11.4.3. Proposition ("Generically finite implies generally finite"). — *Suppose $\pi : X \to Y$ is a generically finite morphism of irreducible k-varieties of dimension n. Then there is a dense open subset $V \subset Y$ above which π is finite.*

(If you wish, you can later relax the irreducibility hypothesis to simply requiring X and Y to be simply of pure dimension n.)

Proof. As in the proof of Proposition 11.4.1, we may assume that Y is affine, and that π is dominant.

11.4.F. EXERCISE. Prove the result under the additional assumption that X is affine. Hint: follow the appropriate part of the proof of Proposition 11.4.1.

For the general case, suppose that $X = \cup_{i=1}^{n} U_i$, where the U_i are affine open subschemes of X. By Exercise 11.4.F, there are dense open subsets $V_i \subset Y$ over which $\pi|_{U_i}$ is finite. By replacing Y by an affine open subset of $\cap V_i$, we may assume that $\pi|_{U_i}$ is finite.

11.4.G. EXERCISE. Show that π is closed. Hint: you will just use that $\pi|_{U_i}$ is closed, and that there are a finite number of U_i.

Then $X \setminus U_1$ is a closed subset, so $\pi(X \setminus U_1)$ is closed.

11.4.H. EXERCISE. Show that this closed subset is not all of Y.

Define $V := Y \setminus \pi(X \setminus U_1)$. Then π is finite above V: it is the restriction of the finite morphism $\pi|_{U_1} : U_1 \to Y$ to the open subset V of the target Y. □

11.4.4. Aside: Other semicontinuities.

Semicontinuity is a recurring theme in algebraic geometry. It is worth keeping an eye out for it. Other examples include the following.

(i) fiber dimension (Theorem 11.4.2 above)
(ii) the rank of a matrix of functions (because rank drops on closed subsets, where various discriminants vanish)
(iii) the rank of a finite type quasicoherent sheaf (Exercise 13.7.J)
(iv) degree of a finite morphism, as a function of the target (§13.7.5)
(v) dimension of tangent space at closed points of a variety over an algebraically closed field (Exercise 21.2.J)
(vi) rank of cohomology groups of coherent sheaves, in proper flat families (Theorem 28.1.1)

All but (ii) are upper semicontinuous; (ii) is a lower semicontinuous function.

11.4.5. $\star\star$ **Generalizing results of §11.4 beyond varieties.** The above arguments can be extended to more general situations than varieties. We remain in the locally Noetherian situation for safety, until the last sentence of §11.4.6. One fact used repeatedly was that codimension is the difference of dimensions (Theorem 11.2.9). This holds much more generally; see Remark 11.2.10 on catenary rings. Extensions of Proposition 11.4.1 should require that π be finite type (which was automatic in the statement of Proposition 11.4.1, by the Cancellation Theorem 10.1.19 for finite type morphisms). In the proof of Proposition 11.4.1, we use that the dimension of the the generic fiber of the morphism $\pi : X \to Y$ of irreducible schemes is