as $U \to M$ was representable), so the magic diagram gives us:

$$\begin{array}{ccc} R & \xrightarrow{\text{rep}} & U \times U \\ {\scriptstyle \text{(rep) cover}}\downarrow & & \downarrow{\scriptstyle \text{(rep) cover}} \\ M & \xrightarrow{\delta_M} & M \times M \end{array}$$

We now base change by $Z \to M \times M$ to get a "fiber cube", whose bottom square is the diagram above, and whose top square is:

$$\begin{array}{ccc} Z_2 & \xrightarrow{\text{rep}} & Z_1 \\ {\scriptstyle\text{(rep) cover}}\downarrow & & \downarrow{\scriptstyle\text{(rep) cover}} \\ Z_3 & \longrightarrow & Z \end{array}$$

Now Z_1 is representable (as $Z_1 \to Z$ is the base change of the representable morphism $U \times U \to M \times M$. Then Z_2 is representable (as $Z_2 \to Z_1$ is the base change of the representable morphism $R \to U \times U$). Then Z_3 is locally representable (as it has a representable cover by $Z_2 \to Z_3$; this is a representable cover becasue it is the base change of the representable cover $R \to M$). □

As promised, we have proved that algebraic spaces have fibered products.

Part 7. **Algebraic stacks**

40. DEFINITION: ORBIFOLDS, DM STACKS, ALGEBRAIC STACKS, COMPLEX ALGEBRAIC STACKS

40.1. **Definition.** *A [blank] is a locally representable stack on the category $\mathcal{G}$ = [blank], with the [blank] topology.* Here are some ways of filling in the blanks.

- orbifold: manifolds, analytic-etale
- needs-a-name: complex analytic varieties, analytic-smooth
- Deligne-Mumford stack: schemes, étale
- algebraic stack: schemes, smooth

By Exercise 36.B, an algebraic stack that is equivalent to a sheaf "is" an algebraic space.

40.A. EXERCISE. Show that an algebraic stack $\mathcal{M}$ is an algebraic space if and only if every object in $\mathcal{M}$ is "asymmetric" (has no nontrivial automorphisms). Translation: for every $X \in \mathcal{G}$, the fiber category $\mathcal{M}(X)$ is equivalent to a set.

40.2. *Caution with orbifolds.* Most people discussing orbifolds (in talks or papers) elide or get confused about the fact that they they form a "2-category" (or that they "are" a category), so pullbacks, morphisms, and fiber products are all slightly subtle.

40.3. You might be used to seeing an additional condition in the definition of Deligne-Mumford stacks and algebraic stacks, which is that the diagonal is supposed to be "representable by algebraic spaces". But this is now a consequence of Proposition 39.4. Reason: the diagonal morphism in both cases are sheaves (in the etale and smooth topologies respectively), which are locally representable, which are automatically algebraic spaces (that was our definition).

(Aside: How did I pull this off? My definition required the cover to be representable! And in all cases I can think of, it *is* representable, when you construct it. And in fact if you take the usual definition, any cover in the usual sense is forced to be representable by algebraic spaces...)

41. G-BUNDLES, AND QUOTIENTS BY G

41.1. **Vague motivating discussion (skip this discussion if you find it unhelpful; it is not logically necessary, at least yet).**

Suppose G is a finite group. We have the notion of an action of G on a set, or even on a "space" $X \in \mathcal{G}$.

What do we mean by X/G? How should we define it? There are different possible reasonable answers, so we should be careful!

41.2. *Motivating example.* For example, if $X \to Y$ is a G-bundle, then Y should certainly deserve the name X/G, under *any* reasonable definition.

41.A. EXERCISE. Define G**-bundle**, or equivalently, G**-torsor**. (Your definition depends on the *topology* you have put in $\mathcal{G}$! Not every étale $(\mathbb{Z}/2)$-bundle is a Zariski $(\mathbb{Z}/2)$-bundle! Rhetorical question: is every smooth $(\mathbb{Z}/2)$-bundle an étale $(Z/2)$-bundle?

So the first example gives the best type of quotient: a map $X \to Y$ expressing X as a G-bundle over Y, so that the G-action on X given in the problem agrees with the G-action coming from the G-bundle structure. Rhetorical question: if such a quotient Y exists, is it unique in a reasonable sense? (What does "in a reasonable sense" mean?)

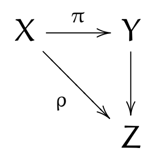
We call this a **geometric quotient**.

41.3. *Motivating example.* Suppose $\mathcal{G}$ is the category of complex finite type schemes, and $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$, and $G = \mathbb{Z}/2$, with the non-identity element acting by sending $x \mapsto -x$.

Then Y should be Spec $\mathbb{C}[x^2]$. In this case, $X \to Y$ is not a G-bundle (in any topology — why?).

This leads to a weaker notion of quotient, the categorical quotient.

41.4. *Definition.* Given a G-action on X (in $\mathcal{G}$). We say $\pi : X \to Y$ is a **categorical quotient** of X by G if π is G-invariant, and π is universal for such maps — any other G-invariant map $\rho : X \to Z$ factors uniquely through Y:



By Yoneda's Lemma, the categorical quotient is unique up to unique isomorphism.

41.B. RHETORICAL QUESTION. Show that if $\pi : X \to Y$ is a G-bundle, then π is a categorical quotient. (This is rhetorical because I haven't thought it through before asking it.) This answers the earlier question: the geometric quotient is indeed unique up to unique isomorphism.

41.5. *Weakest definitions of quotients.*

We could take the sheaf h^X, and quotient it by G. This is a presheaf, not necessarily a sheaf. (It is at least a separated sheaf.) I guess we could call this the presheaf quotient, but is really lame; the lamest possible quotient, but it has the advantage of always existing.

Second, we want something geometric, so at least it should be a sheaf. So this gives us the second-lamest quotient: the sheafification of the presheaf: $(h^X/G)^{sh}$. Again, this exists.

41.C. EXERCISE. If we are in a geometric situation (our objects are in a category, which has a topology in which all objects are sheaves), then any categorical quotient is a sheaf quotient.

41.D. EXERCISE. A geometric quotient (if it exists) is a sheaf quotient.

Thus we have answered rhetorical question 41.B in the geomtric setting in the affirmative.

A fourth definition: here is a different presheaf. Maps to this quotient (i.e., sections of this presheaf) from Z are G-bundles $Z \to Z$, along with G-equivariant maps to X; up to isomorphisms of this diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\text{G-equiv.}} & X \\ {\scriptstyle \text{G-bundle}}\downarrow & & \\ Z & & \end{array}$$

(If such diagrams have no automorphisms, then this is already a sheaf, I think...)

41.6. **Group (objects) and their actions.** Suppose G is a group object in $\mathcal{G}$.

(If you don't know what this means, just ask.)

Suppose $X \in \mathcal{G}$, and we have a G-action on X, i.e. an "action" map $a : G \times X \to X$ satisfying some properties (again, just ask if you haven't seent his before).

41.7. **The case where G acts freely.**

We define when a G-action is **free**. The right notion of G **acting freely on** X is if $G \times X \to X \times X$ is a monomorphism. Translation: for any $T \in \mathcal{G}$, $\mathrm{Hom}(T, G \times X) \to \mathrm{Hom}(T, X \times X)$ is an injection. Translation: for any $T \in \mathcal{G}$, $\mathrm{Hom}(T, G) \times \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, X) \times \mathrm{Hom}(T, X)$ is an injection. Translation: given any $\sigma : T \to X$, the only map $\beta : T \to G$ such that σ is the same as σ acted on by β is if β is the identity.

41.E. EXERCISE. Convince yourself of everything in the previous paragraph.

41.F. MORE PRECISE EXERCSE. Show that the notion of free action above agrees with the usual notion of free action (which is the case where $\mathcal{G} =$ **Sets**).

In what follows, assume G acts freely on X.

Consider diagrams of the form

$$\begin{array}{ccc} Z & \xrightarrow{\text{G-equivariant}} & X \\ {\scriptstyle G}\downarrow & & \\ T & & \end{array}$$

Show that such diagrams have no nontrivial automorphisms. More precisely, show that there is no automorphism of Z preserving everything else in the diagram (the G-action, and the map to X).

Define a presheaf on $\mathcal{G}$, denoted $\boxed{X/G}$, as follows, by sending T to diagrams of this sort, up to isomorphism.

41.G. EXERCISE. Show that X/G is a sheaf (on $\mathcal{G}$).

41.H. EXERCISE. Show that $X \to X/G$ is a G-bundle.

Thus if G is smooth, then $X \to X/G$ shows that X/G is locally representable, so X/G is an algebraic space.

41.I. USEFUL (BUT NOT STRICTLY NEEDED) BONUS EXERCISE. Show that $h_X/h_G \in \underline{\mathbf{PSh}}^+_{\mathcal{G}}$. Show that $(h_X/h_G)^+ \cong h_{X/G}$. (Better: describe a map $\sigma : h_X/h_G \to h_{X/G}$ coming from $h_X \to h_G$. Show that σ satisfies the universal property of the second "+" functor.)

41.8. **What if G doesn't act freely on X?.**

Now let's get rid of the "free action" assumption.

Consider diagrams of the form

$$(3)\qquad \begin{array}{ccc} Z & \xrightarrow{\text{G-equivariant}} & X \\ {\scriptstyle G}\downarrow & & \\ T & & \end{array}$$

where Z is a G-bundle.

(The most interesting case is when the action is the "least free": when X is a point. Then this is just the same as diagrams of G-bundles.)

41.J. EXERCISE. Describe an isomorphism of two such diagrams. Explain then how the collection of such diagrams forms a groupoid.

Define a category fibered in groupoids on $\mathcal{G}$, denoted $\boxed{X/G}$ (more commonly denoted $\boxed{[X/G]}$ to emphasize that it is a stack not a space/sheaf), as follows, by sending diagrams of this sort to T. (What are the morphisms in the category X/G? Pullback diagrams!)

41.K. EXERCISE. Show that X/G is in a stack on $\mathcal{G}$.

41.L. EXERCISE. Show that $X \to X/G$ is a G-bundle.

41.9. **Important consequences.** Thus if G is smooth, then $X \to X/G$ shows that X/G is locally representable, so X/G is an algebraic stack. If G is even discrete, then X/G is a Deligne-Mumford stack.

41.M. EXERCISE. Suppose we have a map $\alpha : T \to [X/G]$, which by Yoneda is the same as a diagram (3). What is the fibered product $T \times_{[X/G]} X$?

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ T & \xrightarrow{\alpha} & [X/G] \end{array}$$

(You should immediately guess the answer!)

In the case where X is a point pt (if you are dealing with schemes over a field; for schemes more generally it will be $\operatorname{Spec} \mathbb{Z}$; in both cases it is the final object), then $[X/G] = BG$, which is the moduli space of G-bundles.

41.N. EXERCISE (CF. EXERCISE 38.H). Explain why $pt \to BG$ is called the **universal** G**-bundle** (or **universal** G**-torsor**).

41.O. EXERCISE. Given a G-action on X, show that there are natural maps $X/G \to BG$. If $\mathcal{G}$ is the category of schemes over $\mathbb{C}$, what are the geometric fibers of this map?

41.P. EXERCISE. If $G \to H$ is a group homomorphism, which way is there a map between BG and BH? What are the fibers of this map? (Add whatever assumptions you need to make this work.)

41.Q. USEFUL BONUS EXERCISE (CF. EXERCISE 41.I). Show that $h_X/h_G \in \underline{\mathbf{CFG}}^+_{\mathcal{G}}$. Show that $(h_X/h_G)^{++} \cong h_{X/G}$.

41.10. **Interesting example.** An elliptic curve in the analytic topology is $\mathbb{C}/\Lambda$, where Λ is a lattice. This (really, $\operatorname{Spec} \mathbb{C}[x]$ modulo the group Λ which acts freely) will soon be an algebraic space, which will be counterintuitive. A simpler variant of this is $(\operatorname{Spec} \mathbb{C}[x])/\mathbb{Z}$, where the action of $\mathbb{Z}$ is by: $n \circ x = x + n$.

42. ALGEBRAIC (AND DELIGNE-MUMFORD) STACKS VIA PRESENTATIONS

The next section will look very much like the previous one. Do you see fully why?

In one of our settings (e.g. algebraic stacks), given a cover $U \to \mathcal{M}$ (e.g. smooth cover of an algebraic stack by a scheme), define $R = U \times_{\mathcal{M}} U$, which is in $\mathcal{G}$ (e.g., is a scheme) because $U \to \mathcal{M}$ is representable. We have a fiber diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ {\scriptstyle t}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{M} \end{array}$$

where R may be informally thought of as the space of relations among the points of U, where s is the source of the relation, and t is the target of the relation. In other words, for $r \in R$, $s(r)$ is identifed with $t(r)$ (speaking informally, so this can apply in a wide variety of setings).

Then the magic (Cartesian) diagram

$$\begin{array}{ccc} U \times_{\mathcal{M}} U & \longrightarrow & U \times U \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\delta} & \mathcal{M} \times \mathcal{M} \end{array}$$

may be rewritten as

(4)
$$\begin{array}{ccc} R & \xrightarrow{(s,t)} & U \times U \\ {\scriptstyle \text{rep. cover}}\downarrow & & \downarrow{\scriptstyle \text{rep. cover}} \\ \mathcal{M} & \xrightarrow{\delta} & \mathcal{M} \times \mathcal{M} \end{array}$$

42.A. EXERCISE. Explain the meaning of $(h_R/h_U)^{st} = \mathcal{M}$, and prove it (where the stackfication shoudl be sheafification if $\mathcal{M}$ is a sheaf). This is usually written $\boxed{[U/R]}$ even though this isn't great (as it is in conflict with the notation $[X/G]$).

As a special case, we obtain a diagram from the section on G-quotients:

$$\begin{array}{ccccc} R & \overset{\sim}{=\!=} & X \times G & \xrightarrow{a} & X \\ & & {\scriptstyle pr_1}\downarrow & & \downarrow \\ & & X & \longrightarrow & [X/G] \end{array}$$

43. THE DIAGONAL MORPHISM FOR ALGEBRAIC STACKS, AND THE ISOTROPY/INERTIA STACK

If $\mathcal{M}$ is an algebraic sgtack, we have shown that $\delta_{\mathcal{M}}$ is a loally representable relative sheaf, i.e., a "relative algebraic space".

43.A. EXERCISE. Explain what that means!

Our Accidental Theorem 34.3 saysthat if $\mathcal{M}$ is an algebraic space, then in fact $\delta_{\mathcal{M}}$ is a relative *scheme*, but we don't really care.

43.1. **Proposition. —** *$\delta_{\mathcal{M}}$ is locally of finite type.*

Proof. $R \to U$ is locally of finite type, hence $R \to U \times U$ is, which means that $\delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is (from (4)). □

We now interpret autmorphisms of objects of $\mathcal{M}$ in terms of the diagonal. More precisely, consider an algebraic stack $\mathcal{M}$, and a map $X \to \mathcal{M}$, which is the same information as an object α of the fiber category $\mathcal{M}(X)$ (by 2-Yoneda). I will now make sense of the automorphisms of the object $\mathcal{M}(X)$ in the fiber category.

43.B. EXERCISE. Interpret the (2-)fibrered product

(5)
$$\begin{array}{ccc} & & \mathcal{M} \\ & & \downarrow{\scriptstyle \delta} \\ X & \xrightarrow{(\alpha,\alpha)} & \mathcal{M} \times \mathcal{M} \end{array}$$

as a sheaf on X, where the sections over an open $\rho : U \to X$ are the automorphisms of $\rho^* \alpha \in \mathcal{M}(U)$. Explain how it is a sheaf of groups.

It is almost a finite type group scheme over X — it is a finite type group "algebraic space" over X (because $\delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is a "relative algebraic space").

The automorphisms of α are just the global sections of this sheaf!

This construction works for any X, so we can "universalize" this construction by considering instead the (2-)fibrered product

$$\begin{array}{ccc} I_{\mathcal{M}} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle \delta} \\ \mathcal{M} & \xrightarrow{(\alpha,\alpha)} & \mathcal{M} \times \mathcal{M} \end{array}$$

Then for any specific $\alpha : X \to \mathcal{M}$, we just pull this diagram back by α

$$\begin{array}{ccccc} & & I_{\mathcal{M}} & \longrightarrow & \mathcal{M} \\ & & \downarrow & & \downarrow{\scriptstyle \delta} \\ X & \xrightarrow{\alpha} & \mathcal{M} & \xrightarrow{(\alpha,\alpha)} & \mathcal{M} \times \mathcal{M} \end{array}$$

to get (5).

43.2. *Definition.* This "double diagonal" is called the **inertia stack** $\boxed{I_{\mathcal{M}} \to \mathcal{M}}$, which by the above discussion, is a locally-finite-type group algebraic space over $\mathcal{M}$. I prefer to call it the **isotropy (inertia) stack**.

43.3. *Observation.* The isotropy (intertia) stack $I_{\mathcal{M}} \to \mathcal{M}$ is locally of finite type over $\mathcal{M}$, because it is obtained by base change from $\delta_{\mathcal{M}}$, which is locally of finite type by Proposition 43.1. Hence $\Omega_{I_{\mathcal{M}}/\mathcal{M}}$ is a finite type quasicoherent sheaf on $I_{\mathcal{M}}$.

I gave a picture of the isotropy/inertia stack over $\mathcal{M}_2$, which had two components mapping isomorphically onto $\mathcal{M}_2$ (the identity and the hyperelliptic involution), as well as some sporadic additional bits.

43.4. **Accidental facts.** A "group algebraic space" over a field is a scheme, but a "group algebraic space" over a more general scheme need not be a scheme. See

`https://mathoverflow.net/questions/8918/`

for some discussion.

43.5. **Proposition (generalized in Thm. 43.8). —** *If $\mathcal{M}$ is a Deligne-Mumford stack, then $I_{\mathcal{M}} \to \mathcal{M}$ is unramified.*

Proof. This is obtained by base-change from $\delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$, so we need to show that δ is unramified. But from the Cartesian diagram (4) it suffices to show that $R \to U \times U$ is unramified, which we showed earlier.

43.C. EXERCISE. Show how this translates to show that the automorphism group scheme of every $\alpha \in \mathcal{M}(\operatorname{Spec} k)$ for all fields k have no infinitesimal tangent vectors.

43.6. **Proposition. —** *Suppose $\mathcal{M}$ is an algebraic stack, with unramified diagaonal. Then $\mathcal{M}$ is Deligne-Mumford.*

This will give a clean characterization of which algebraic stacks are Deligne-Mumford! Precisely:

43.7. **Proposition. —** *If an algebraic stack has unramified diagonal, i.e., if the objects have unramified automorphism groups, then it is Deligne-Mumford. If the objects have only* trivial *automorphism groups, then it is in fact an algebraic space.*

We have a smooth representable cover $\mathcal{U} \to \mathcal{M}$, and we want an étale representable cover $\mathcal{U}' \to \mathcal{M}$.

This is a conequence of the following more precise statement.

Explained May 11. Added to these note May 14.

43.8. **Theorem. —** *Suppose $\mathcal{M}$ is an algebraic stack. The following are equivalent.*

(a) *$\mathcal{M}$ has a representable étale cover by a scheme (ie., $\mathcal{M}$ is a Deligne-Mumford stack)*
(b) $\Omega_{\delta_{\mathcal{M}}} = 0$. *(Since $\delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is automatically locally of finite type, this condition is equivalent to saying that $\delta_{\mathcal{M}}$ is unramified.)*
(c) $\Omega_{I/\mathcal{M}} = 0$. *(Since $I \to \mathcal{M}$ is automatically locally of finite type, this condition is equivalent to saying that $I \to \mathcal{M}$ is unramified.)*

43.9. *Remark.* I discussed how (c) is the same as the fact that "objects" over algebraically closed fields have non nontrivial infinitesimal automorphisms, as $\Omega_{I/\mathcal{M}}$ restricted to the identity of the automorphism group scheme of that object is the Zariski tangent space of the automorphism tangent space at the identity, which in turn can be interpreted as maps of Spec of the dual numbers $\kappa[x]/(x^2)$ to the automorphism group scheme, which in turn can be interpreted as infinitesimal automorphisms.

43.10. **"The DM locus of an algebraic stack is Zariski-open".** Suppose $\mathcal{M}$ is an algebraic stack. Then $\Omega_{I_{\mathcal{M}}/\mathcal{M}}$ is a finite type quasicoherent sheaf on $I_{\mathcal{M}}$. Let $s : \mathcal{M} \to I_{\mathcal{M}}$ that is the "identity" section of the group space $I_{\mathcal{M}}/\mathcal{M}$. Then $s^*\Omega_{I_{\mathcal{M}}/\mathcal{M}}$ is a finite type quasicoherent sheaf on $\mathcal{M}$. Its support is Zariski-closed, so the complement of the support is Zariski-open.

43.D. EXERCISE. Explain why this Zariski-open substack should be called the Deligne-Mumford locus of $\mathcal{M}$.

Proof of Theorem 43.8. I first showed that that (b) and (c) are equivalent. (b) implies (c) because we have the following pullback diagram:

$$\begin{array}{ccc} I & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{M} \times \mathcal{M} \end{array}$$

To show (c) implies (b), suppose $\mathcal{M} \to I$ is the "identity section" of the automorphism group space I. Then show that the pullback of $\Omega_{I/\mathcal{M}}$ by this section is $\Omega_{\delta_{\mathcal{M}/\mathcal{M}^2}}$.

43.E. EXERCISE. Prove this.

We proved (a) implies (b) earlier, in Proposition 43.5.

It remains to show that (b)=(c) implies (a). Suppose we have a schematic smooth cover $\pi : U \to M$. Our goal is to find a schematic étale cover of M. We will find an étale map to M factoring through each $p \in U$, which will thus cover M.

Suppose $\pi(p) = q \in M$.

Suppose π (which is smooth) has relative dimension m at p. I'll give you a cover that has relative dimension $m-1$ at p.

Now δ_M is unramified at q by (b).

We have

$$\begin{array}{ccc} R & \xrightarrow[\text{rel dim } m]{\text{smooth}} & U \\ {\scriptstyle \text{smooth rel dim } m}\downarrow & & \downarrow{\scriptstyle \text{smooth rel dim } m} \\ U & \xrightarrow[\text{rel dim } m]{\text{smooth}} & M \end{array}$$

and

$$\begin{array}{ccc} R & \xrightarrow{\therefore \text{unram.}} & U \\ {\scriptstyle \text{smooth rel dim } 2m}\downarrow & & \downarrow{\scriptstyle \text{smooth rel dim } 2m} \\ U & \xrightarrow{\text{unram.}} & M \end{array}$$

From the relative cotangent sequence, we have the exactness of

$$\begin{array}{ccccccc} \Omega_{U\times U/M} & \longrightarrow & \Omega_{R/M} & \longrightarrow & \Omega_{R/U\times U} & \longrightarrow & 0 \\ \| & & & & \| & & \\ \pi_1^*\Omega_{U/M} \oplus \pi_1^*\Omega_{U/M} & & & & 0 & & \end{array}$$

And we also have

$$0 \longrightarrow \pi_1^*\Omega_{U/M} \longrightarrow \Omega_{R/M} \longrightarrow \text{loc. free rank } n \longrightarrow 0$$

Now take $df \in (\Omega_{U/M})_p$ such that $\pi_2 * df$ is *not* in the image of $\pi_1^*\Omega_{R/M}$.

So we can choose an f with $f(p) = 0$ whose derivation is this df.

Then $f = 0$ cuts out a locus that is smooth of relative dimension $m-1$ at p (hence near p). Reason:

43.F. EXERCISE. Show that if $X \to Y$ is smooth of relative dimension m, and $p \in U$, and $f(p) = 0$, and in the fiber containing p, $V(f)$ is smooth, then $V(f)$ is smooth of relative dimension $m - 1$. (Hint: show $V(f)$ is flat using a local criterion. Show the geometric fibers are smooth of dimension $m - 1$ near p.)

$\square$

43.G. EXERCISE. More generally, if you have an algebraic stack, and all of the isotropy groups have tagent spaces with dimension at most m, show that you can find a cover that is smooth of relative dimension m.

43.H. REALITY CHECK QUESTION. More generally, if $\mathcal{M}$ is a stack on a category with topology, not in any algebraic situation, what can you say about $I \to \mathcal{M}$? Answer: it is a group sheaf.

43.I. REALITY CHECK QUESTION. Fill in the blank. $I \to \mathcal{M}$ is a [blank], iff $\mathcal{M}$ is a sheaf. (What does that last "is" mean in the previous sentence?)

Part 8. Showing lots of moduli spaces are algebraic stacks (or even DM stacks or orbifolds or algebraic spaces)

44. BLACK BOXES WE WILL USE REPEATEDLY

We will use a very small number of black boxes repeatedly. All of these were actually proved last quarter in the "moduli spaces" course in the Noetherian setting, so you can see proofs in the slides on the webpage (although I make no promises as to how comprehensible they are). A discussion on removing Noetherian hypotheses is in §44.6; even this is mostly reasonable.

44.1. **Uppersemicontinuity results.**

We will use uppersemicontinuity of fiber dimension; uppersemicontinuity of cohomology groups in flat families; uppersemicontinuty of ranks of finite type quasicoherent sheaves.

44.2. **Representability of various moduli spaces.**

We will use the representability of the Hilb, Quot, Isom, and Mor functors in various settings.

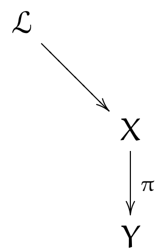
44.3. **Cohomology and base change theorem.**

We will use the Cohomology and Base Change Theorem.

44.4. **See-saw theorem.**

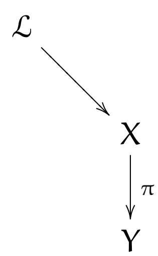
We will use Theorem 44.5, which we first describe informally.

Suppose we have the following situation.

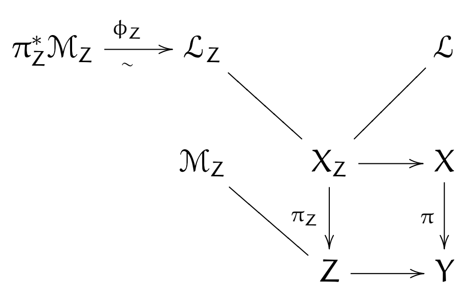


where π is proper, flat, and has geometric fibers that are connected and reduced. Suppose Y is locally Noetherian (but see §44.6 to exchange this for finitely presented hypotheses on π). For some of the points of Y, the restrict of $\mathcal{L}$ to Y might be isomorphic to the trivial bundle. From examples, you might convince yourself that this will form a locally closed subset. If you want to show this, after some thought and discussion, you'll realize that the right statement to prove is the following.

44.5. **Theorem. —** *Suppose we have the following situation.*



where Y is locally Noetherian, and π is proper, flat, and has geometric fibers that are connected and reduced. The contravariant functor $\underline{\textbf{Sch}}_Y \to \underline{\textbf{Sets}}$ *given (on objects) by sending* $(Z \to Y)$ *to*



(where everything is defined as you think, by pullbacks) is a locally closed subsheaf (i.e., locally closed subscheme) of Y.

If furthermore the fibers of π are integral, then this functor is a closed subpresheaf (i.e., closed subscheme) of Y.

Note that this construction "commutes with base change". Also the Noetherian conditions can be exchanged for local finite presentation conditions on π (see §44.6). So this works just fine if Y is an algebraic stack on **<u>Sch</u>**, where we assume in addition that π is schematic and finitely presented with geometric fibers that are connected and reduced, then we can cover Y with affine open subschemes so that on each open the situation is pulled back from

$$\mathcal{L}' \longrightarrow X' \xrightarrow{\pi'} \operatorname{Spec} A$$

where A is Noetherian, π' is proper, flat, with geometric fibers that are connected and reduced.

I think I can explain easily how to do this with everything but flatness. If π is projective, then we can arrange for π' to be projective, and then use the flattening stratification (proved last quarter). If π is proper, there is still a flattening stratification (stacks tag 05PS), but I've not thought about the proof.

44.6. **Removing Noetherian hypotheses.**

Most of the statements above were proved in Noetherian situations, but once that is done, they Noetherian conditions can be fairly cheaply removed, because the results then apply to any situation base-changed (and hence "locally base-changed") from the Noetherian situation, which means that they will hold for any situation locally base changed from a Noetherian situation where the hypotheses hold. So to make this work, we need to show that given a non-Noetherian situation with the desired hypotheses, we can locally pull it back from a Noetherian situation with the desired hypotheses. For example, for the Seesaw Theorem, we need to show something like the following.

If $\mathcal{L}$ is a line bundle on X, and $\pi : X \to Y$ is a finitely presented, proper, flat morphism. Then there is a cover of Y by opens Y', so that the restriction $\mathcal{L}'$ on $X' \to Y'$ is pulled back from a Noetherian situation $\mathcal{L}''$ on $X'' \to Y''$ (where of course we can take Y'' to be affine).

45. Various interesting moduli spaces are algebraic stacks

To be precise, we are saying that various interesting moduli CFG's form algebraic stacks. But that is a mouthful!

45.1. **Warm-up: the moduli of smooth curves of genus $g > 1$.**

In class I showed that if $g > 1$, then $\mathcal{M}_g$ is an algebraic stack.

Then more generally, I showed that the modlui space of dimension n proper smooth varieties X with $\det \Omega_X$ ample also forms an algebraic stack, locally of finite type.

> Some discussion from May 13 on why there are not more "general" versions of algebraic spaces are now back in the "Fancy facts about algebraic spaces we won't really use" section 34.

In both cases, we see that these are locally finite type over $\operatorname{Spec} \mathbb{Z}$; and if we fix "numerical data" (so we have a single Hilbert polynomial coming up in the construction), they are even finite type over $\operatorname{Spec} \mathbb{Z}$.

We also see that $I_{\mathcal{M}_g} \to \mathcal{M}_g$ is an affine group scheme over $\mathcal{M}_g$, and a closed subgroup scheme of $\operatorname{Aut} \mathbb{P}(\Omega_C^\vee)$ (a $PGL(g)$-bundle, which is an affine bundle, as pointed out by Spencer).

Thus $\mathcal{M}_g$ has "affine diagonal": $\delta_{\mathcal{M}_g}$ is an affine morphism. (And similarly for the higher-dimensional analogue.)

Moreoever, the fact that curves of genus $g > 1$ have no infinitesimal automorphisms (this requires showing that infinitesimal automorhpisms are given by $H^0(C, T_C)$, which we haven't shown — then T_C is a negative degree line bundle, which thus has no nonzero sections) implies that the fibers of $I_{\mathcal{M}_g} \to \mathcal{M}_g$ has fibers with 0-dimensional Zariski tangent spaces, thus the fibers are zero-dimensional affine schemes, hence curves of genus $g > 1$ have finite automorphism groups. This is a hard fact! (Once you know that the automorphism groups are finite, it is *much* easier to bound the size of the automorphism group in characteristic 0 — but you need the finiteness to get the argument started.)

Spencer pointed out that this argument only used what we did last quarter (about automorphisms of projective varieties, not about moduli spaces).

45.A. EXERCISE. Suppose X is a smooth projective variety (of any dimension), with $\det \Omega_X$ ample, and $H^0(X, T_X) = 0$. Prove that X has finite automorphism group.

45.2. **Moduli of very nice dimension at most 1 curves with marked points.**

We now define a useful stack.

First, you should verify that the following is a category fibered in groupoids over <u>**Sch**</u>, in the fpqcK topology. We denote this $\mathcal{V}_{\chi,n}$ for "very nice curves, with euler characteristic χ, and n marked points".

The objects over B are flat spatial morphisms $\pi : C \to B$, of relative dimension at most 1, finitely presented, such that the fibers of π have Euler characteristic (of the structure sheaf) χ; along with n sections $\sigma_i : B \to C$ of π (for $i = 1, \dots, n$), pairwise disjoint, all landing in the "smooth relative dimension 1" locus, ... [we interrupt this definition for an exercise]

45.B. NONTRIVIAL EXERCISE. Show that that the n sections $s_1 = \sigma_1(B), \dots, s_n = \sigma_n(B)$ are each effective Cartier divisors on C, so $\mathcal{L} := \mathcal{O}_C(s_1 + \cdots + s_n)$ is a line bundle on C.

[We continue the definition now] ... such that $\mathcal{L}$ on the fibers is very ample and has no h^1.

45.C. EXERCISE. Show that $C \to B$ is actually schematic (because it is relatively projective).

45.D. EXERCISE. Show that $\mathcal{V}_{\chi,n}$ is a stack in the fpqcK topology.

45.3. **Theorem. —** *$\mathcal{V}_{\chi,n}$ is an algebraic stack, finite type over $\mathbb{Z}$.*

Proof. I described a smooth cover of it. Consider the Hilbert scheme in $\mathbb{P}^{\chi-1}$ of subschemes with Hilbert polynomial $p(m) = nm + \chi$ (so, of curves with Euler characteristic χ, and degree n).

45.E. EXERCISE. Show that the locus corresponding to those schemes meeting the hyperplane $H := \{x_0 = 0\}$ at n distinct geometric points forms an open subset U, and that in this case the n points are all smooth points of the curve.

I explained how to label these n points, taking the n-fold fibered product of $H \cap C$ over U, and removing the "big diagonal", to obtain a scheme V.

45.F. EXERCISE. Show that $V \to \mathcal{V}_{\chi,n}$ is a G-bundle, where G is the group of automorphisms of $\mathbb{P}^{\chi(\mathcal{L})-1}$ fixing H as a set.

Thus $\mathcal{V}_{\chi,n} = [V/G]$, completing the proof. □

45.4. *First example: moduli of schemes of dimension* 0. If $n = 0$, then in fact C must be relative dimension 0. We have constructed the moduli space of schemes of relative dimension 0!

45.5. *Various moduli of curves.*

45.6. **Theorem. —** *The following locus are open in $\mathcal{V}_{X,n}$.*

(a) *The locus where $C \to B$ is smooth of pure relative dimension 1.*
(b) *The locus where $C \to B$ is flat of pure relative dimension 1, and the fibers are at worst nodal.*
(c) *The locus where $C \to B$ is smooth of pure relative dimension 1, and the fibers are geometrically connected.*
(d) *The locus where $C \to B$ is flat of pure relative dimension 1, and the fibers are at worst nodal, and geometrically connected.*

Proof. (a) The locus where $C \to B$ is smooth of relative dimension 1 is open (see The Rising Sea for example), so its complement is closed, so the image of that complement is closed in B, so the complemenet of *that* is open.

(b) Add in the requirement that $h^0(\mathcal{O}_F) = 1$ for the fiber, and use uppersemicontinuity of h^0.

(c) Because the stack is already finite type over $\operatorname{Spec} \mathbb{Z}$, it suffices to show the following.

45.G. EXERCISE. Suppose you have a family over a discrete valuation ring, whose central fiber is at worst nodal. Show that the general fiber is at worst nodal.

(d) Add in the requirement that $h^0(\mathcal{O}_F) = 1$ for the fiber, and use uppersemicontinuity of h^0. □

We can use this to make all sorts of spaces.

45.H. EXERCISE. Show that the following is a category fibered in groupoids, denoted $\mathfrak{M}_{g,n}$. To B, we have flat algebraic-spatial families of at worst nodal curves, along with n sections, all to the smooth locus, and pairwise distinct. (We have *no* ample conditions.)

Then $\mathfrak{M}_{g,n}$ is an algebraic stack: near any point corresponding to a nodal genus g, n-pointed curve, add in lots more smooth points [What is you are over a finite field, and there *aren't* any other smooth points?], so that the union of all the points is very ample, with no h^1. Let N be the new number of marked points. Then we have a map from an open subset of $\mathcal{V}_{1-g,N}$ to $\mathfrak{M}_{g,n}$ which is smooth, and hits the point in question.

The stack $\mathfrak{M}_{g,n}$ is useful; it is often called the stack of n-pointed nodal genus g curves.

The stack $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford locus in $\mathfrak{M}_{g,n}$.

45.I. EXERCISE. If you've heard a different definition of $\overline{\mathcal{M}}_{g,n}$: show that the points of $\overline{\mathcal{M}}_{g,n}$ parametrize what you think they do.

If X is a projective variety, we define the moduli space of stable maps (of genus g curves with n marked points) to X as follows. Over $\mathfrak{M}_{g,n}$, we have the "universal curve" $\mathcal{U}\mathfrak{M}_{g,n} \to_{g,n}$. The Hom scheme of something projective to something projective is representable, so $\mathrm{Hom}(\mathcal{U}\mathfrak{M}_{g,n}, X) \to \mathfrak{M}_{g,n}$ is schematic and representable. Thus $\mathrm{Hom}(\mathcal{U}\mathfrak{M}_{g,n}, X)$. The space of stable maps is the Deligne-Mumford locus $\mathrm{Hom}(\mathcal{U}\mathfrak{M}_{g,n}, X)^{DM}$.

45.J. EXERCISE. Show that this agrees with any other definition you have heard.

45.K. EXERCISE. The above construction only works if $\mathcal{U}\mathfrak{M}_{g,n} \to \mathfrak{M}_{g,n}$ is schematic; but a priori it is only algebraic spatial! How do you fix this?

45.7. **Hurwitz spaces.** Various things go by the name "Hurwitz spaces". They are basically variants of finite covers of $\mathbb{P}^1$ by smooth curves, perhaps with marked points, and perhaps with conditions on branching. All will be built out of $\mathrm{Hom}(\mathcal{U}\mathcal{M}_{g,n}, \mathbb{P}^1)^{DM}$ and some of our standard tools.

45.L. EXERCISE. Show that the moduli space of degree 10 simply branched covers of $\mathbb{P}^1$, by smooth genus 100 curves, is a Deligne-Mumford stack.

45.8. **What next?.**

We now can build lots of these spaces. What might you want to know about them? Possibilities:

- dimension
- smoothness
- irreducibility
- separatedness (not yet defined for DM stacks; problematic for algebraic stacks)
- properness (not yet defined for DM stacks; problematic for algebraic stacks)
- coarse moduli space (not yet defined)

46. WHERE TO NEXT?

In the last couple of weeks, there are a few possible ways we can go.

Possible goals:

- The Picard "stack" in reasonable generality. (We'll start with this.)
- $\overline{\mathcal{M}}_{g,n}$ is an irreducible smooth proper DM stack of relative dimension $3g-3+n$ over $\mathbb{Z}$.
- Existence of coarse moduli spaces (Keel-Mori, at least in some form)
- towards tags 04S6 and 06DC (Artin's "flat is enough" theorems), or at least "smooth is enough" `https://www.math.columbia.edu/~dejong/wordpress/?p=1584`

47. THE PICARD STACK

As a warm-up, I discussed the representability of the Picard stack and the Picard group of a smooth projective curve C over a field k, which had a chosen point $p \in C$. We noticed that a good deal didn't depend on p, and didn't depend on C being a smooth curve.

I'll try to type some in next, or possibly we'll just go to the general case next.