24.3.C. EXERCISE. Show that you can also compute the derived functors of an objects B of $\mathcal{A}$ using **acyclic resolutions**, i.e. by taking a resolution

$$\cdots \to A_2 \to A_1 \to A_0 \to B \to 0$$

by F-acyclic objects A_i, truncating, applying F, and taking homology. Hence $\mathrm{Tor}_i(M, N)$ can be computed with a flat resolution of M or N.

24.3.3. *Hint for Exercise 24.3.C (and a useful trick: building a "projective resolution of a complex").* Show that you can construct a double complex

(24.3.3.1)
$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & P_{2,1} \arrow[r] \arrow[d] & P_{1,1} \arrow[r] \arrow[d] & P_{0,1} \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_{2,0} \arrow[r] \arrow[d] & P_{1,0} \arrow[r] \arrow[d] & P_{0,0} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A_2 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where the rows and columns are exact and the $P_?$'s are projective. Do this by constructing the $P_?$'s inductively from the bottom right. Remove the bottom row, and the right-most nonzero column, and then apply F, to obtain a new double complex. Use a spectral sequence argument to show that (i) the double complex has homology equal to $L_i F\, B$, and (ii) the homology of the double complex agrees with the construction given in the statement of the exercise. If this is too confusing, read more about the Cartan-Eilenberg resolution below.

24.3.4. The Grothendieck composition-of-functors spectral sequence.

Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are abelian categories, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are a left-exact additive covariant functors, and $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Thus right derived functors of F, G, and $G \circ F$ exist. A reasonable question is: how are they related?

24.3.5. Theorem (Grothendieck composition-of-functors spectral sequence). — *Suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are left-exact additive covariant functors, and $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Suppose further that F sends injective elements of $\mathcal{A}$ to G-acyclic elements of $\mathcal{B}$. Then for each $X \in \mathcal{A}$, there is a spectral sequence with $E^2_{p,q} = R^q G(R^p F(X))$ converging to $R^{p+q}(G \circ F)(X)$.*

We will soon see the Leray spectral sequence as an application (Exercise 24.4.E).

We will have to work to establish Theorem 24.3.5, so the proof is possibly best skipped on a first reading.

24.3.6. $\star$ **Proving Theorem 24.3.5.**

Before we give the proof (in §24.3.8), we begin with some preliminaries to motivate it. In order to discuss derived functors applied to X, we choose an injective resolution of X:

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots .$$

To compute the derived functors $R^pF(X)$, we apply F to the injective resolution $I^\bullet$:

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots .$$

Note that $F(I^p)$ is G-acyclic, by hypothesis of Theorem 24.3.5. If we were to follow our nose, we might take an injective resolution $I^{\bullet,\bullet}$ of the above complex $F(I^\bullet)$ (the "dual" of Hint 24.3.3 — note that the rows and columns are both exact), and apply G, and consider the resulting double complex:

(24.3.6.1)

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & G(I^{0,2}) \arrow[r] \arrow[u] & G(I^{1,2}) \arrow[r] \arrow[u] & G(I^{2,2}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & G(I^{0,1}) \arrow[r] \arrow[u] & G(I^{1,1}) \arrow[r] \arrow[u] & G(I^{2,1}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & G(I^{0,0}) \arrow[r] \arrow[u] & G(I^{1,0}) \arrow[r] \arrow[u] & G(I^{2,0}) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

24.3.D. EXERCISE. Consider the spectral sequence with upward orientation, starting with (24.3.6.1) as page E_0. Show that $E_2^{p,q}$ is $R^p(G \circ F)(X)$ if $q = 0$, and 0 otherwise.

We now see half of the terms in the conclusion of Theorem 24.3.5; we are halfway there. To complete the proof, we would want to consider another spectral sequence, with rightward orientation, but we need to know more about (24.3.6.1); we will build it more carefully.

24.3.7. *Cartan-Eilenberg resolutions.*

Suppose $\cdots \to C^{p-1} \to C^p \to C^{p+1} \to \cdots$ is a complex in an abelian category $\mathcal{B}$. We will build an injective resolution of $C^\bullet$

(24.3.7.1)
$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & I^{0,2} \arrow[r] \arrow[u] & I^{1,2} \arrow[r] \arrow[u] & I^{2,2} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0,1} \arrow[r] \arrow[u] & I^{1,1} \arrow[r] \arrow[u] & I^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & C^0 \arrow[r] \arrow[u] & C^1 \arrow[r] \arrow[u] & C^2 \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

satisfying some further properties.

We first define some notation for functions on a complex.

- Let $Z^p(K^\bullet)$ be the kernel of the pth differential of a complex $K^\bullet$.
- Let $B^{p+1}(K^\bullet)$ be the image of the pth differential of a complex $K^\bullet$. (The superscript is chosen so that $B^{p+1}(K^\bullet) \subset K^{p+1}$.)
- As usual, let $H^p(K^\bullet)$ be the homology at the pth step of a complex $K^\bullet$.

For each p, we have complexes

(24.3.7.2)
$$0 \longrightarrow Z^p(C^\bullet) \longrightarrow Z^p(I^{\bullet,0}) \longrightarrow Z^p(I^{\bullet,1}) \longrightarrow \cdots$$

$$0 \longrightarrow B^p(C^\bullet) \longrightarrow B^p(I^{\bullet,0}) \longrightarrow B^p(I^{\bullet,1}) \longrightarrow \cdots$$

$$0 \longrightarrow H^p(C^\bullet) \longrightarrow H^p(I^{\bullet,0}) \longrightarrow H^p(I^{\bullet,1}) \longrightarrow \cdots$$

We will construct (24.3.7.1) so that the three complexes (24.3.7.2) are all injective resolutions (of their first nonzero terms). We begin by choosing injective resolutions $B^{p,*}$ of $B^p(C^\bullet)$ and $H^{p,*}$ of $H^p(C^\bullet)$; these will eventually be the last two lines of (24.3.7.2).

24.3.E. EXERCISE. Describe an injective resolution $Z^{p,*}$ of $Z^p(C^\bullet)$ (the first line of (24.3.7.2)) making the following diagram a short exact sequence of complexes.

(24.3.7.3)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^p(C^\bullet) \arrow[r] \arrow[d] & B^{p,0} \arrow[r] \arrow[d] & B^{p,1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & Z^p(C^\bullet) \arrow[r] \arrow[d] & Z^{p,0} \arrow[r] \arrow[d] & Z^{p,1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & H^p(C^\bullet) \arrow[r] \arrow[d] & H^{p,0} \arrow[r] \arrow[d] & H^{p,1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hint: the "dual" problem was solved in (24.1.2.1), by a "horseshoe construction".

24.3.F. EXERCISE. Describe an injective resolution $I^{p,*}$ of C^p making the following diagram a short exact sequence of complexes.

(24.3.7.4)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z^p(C^\bullet) \arrow[r] \arrow[d] & Z^{p,0} \arrow[r] \arrow[d] & Z^{p,1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & C^p \arrow[r] \arrow[d] & I^{p,0} \arrow[r] \arrow[d] & I^{p,1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B^{p+1}(C^\bullet) \arrow[r] \arrow[d] & B^{p+1,0} \arrow[r] \arrow[d] & B^{p+1,1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(The hint for the previous problem applies again. We remark that the first nonzero columns of (24.3.7.3) and (24.3.7.4) appeared in (2.6.5.3).)

24.3.G. EXERCISE/DEFINITION. Build an injective resolution (24.3.7.1) of $C^\bullet$ such that $Z^{p,*} = Z^p(I^{\bullet,*})$, $B^{p,*} = B^p(I^{\bullet,*})$, $H^{p,*} = H^p(I^{\bullet,*})$, so the three complexes (24.3.7.2) are injective resolutions. This is called a **Cartan-Eilenberg resolution** of $C^\bullet$.

24.3.8. *Proof of the Grothendieck spectral sequence, Theorem 24.3.5.* We pick up where we left off before our digression of Cartan-Eilenberg resolutions. Choose an injective resolution $I^\bullet$ of X. Apply the functor F, then take a Cartan-Eilenberg resolution $I^{\bullet,\bullet}$ of $FI^\bullet$, and then apply G, to obtain (24.3.6.1).

Exercise 24.3.D describes what happens when we take (24.3.6.1) as E_0 in a spectral sequence with upward orientation. So we now consider the rightward orientation.

From our construction of the Cartan-Eilenberg resolution, we have injective resolutions (24.3.7.2), and short exact sequences

$$0 \longrightarrow B^p(I^{\bullet,q}) \longrightarrow Z^p(I^{\bullet,q}) \longrightarrow H^p(I^{\bullet,q}) \longrightarrow 0 \tag{24.3.8.1}$$

$$0 \longrightarrow Z^p(I^{\bullet,q}) \longrightarrow I^{p,q} \longrightarrow B^{p+1}(I^{\bullet,q}) \longrightarrow 0 \tag{24.3.8.2}$$

of *injective* objects (from the columns of (24.3.7.3) and (24.3.7.4)). This means that both are *split* exact sequences (the central term can be expressed as a direct sum of the outer two terms), so upon application of G, both exact sequences remain exact.

Applying the left-exact functor G to

$$0 \longrightarrow Z^p(I^{\bullet,q}) \longrightarrow I^{p,q} \longrightarrow I^{p+1,q},$$

we find that $GZ^p(I^{\bullet,q}) = \ker(GI^{p,q}, GI^{p+1,q})$. But this kernel is the *definition* of $Z^p(GI^{\bullet,q})$, so we have an induced isomorphism $GZ^p(I^{\bullet,q}) = Z^p(GI^{\bullet,q})$ ("G and Z^p commute"). From the exactness of (24.3.8.2) upon application of G, we see that $GB^{p+1}(I^{\bullet,q}) = B^{p+1}(GI^{\bullet,q})$ (both are $\mathrm{coker}(GZ^p(I^{\bullet,q}) \to GI^{p,q})$). From the exactness of (24.3.8.1) upon application of G, we see that $GH^p(I^{\bullet,q}) = H^p(GI^{\bullet,q})$ (both are $\mathrm{coker}(GB^p(I^{\bullet,q}) \to GZ^p(I^{\bullet,q}))$ — so "G and H^p commute").

We return to considering the rightward-oriented spectral sequence with (24.3.6.1) as E_0. Taking cohomology in the rightward direction, we find $E_1^{p,q} = H^p(GI^{\bullet,q}) = GH^p(I^{\bullet,q})$ (as G and H^p commute). Now $H^p(I^{\bullet,q})$ is an injective resolution of $(R^pF)(X)$ (the last resolution of (24.3.7.2)). Thus when we compute E_2 by using the vertical arrows, we find $E_2^{p,q} = R^qG(R^pF(X))$.

You should now verify yourself that this (combined with Exercise 24.3.D) concludes the proof of Theorem 24.3.5. $\square$

24.4 ⋆ Derived functor cohomology of $\mathscr{O}$-modules

We wish to apply the machinery of derived functors to define cohomology of quasicoherent sheaves on a scheme X. Sadly, this category $QCoh_X$ usually doesn't have enough injectives! Fortunately, the larger category $Mod_{\mathscr{O}_X}$ does.

24.4.1. Theorem. — *Suppose $(X, \mathscr{O}_X)$ is a ringed space. Then the category of $\mathscr{O}_X$-modules $Mod_{\mathscr{O}_X}$ has enough injectives.*

As a side benefit (of use to others more than us), taking $\mathscr{O}_X = \underline{\mathbb{Z}}$, we see that the category of sheaves of abelian groups on a fixed topological space have enough injectives.

We prove Theorem 24.4.1 in a series of exercises. Suppose $\mathscr{F}$ is an $\mathscr{O}_X$-module. We will exhibit an injection $\mathscr{F} \hookrightarrow \mathscr{Q}'$ into an injective $\mathscr{O}_X$-module. For each $x \in X$, choose an inclusion $\mathscr{F}_x \hookrightarrow Q_x$ into an injective $\mathscr{O}_{X,x}$-module (possible as the category of $\mathscr{O}_{X,x}$-modules has enough injectives, Exercise 24.2.J).