open sets containing \( f(U) \), with an equivalence relation. Note that \( f(U) \) won’t be an open set in general.

**3.6.A. Exercise.** Show that this defines a presheaf on \( X \).

Now define the inverse image of \( \mathcal{G} \) by \( f^{-1}\mathcal{G} := (f^{-1}\mathcal{G})^{\text{pre}} \). The next exercise shows that this satisfies the universal property. But you may wish to try the later exercises first, and come back to Exercise 3.6.B later. (For the later exercises, try to give two proofs, one using the universal property, and the other using the explicit description.)

**3.6.B. Important Tricky Exercise.** If \( f : X \to Y \) is a continuous map, and \( \mathcal{F} \) is a sheaf on \( X \) and \( \mathcal{G} \) is a sheaf on \( Y \), describe a bijection \( \text{Mor}_{X}(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \text{Mor}_{Y}(\mathcal{G}, f_{*}\mathcal{F}) \).

Observe that your bijection is “natural” in the sense of the definition of adjoints (i.e. functorial in both \( \mathcal{F} \) and \( \mathcal{G} \)). Thus Construction 3.6.2 satisfies the universal property of Definition 3.6.1. Possible hint: Show that both sides agree with the following third construction, which we denote \( \text{Mor}_{XY}(\mathcal{G}, \mathcal{F}) \). A collection of maps \( \phi_{UV} : \mathcal{G}(V) \to \mathcal{F}(U) \) (as \( U \) runs through all open sets of \( X \), and \( V \) runs through all open sets of \( Y \) containing \( f(U) \)) is said to be compatible if for all open \( U' \subset U \subset X \) and all open \( V' \subset V \subset Y \) with \( f(U) \subset V \), \( f(U') \subset V' \), the diagram

\[
\begin{array}{ccc}
\mathcal{G}(V) & \xrightarrow{\phi_{UV}} & \mathcal{F}(U) \\
\text{res}_{V',V} \downarrow & & \downarrow \text{res}_{U',U} \\
\mathcal{G}(V') & \xrightarrow{\phi_{V'U'}} & \mathcal{F}(U')
\end{array}
\]

commutes. Define \( \text{Mor}_{XY}(\mathcal{G}, \mathcal{F}) \) to be the set of all compatible collections \( \phi = \{ \phi_{UV} \} \).

**3.6.3. Remark.** As a special case, if \( X \) is a point \( p \in Y \), we see that \( f^{-1}\mathcal{G} \) is the stalk \( \mathcal{G}_p \) of \( \mathcal{G} \), and maps from the stalk \( \mathcal{G}_p \) to a set \( S \) are the same as maps of sheaves on \( Y \) from \( \mathcal{G} \) to the skyscraper sheaf with set \( S \) supported at \( p \). You may prefer to prove this special case by hand directly before solving Exercise 3.6.B, as it is enlightening. (It can also be useful — can you use it to solve Exercises 3.4.M and 3.4.O?)

**3.6.C. Exercise.** Show that the stalks of \( f^{-1}\mathcal{G} \) are the same as the stalks of \( \mathcal{G} \). More precisely, if \( f(p) = q \), describe a natural isomorphism \( \mathcal{G}_q \cong (f^{-1}\mathcal{G})_p \). (Possible hint: use the concrete description of the stalk, as a colimit. Recall that stalks are preserved by sheafification, Exercise 3.4.M. Alternatively, use adjointness.) This, along with the notion of compatible stalks, may give you a way of thinking about inverse image sheaves.

**3.6.D. Exercise (Easy But Useful).** If \( U \) is an open subset of \( Y \), \( i : U \to Y \) is the inclusion, and \( \mathcal{G} \) is a sheaf on \( Y \), show that \( i^{-1}\mathcal{G} \) is naturally isomorphic to \( \mathcal{G}|_U \).

**3.6.E. Exercise.** Show that \( f^{-1} \) is an exact functor from sheaves of abelian groups on \( Y \) to sheaves of abelian groups on \( X \) (cf. Exercise 3.5.D). (Hint: exactness can be checked on stalks, and by Exercise 3.6.C, the stalks are the same.) The identical