**12.3.9. Algebraic Hartogs' Lemma for Noetherian normal schemes.**

Hartogs' Lemma in several complex variables states (informally) that a holomorphic function defined away from a codimension two set can be extended over that. We now describe an algebraic analog, for Noetherian normal schemes. We will use this repeatedly and relentlessly when connecting line bundles and divisors.

**12.3.10. Algebraic Hartogs' Lemma. —** *Suppose $A$ is a Noetherian normal integral domain. Then*

$$A = \cap_{\mathfrak{p} \text{ codimension } 1} A_{\mathfrak{p}}.$$

The equality takes place in $K(A)$; recall that any localization of an integral domain $A$ is naturally a subset of $K(A)$ (Exercise 2.3.C). Warning: few people call this Algebraic Hartogs' Lemma. I call it this because it parallels the statement in complex geometry.

One might say that if $f \in K(A)$ does not lie in $A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then $f$ has a pole at $[\mathfrak{p}]$, and if $f \in K(A)$ lies in $\mathfrak{p}A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then $f$ has a zero at $[\mathfrak{p}]$. It is worth interpreting Algebraic Hartogs' Lemma as saying that *a rational function on a normal scheme with no poles is in fact regular* (an element of $A$). Informally: *"Noetherian normal schemes have the Hartogs property."* (We will properly define zeros and poles in §13.4.8, see also Exercise 13.4.H.)

One can state Algebraic Hartogs' Lemma more generally in the case that Spec $A$ is a Noetherian normal scheme, meaning that $A$ is a product of Noetherian normal integral domains; the reader may wish to do so.

**12.3.11.** ⋆ *Proof.* (This proof may be stated completely algebraically, but we state it as geometrically as possible, at the expense of making it longer.) The left side is obviously contained in the right, so assume some $x$ lies in every $A_{\mathfrak{p}}$ but not in $A$. As in the proof of Proposition 6.4.2, we measure the failure of $x$ to be a function (an element of Spec $A$) with the "ideal of denominators" $I$ of $x$:

$$I := \{r \in A \ : \ rx \in A\}.$$

As $1 \notin I$, we have $I \neq A$. Choose a minimal prime $\mathfrak{q}$ containing $I$.

Our second step in obtaining a contradiction is to focus near the point $[\mathfrak{q}]$, i.e. focus attention on $A_{\mathfrak{q}}$ rather than $A$, and as a byproduct notice that $\operatorname{codim} \mathfrak{q} > 1$. The construction of the ideal of denominators behaves well with respect to localization — if $\mathfrak{p}$ is any prime, then the ideal of denominators of $x$ in $A_{\mathfrak{p}}$ is $I_{\mathfrak{p}}$, and it again measures "the failure of Algebraic Hartogs' Lemma for $x$," this time in $A_{\mathfrak{p}}$. But Algebraic Hartogs' Lemma is vacuously true for dimension 1 rings, so no codimension 1 prime contains $I$. Thus $\mathfrak{q}$ has codimension at least 2. By localizing at $\mathfrak{q}$, we can assume that $A$ is a local ring with maximal ideal $\mathfrak{q}$, and that $\mathfrak{q}$ is the *only* prime containing $I$.

In the third step, we construct a suitable multiple $z$ of $x$ that is still not a function on Spec $A$, so that multiplying $z$ by anything vanishing at $[\mathfrak{q}]$ results in a function. (Translation: $z \notin A$, but $z\mathfrak{q} \subset A$.) As $\mathfrak{q}$ is the only prime containing $I$, $\sqrt{I} = \mathfrak{q}$ (Exercise 4.4.F), so as $\mathfrak{q}$ is finitely generated, there is some $n$ with $I \supset \mathfrak{q}^n$ (do you see why?). Take the minimal such $n$, so $I \not\supset \mathfrak{q}^{n-1}$, and choose any $y \in \mathfrak{q}^{n-1} - I$. Let $z = yx$. As $y \notin I$, $z \notin A$. On the other hand, $\mathfrak{q}y \subset \mathfrak{q}^n \subset I$, so $\mathfrak{q}z \subset Ix \subset A$, so $\mathfrak{q}z$ is an ideal of $A$, completing this step.

Finally, we have two cases: either there is function vanishing on $[\mathfrak{q}]$ that, when multiplied by $z$, doesn't vanish on $[\mathfrak{q}]$; or else every function vanishing on $[\mathfrak{q}]$, multiplied by $z$, still vanishes on $[\mathfrak{q}]$. Translation: (i) either $\mathfrak{q}z$ is not contained in $\mathfrak{q}$, or (ii) it is.

(i) If $\mathfrak{q}z \subset \mathfrak{q}$, then we would have a finitely-generated $A$-module (namely $\mathfrak{q}$) with a faithful $A[z]$-action, forcing $z$ to be integral over $A$ (and hence in $A$, as $A$ is integrally closed) by Exercise 8.2.J, yielding a contradiction.

(ii) If $\mathfrak{q}z$ is an ideal of $A$ not contained in the unique maximal ideal $\mathfrak{q}$, then it must be $A$! Thus $\mathfrak{q}z = A$ from which $\mathfrak{q} = A(1/z)$, from which $\mathfrak{q}$ is principal. But then $\operatorname{codim} \mathfrak{q} = \dim A \leq \dim_{A/\mathfrak{q}} \mathfrak{q}/\mathfrak{q}^2 \leq 1$ by Nakayama's Lemma 8.2.H, contradicting $\operatorname{codim} \mathfrak{q} \geq 2$. □

## 12.4 Dimensions of fibers of morphisms of varieties

In this section, we show that the dimensions of fibers of morphisms of varieties behaves in a way you might expect from our geometric intuition. What we need about varieties is Theorem 12.2.8 (codimension is the difference of dimensions). We discuss generalizations in §12.4.3.

We begin with an inequality that holds more generally in the locally Noetherian setting.

**12.4.A.** EXERCISE (CODIMENSION BEHAVES AS YOU MIGHT EXPECT FOR A MORPHISM). Suppose $\pi : X \to Y$ is a morphism of locally Noetherian schemes, and $p \in X$ and $q \in Y$ are points such that $q = \pi(p)$. Show that

$$\operatorname{codim}_X p \leq \operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1} q} p.$$

(Does this agree with your geometric intuition? You should be able to come up with enlightening examples where equality holds, and where equality fails. We will see that equality always holds for sufficiently nice — flat — morphisms, see Proposition 25.5.5.) Hint: take a system of parameters for $q$ "in $Y$", and a system of parameters for $p$ "in $\pi^{-1} q$", and use them to find $\operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1} q} p$ elements of $\mathscr{O}_{X,p}$ cutting out $\{[\mathfrak{m}]\}$ in $\operatorname{Spec} \mathscr{O}_{X,p}$. Use Exercise 12.3.G (where "system of parameters" was defined).

We now show that the inequality of Exercise 12.4.A is actually an equality over "most of Y" if Y is an irreducible variety.

**12.4.1. Proposition. —** *Suppose $\pi : X \to Y$ is a (necessarily finite type) morphism of irreducible $k$-varieties, with $\dim X = m$ and $\dim Y = n$. Then there exists a nonempty open subset $U \subset Y$ such that for all $y \in U$, $f^{-1}(y) = \varnothing$ or $\dim f^{-1}(y) = m - n$.*

*Proof.* By shrinking $Y$ if necessary, we may assume that $Y$ is affine, say $\operatorname{Spec} B$. We may also assume that $X$ is affine, say $\operatorname{Spec} A$. (Reason: cover $X$ with a finite number of affine open subsets $X_1, \dots, X_a$, and take the intersection of the $U$'s for each of the $\pi|_{X_i}$.) If $\pi$ is not dominant, then we are done, as by Chevalley's Theorem 8.4.2, the image misses a dense open subset $U$ of $\operatorname{Spec} A$. So assume now that $\pi$ is dominant.